Theory of Disordered Systems

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Lecture 7: Properties of percolation clusters.

Let us recuperate what we have learned about percolation clusters (for simplicity we adopt the language of the site model here). Below the percolation concentration there are only finite clusters of intact sites, and their size distribution $n_s(p)$ (the probability to find a cluster of s sites among *all clusters*) can be obtained at least numerically. Above the critical concentration an infinite cluster appears. The density of this cluster P (i.e. the probability to find a site belonging to it among all sites) scales as $P_\infty \propto (p-p_c)^\beta$ with $0 < \beta \leq 1$. The typical size (gyration radius) of a finite cluster behaves as $\xi \propto |p-p_c|^{-\nu}$ on both sides of the transition. This ξ defines the scale above which the whole system can be considered as homogeneous, i.e. above p_c it also gives the size of the largest holes in the percolation cluster.

At length scales smaller than ξ the mean density of the percolation cluster measured as the number of sites N within a domain of radius L divided by L^d explicitly depends on L. It was (numerically) found that this dependence corresponds to a power law

$$N \propto L^{d_f},$$

which allows us to define the fractal (mass) dimension of the infinite cluster d_f which is smaller than d. This d_f is connected with critical indices β and via a simple scaling argument: The density of the cluster grows at $L < \xi$ as

$$\rho(L) \propto N/L^d = L^{d_f - d}.$$

At $L > \xi$ it stagnates and is equal to $P\infty(p)$. Exactly a ξ we thus have $\xi^{d_f - d} \simeq P_\infty$. Expressing both ξ and P_∞ as functions of $p - p_c$ we get $(p - p_c)^{-\nu(d_f - d)} = (p-p_c)^\beta$ so that

$$d_f = d - \frac{\beta}{\nu}.$$

We thus see that just at percolation threshold the infinite cluster is fractal (has a dimension smaller than the one of the embedding space), i.e. is an extremely inhomogeneous structure characterized by holes on all scales. Such structures are quite peculiar and worth discussing (see next lecture). In the present lecture we will find the connection between the geometrical properties of the clusters and the properties of the distribution of finite clusters $n_s(p)$. To gain some intuition on what can happen close to the transition we first revisit our tree model discussed in the previous lecture.

1 Percolation on a tree

1.1 The correlation length

Let us recall the situation with a tree when

$$P_\infty \propto (p - p_c)^1$$

with $p_c = 1/(Z-1)$. In this case the critical exponent β is $\beta = 1$. The correlation function $g(r)$, giving the probability that two sites at distance r from each other are connected by an intact path, for the tree can be easily calculated in the *chemical space* (where the distance r is measured as the number of steps (bonds) of the path connecting each two points; on a tree there is exactly one such path):

$$g(r) = Z(Z-1)^{r-1}p^r$$

where $Z(Z-1)^{r-1}$ is exactly the number of paths of length r. This gives us the possibility to calculate the correlation length in the chemical space

$$\tilde{\xi}^2(p) = \frac{\sum_{r=0}^{\infty} r^2 g(r)}{\sum_{r=0}^{\infty} g(r)}$$

(at least below p_c when both sums converge). The trick used for such a calculation is quite standard: we note that

$$\sum_{r=0}^{\infty} g(r) = \frac{Z}{Z-1} \sum_{r=0}^{\infty} [(Z-1)p]^r = \frac{Z}{Z-1} \frac{1}{1-(Z-1)p}$$

and denote $(Z-1)p = x$ so that

$$\sum_{r=0}^{\infty} g(r) = \frac{Z}{Z-1} \frac{1}{1-x}$$

We moreover note that

$$\sum_{r=0}^{\infty} r^2 g(r) = \frac{Z}{Z-1}\sum_{r=0}^{\infty} r^2[(Z-1)p]^r$$

so that

$$\begin{aligned}\sum_{r=0}^{\infty} r^2 g(r) = \sum_{r=0}^{\infty} r^2 x^r &= \frac{d}{dx}x\frac{d}{dx}\frac{Z}{Z-1}\sum_{r=0}^{\infty}[x]^r \\ = \frac{Z}{Z-1}\frac{d}{dx}x\frac{d}{dx}\frac{1}{1-x} &= \frac{Z}{Z-1}\frac{1+x}{(1-x)^3}.\end{aligned}$$

We finally get

$$\begin{aligned}\tilde{\xi}^2(p) &= \frac{1+x}{(1-x)^2} = \frac{1+(Z-1)p}{[1-(Z-1)p]^2} \\ &= (Z-1)^{-1}\frac{(Z-1)^{-1}+p}{[(Z-1)^{-1}-p^2]} = p_c\frac{p_c+p}{(p_c-p)^2}.\end{aligned}$$

The same expression holds essentially also on the other side of transition (note that in this case the infinite cluster has to be disregarded). Thus, $\tilde{\xi} \propto |p-p_c|^{-1}$. In the (infinitely dimensional) Euclidean space in which the tree leaves each such path corresponds to a trajectory of a random walk, and its "real" length ξ scales as a root of its chemical length and therefore $\xi \propto \tilde{\xi}^{1/2} \propto (p-p_c)^{1/2}$, so that $\nu = 1/2$.

