Elektrostatik / Magnetostatik SS 2013

Übungsblatt 10

Aufgabe 10.1 $\quad \operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}, \quad \vec{E} = -\nabla\phi \qquad \rho = \text{const} = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3}$

$$\Delta\phi = -\frac{\rho}{\varepsilon_0} \qquad \Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\underbrace{\cancel{\frac{\partial}{\partial\theta}}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)}_{\to 0} + \frac{1}{r^2\sin^2\theta}\underbrace{\cancel{\frac{\partial^2\phi}{\partial\varphi^2}}}_{\to 0} = -\frac{\rho}{\varepsilon_0}$$

$$\Phi = \Phi(r): \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = -\frac{\rho}{\varepsilon_0}$$

Aussen: $r > R: \rho = 0 \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_a}{dr}\right) = 0 \qquad r^2\frac{d\Phi_a}{dr} = C_1 \qquad \frac{d\Phi_a}{dr} = \frac{C_1}{r^2}$

$$\boxed{\Phi_a = C_1\int\frac{dr}{r^2} = -\frac{C_1}{r} + C_2} \quad (r > R)$$

Innen: $r < R: \rho \neq 0 \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_i}{dr}\right) = -\frac{\rho}{\varepsilon_0} \qquad \frac{d}{dr}\left(r^2\frac{d\Phi_i}{dr}\right) = -\frac{\rho}{\varepsilon_0}r^2 \qquad r^2\frac{d\Phi_i}{dr} = -\frac{\rho}{\varepsilon_0}\left(\frac{r^3}{3} + D_1\right)$

$$\frac{d\Phi_i}{dr} = -\frac{\rho}{\varepsilon_0}\left(\frac{r}{3} + \frac{D_1}{r^2}\right) \qquad \boxed{\Phi_i = -\frac{\rho}{\varepsilon_0}\left[\frac{r^2}{6} - \frac{D_1}{r} + D_2\right]} \quad (r < R)$$

(4 Konstanten: C_1, C_2, D_1, D_2 $\Rightarrow$ 4 Randbedingungen)

Randbedingungen

① $r \to 0: |\Phi_i| < \infty \Rightarrow \boxed{D_1 = 0}$ ② $r \to \infty: \Phi_a \to 0 \Rightarrow \boxed{C_2 = 0}$

③ Stetigkeit von Φ: $\Phi_i(r=R) = \Phi_a(r=R)$

④ Stetigkeit von Feld: $E_i(r=R) = E_a(r=R) \Rightarrow \left.\frac{d\Phi_i}{dr}\right|_{r=R} = \left.\frac{d\Phi_a}{dr}\right|_{r=R}$

$$\Phi_a = -\frac{C_1}{r} \qquad \Phi_i = -\frac{\rho}{\varepsilon_0}\left(\frac{r^2}{6} + D_2\right)$$

$$\Phi_a' = \frac{C_1}{r^2} \qquad \Phi_i' = -\frac{\rho}{\varepsilon_0}\frac{r}{3}$$

③: $-\frac{C_1}{R} = -\frac{\rho}{\varepsilon_0}\left[\frac{R^2}{6} + D_2\right] \Rightarrow -\frac{\rho}{\varepsilon_0}D_2 = \frac{\rho R^2}{6\varepsilon_0} - \frac{C_1}{R} = \frac{\rho R^2}{6\varepsilon_0} + \frac{\rho R^2}{3\varepsilon_0} = \frac{\rho R^2}{2\varepsilon_0}$

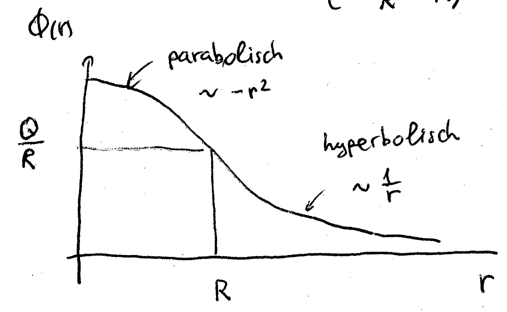
④: $\frac{C_1}{R^2} = -\frac{\rho}{\varepsilon_0}\frac{R}{3} \Rightarrow \boxed{C_1 = -\frac{\rho R^3}{3\varepsilon_0}}$

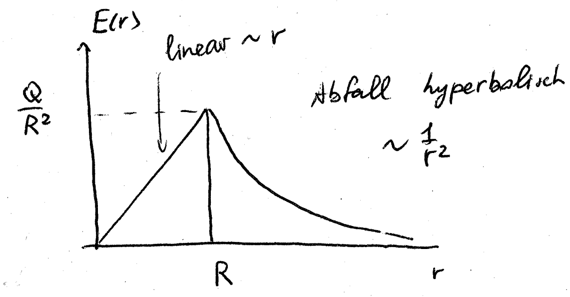
$\Phi_a(r) = \frac{\rho}{\varepsilon_0}\frac{R^3}{3r} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ — Coulomb-Gesetz $(r > R)$ $\left(\rho = \frac{Q}{\frac{4}{3}\pi R^3}\right)$

$\Phi_i(r) = \frac{\rho}{\varepsilon_0}\left[\frac{R^2}{2} - \frac{r^2}{6}\right] = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right] \quad (r < R)$

$$\Phi(r) = \begin{cases} \frac{\rho}{\varepsilon_0}\frac{R^3}{3r}, & r > R \\ \frac{\rho}{\varepsilon_0}\left[\frac{R^2}{2} - \frac{r^2}{6}\right], & r \leq R \end{cases} = \left(\frac{1}{4\pi\varepsilon_0}\right)\begin{cases} \frac{Q}{r}, & r > R \\ \frac{Q}{2R}\left[3 - \left(\frac{r}{R}\right)^2\right], & r \leq R \end{cases}$$

$$E = -\frac{d\Phi}{dr} = \left(\frac{1}{4\pi\varepsilon_0}\right)\begin{cases} \frac{Q}{r^2}, & r > R \\ \frac{Q}{R^2}\left(\frac{r}{R}\right), & r \leq R \end{cases}$$





b) Energiedichte des E-Feldes (Vakuum)

$$w = \frac{\varepsilon_0}{2}\vec{E}^2 \qquad \vec{E}(r) = E(r)\vec{e}_r \;;\quad E(r) = \left(\frac{Q}{4\pi\varepsilon_0}\right)\begin{cases}\frac{1}{r^2}, & r > R\\ \frac{\cancel{Q}}{R^3}r, & r \leq R\end{cases}$$

$$w = \frac{Q^2}{16\pi^2\varepsilon_0^2}\cdot\frac{\varepsilon_0}{2}\cdot\begin{cases}\frac{1}{r^4}, & r > R\\ \frac{r^2}{R^6}, & r \leq R\end{cases} = \frac{Q^2}{32\pi^2\varepsilon_0}\begin{cases}\frac{1}{r^4}, & r > R\\ \frac{r^2}{R^6}, & r \leq R\end{cases}$$

Gesamtenergie des E-Feldes

$$W = \int w\,dV = \frac{Q^2}{32\pi^2\varepsilon_0}\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\underbrace{\int_0^{\pi}\sin\vartheta\,d\vartheta}_{2}\left\{\int_0^R \frac{r^2}{R^6}r^2dr + \int_R^\infty \frac{1}{r^4}r^2dr\right\} =$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left\{\frac{1}{5}\frac{r^5}{R^6}\Big|_{r=0}^{r=R} - \frac{1}{r}\Big|_{r=R}^{r=\infty}\right\} = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{5R} + \frac{1}{R}\right) = \frac{Q^2}{8\pi\varepsilon_0}\frac{6}{5R} = \frac{1}{4\pi\varepsilon_0}\frac{3}{5}\frac{Q^2}{R} = \frac{3Q^2}{20\pi\varepsilon_0 R}$$

„Elektron"

$$W = mc^2 = \frac{3Q^2}{20\pi\varepsilon_0 R} \qquad R_e = \frac{3}{20}\frac{e^2}{\pi\varepsilon_0 m_e c^2}$$

$$e \approx -1.6\cdot 10^{-19}\,C \qquad c \approx 3.0\cdot 10^{8}\,m/s$$

$$m_e \approx 9.1\cdot 10^{-31}\,kg \qquad \varepsilon_0 \approx 8.9\cdot 10^{-12}\,\frac{C^2}{Nm^2}$$

$$N = kg\cdot m\,\frac{1}{s^2}$$

$$[\varepsilon_0] = \frac{C^2}{Nm^2} = \frac{C^2 s^2}{kg\cdot m^3}$$

$$\frac{[e^2]}{[\varepsilon_0][m_e][c^2]} = \frac{\cancel{C^2}}{\frac{\cancel{C^2}\cancel{s^2}}{\cancel{kg}\cdot m^3}\cdot \cancel{kg}\cdot\frac{m^2}{\cancel{s^2}}} = m \quad ✓$$

$$R_e \approx \frac{3}{20}\frac{(1.6)^2\cdot 10^{-38}}{(3.1)(8.9\cdot 10^{-12})(9.1\cdot 10^{-31})(9\cdot 10^{16})} = \overbrace{\boxed{\frac{(0.15)(2.6)}{(3.1)(8.9)(9.1)\cdot 9}}}^{1.7\cdot 10^{-4}}\ \frac{10^{\overbrace{-38+31}^{-7}}}{10^{\underbrace{16-12}_{4}}} =$$

$$\approx 1.7\cdot 10^{-4}\cdot 10^{-7}\cdot 10^{-4} \approx 1.7\cdot 10^{-15}\,(m)$$