1.2 Cluster size distribution

Let us consider a cluster with s sites. For this cluster its t *perimeters sites* (i.e. immediate neighbors of the cluster sites not belonging to the cluster) has to be blocked, otherwise the cluster would be larger. Therefore the probability to find such a cluster is

$$n_s = N_{s,t}p^s(1-p)^t,$$

where $N_{s,t}$ is the number of different geometric configurations of a cluster with s sites and t perimeter sites. On a whatever lattice other than a tree calculation of t and $N_{s,t}$ is a hard job. On a tree t is a simple function of s:

$$t(s) = Z + (Z-2)(s-1).$$

To see this it is enough to note that $t(1) = Z$ and that

$$t(s+1) = t(s) - 1 + (Z-1) = t(s) + (Z-2)$$

(passing from a cluster with s sites to a one with $s+1$ sites we change one of the perimeter sites for an internal one and add $Z-1$ new perimeter sites to block its neighbors), and to use mathematical induction.

Therefore

$$n_s = N_s p^s (1-p)^{Z+(Z-2)(s-1)} = N_s (1-p)^2 [p(1-p)^{Z-2}]^s.$$

Let us assume that exactly at p_c n_s

$$n_s(p_c) = N_s (1-p_c)^2 [p_c(1-p_c)^{Z-2}]^s$$

is known (and follows for large s a power law $n_s(p_c) \propto s^{-\tau}$, which can be verified later). Then, at any other concentration than p_c we can write

$$n_s(p) = N_s(1-p)^2[p(1-p)^{Z-2}]^s = n_s(p_c)\frac{(1-p)^2[p(1-p)^{Z-2}]^s}{(1-p_c)^2[p_c(1-p_c)^{Z-2}]^s}$$

i.e. we see that $n_s(p) = n_s(p_c) f_s(p)$ where the function

$$f_s(p) = \frac{(1-p)^2}{(1-p_c)^2}\left[\frac{p(1-p)^{Z-2}}{p_c(1-p_c)^{Z-2}}\right]^s$$

tends to unity for $r \to p_c$ for any s. For $p = p_c + \delta p = (Z-1)^{-1} + \delta p$ we can expand the function in the square brackets and get

$$f_s(p) \simeq 1 - \frac{1}{2}\frac{(Z-1)^3}{Z-2}(\delta p)^2 = 1 - \frac{1}{2p_c^2(1-p_c)}(p-p_c)^2.$$

(the first term of the expansion vanishes at p_c). Therefore close to p_c and for s large

$$f_s(p) \simeq \left[1 - \frac{1}{2p_c^2(1-p_c)}(p-p_c)^2\right]^s \approx \exp\left[-\frac{(p-p_c)^2}{2p_c^2(1-p_c)}s\right],$$

a strongly decaying (exponential) function. This shows that the cluster size distribution at p follows the one at p_c for smaller clusters, but has a cutoff at the maximal cluster size $s_c \propto |p-p_c|^{-2}$.

Now we use the fact that the mean size of a finite cluster

$$S(p) = \frac{1}{p}\sum_{s=1}^{\infty} s \cdot s n_s(p)$$

and calculate it using our expression for $n_s(p) \propto s^{-\tau} \exp[-\mathrm{const}\cdot(|p-p_c|^2 s)]$:

$$\begin{aligned} S(p) &\simeq \int_1^\infty s^{2-\tau} e^{-\mathrm{const}\cdot(|p-p_c|^2 s)} \approx \int_0^\infty s^{2-\tau} e^{-\mathrm{const}\cdot(|p-p_c|^2 s)} \\ &\simeq \int_0^{|p-p_c|^{-2}} s^{2-\tau} \propto |p-p_c|^{2(\tau-3)}. \end{aligned}$$

On the other hand,

$$S(p) = 1 + \sum_{r=1}^{\infty} g(r) = p_c \frac{1+p}{p_c - p} \propto (p_c - p)^{-1}$$

(at least below the percolation concentration). Therefore $2(\tau - 3) = 1$ i.e.

$$\tau = \frac{5}{2},$$

so that for larger clusters

$$n_s(p) \propto s^{-5/2} e^{-\mathrm{const}\cdot(|p-p_c|^2 s)}.$$

There is strong numerical evidence that also in general

$$n_s(p) = s^{-\tau} f_{\pm}\left(|p-p_c|^{1/\sigma} s\right) \qquad (1)$$

where the values of critical exponents τ and σ may depend on the lattice (essentially only on its dimensionality), and the functions f may differ below and above transition. In our case

$$\sigma = \frac{1}{2}.$$

The total balance of probabilities is given by

$$1 = (1-p) + P_\infty + p \sum_s s n_s(p)$$

(the first term in the r.h.s. is the probability that the site is broken, the second one is the probability that it is intact and belongs to an infinite cluster, and the third one corresponds to the probability that it is intact and belongs to a finite cluster. Therefore

$$P_\infty = p - p \sum_s s n_s(p) \tag{2}$$

Exactly at p_c the density P_∞ vanishes, and

$$0 = p_c - p_c \sum_s s n_s(p_c).$$

Using expressions from our previous discussion we get

$$\begin{aligned} \frac{P_\infty}{p} &= \sum_s s[n_s(p_c) - n_s(p)] = \sum_s s n_s(p_c)[1 - f_s(p)] \\ &\simeq \int_0^\infty s s^{-\tau}(1 - e^{-\text{const}\cdot(|p-p_c|^2 s)}) ds. \end{aligned}$$

Introducing the new integration variable $z = |p - p_c|^2 s$ and assuming that the integral in

$$\frac{P_\infty}{p} \simeq |p - p_c|^{2(\tau-2)} \int_0^\infty s s^{-\tau}(1 - e^{-\text{const}\cdot(|p-p_c|^2 s)}) ds.$$

converges, we compare now this expression with our previous one for $P_\infty \propto (p - p_c)^1$, to obtain the same value of tau as before: $2\tau - 4 = 1$, i.e. $\tau = 5/2$. In general for σ different from 1/2 and for β different from one we will get

$$\beta = \frac{\tau - 2}{\sigma},$$

an expression of general validity. Haven learned about cluster properties in a tree and assuming that the overall behavior of the cluster size distribution is always given by Eq.(1) we can turn to systems other then the tree and obtain general relations between the critical exponents.

2 Scaling theory of percolation clusters

The expression for $n_s(p)$, Eq.(1) allows to obtain the moments of the cluster sizes:

$$\begin{aligned} M_k &= \sum_{s=1}^{\infty} s^k n_s(p) = \sum_{s=1}^{\infty} s^{k-\tau} f_{\pm}(|p-p_c|^{\frac{1}{\sigma}} s) \\ &\sim \sum_{s=1}^{|p-p_c|^{-\frac{1}{\sigma}}} s^{k-\tau} \simeq \int_1^{|p-p_c|^{-\frac{1}{\sigma}}} s^{k-\tau} ds \\ &\simeq s^{k+1-\tau}\Big|_1^{|p-p_c|^{-\frac{1}{\sigma}}} . \end{aligned}$$

Depending on k (which does not have to be a whole number) this expression can either diverge for $p \to p_c$ or converge to a constant. Thus, for $k > \tau - 1$

$$M_k \propto |p-p_c|^{-\frac{k+1-\tau}{\sigma}}$$

and for $k < \tau - 1$

$$M_k = \text{const} - A|p-p_c|^{-\frac{k+1-\tau}{\sigma}}$$

with A being some prefactor. The zeroth moment $M_0 = \sum_{s=1}^{\infty} n_s(p)$ is evidently unity, due to normalization. The first moment

$$M_1 = \sum_{s=1}^{\infty} s n_s(p) = \text{const} - A|p-p_c|^{-\frac{2-\tau}{\sigma}}$$

Comparing this with the expression (2) from the previous section leading us to

$$\sum_s s n_s(p) = 1 - P_\infty$$

gives us the value of the constant and the expression for β already discussed. The second moment M_2 giving

$$M_2 = \sum_{s=1}^{\infty} s^2 n_s(p) \propto |p-p_c|^{-\frac{3-\tau}{\sigma}}$$

defines the value of the critical exponent γ:

$$\gamma = \frac{3-\tau}{\sigma}.$$

Our nest aim will be to connect the exponent ν of the correlation length with τ and σ. On one hand, $\xi(atleastbelow \mathrm{p}_c)$ is defined as

$$\xi^2(p) = \frac{\sum_{r=0}^{\infty} r^2 g(r)}{\sum_{r=0}^{\infty} g(r)},$$

on the other hand, as the mean gyration radius of the finite cluster, it can be obtained via

$$\xi^2 = \frac{\sum_{r=0}^{\infty} r^2(s) s^2 n(s)}{\sum_{r=0}^{\infty} s^2 n(s)}$$

where $r^2(s)$ is a typical distance between the two sites of a cluster of s sites, and $s^2 n(s)$ is proportional to the probability that these two sites belong to the same cluster. Assuming that large finite clusters have the same fractal geometry as the infinite one we can take

$$r(s) \propto s^{1/d_f}$$

so that

$$\xi^2 = \frac{M_{2+2/d_f}}{M_2} \simeq |p - p_c|^{-\frac{2}{d_f \sigma}}$$

Therefore

$$\nu = \frac{2}{d_f \sigma} = \frac{2}{(d - \beta/\nu)\sigma}.$$

In the last equation the relation $d_f = d - \beta/\nu$ was used. Resolving this last equation as an equation for ν we get

$$\nu = \frac{\tau - 1}{d\sigma}.$$

Combining this relation with the ones for β and γ we get

$$d\nu = 2\beta + \gamma.$$

Introducing the “classical” values of critical exponents (the one for the tree) $\beta = 1$, $\gamma = 1$ and $\nu = 1/2$ we obtain that these ones correspond to $d = 6$ which is the upper critical dimension for percolation problems.