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NONLINEAR EVOLUTION OF THE SKEWED VARICOSE INSTABILITY IN THERMAL CONVECTION

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Rayleigh-Bénard convection in a horizontal layer of incompressible fluid has become a standard illustration for the generation of spatio-temporal structures. The first and simplest pattern which arises, when the increase of the thermal gradient destabilizes the basic homogeneous state, consists of straight convective rolls. The stability of this pattern depends on the Rayleigh number Ra, the Prandtl number Pr and on the wavenumber of the rolls. In this parameter space the domain of stability with respect to various types of perturbations was investigated in a series of papers by Busse and coworkers [1]. In particular, it has turned out [2] that in case of an infinite layer with stress-free boundaries the rolls in a fluid with sufficiently low Prandtl number, $Pr \leq 0.543$, are unstable with respect to skewed varicose perturbations already at the very onset of convection.

In this paper we take into account the finite horizontal extent of the layer through the imposition of periodic lateral conditions. Thus upper bounds for the wavelength of disturbances exist and a narrow region of stability for the steady rolls is obtained. In an earlier paper [3] it was found numerically, that within the range of Ra where the transitions to time-dependency and chaos take place, the dynamics of a high-order finite-dimensional system, obtained from the Boussinesq equations with the help of projection methods, is reasonably well approximated by behavior of only four variables. When the velocity field is decomposed into the poloidal and toroidal components: $\boldsymbol{v} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{g}\phi) + \boldsymbol{\nabla} \times \boldsymbol{g}\psi$ (where $\boldsymbol{g}$ is the vertical unit vector), then three of these variables, $a_j (j = 1, 2, 3)$, correspond to the time-dependent amplitudes of the basic wave components of ϕ. The corresponding 2-dimensional wavevectors may be written in the form $(k_c/\sqrt{2},\, k_c/\sqrt{2} + (j-2)\delta)$ where $k_c = \pi/\sqrt{2}$ is the critical wavenumber, and $\delta > 0$ corresponds to the smallest wavenumber allowed by the periodicity condition. The fourth variable u corresponds to the component of the toroidal field with the weakest linear damping. In the interval of Ra that we are considering, the other modes can be regarded as "enslaved". Their influence is weak and results in insignificant quantitative corrections, as can be tested by comparison with the numerical approach [3].

Here we present the results of a detailed bifurcation analysis of this low-dimensional model. The equations, governing the time evolution of the amplitudes a_j, u are:

$$\begin{array}{rcl} \dot{a}_1 & = & \gamma_1(Ra - Ra_1)a_1 + a_1\left(\sum_{i=1}^3 \alpha_{1i}a_i^2\right) + \alpha_{14}a_2^2a_3 + q_1a_2u \\ \dot{a}_2 & = & \gamma_2(Ra - Ra_2)a_2 + a_2\left(\sum_{i=1}^3 \alpha_{2i}a_i^2\right) + \alpha_{24}a_1a_2a_3 + q_2a_1u + q_3a_3u \\ \dot{a}_3 & = & \gamma_3(Ra - Ra_3)a_3 + a_3\left(\sum_{i=1}^3 \alpha_{3i}a_i^2\right) + \alpha_{34}a_2^2a_1 + q_4a_2u \\ \dot{u} & = & -\delta^{-2}u + q_5a_1a_2 + q_6a_2a_3 \end{array} \quad (1)$$

where Ra_j are the values of Ra corresponding to the onset of the j-th mode, with $Ra_2 = Ra_c = 27\pi^4/4$. The derivation of the equations (1) and the expressions relating the factors γ_j, α_{ji} and q_i to the parameters Ra, Pr and δ can be found in [3].

The system (1) has two important symmetry properties: the equations remain invariant if either all variables a_j, or two variables a_2 and u, change their signs. Our numerical investigations show that for $\delta < 0.3$ a decrease of δ does not produce visible alterations in the structure of the bifurcation diagram for the Eq.(1) except for a rescaling: the characteristic differences of the Ra values are inversely proportional to the second power of δ. The typical bifurcation diagram in the plane spanned by the parameters Pr and Ra for the fixed value $\delta = 0.1$ is presented in Figure 1, where the domains of stability of steady patterns are dashed, the regions with temporally periodic motions are left blank, and the regions with chaotic dynamics are shown in black.

The amplitude a_2, corresponding to the roll pattern with the critical wavelength, is the first one to be excited with the increase of Ra. The finite extent of the layer ensures the stability of the steady roll solutions in a narrow range of Ra values, which gets narrower with decreasing Pr and vanishes for $Pr \to 0$. With increasing Ra the pure rolls become unstable and are replaced by a steady mixed-mode pattern composed of deformed rolls (curve v in Fig.1); in their turn the latter loses its stability as a result of a Hopf bifurcation (curve H). The time dependency assumes the form of stable regular oscillations. Our numerical results show that for $Pr < 1.5$ the further evolution of time-dependent patterns is associated with the formation and subsequent destruction of heteroclinic contours in the phase space of the model. These contours connect the unstable roll solutions to each other and to the unstable equilibrium. Due to the symmetries of the system, the formation of this saddle connection is a codimension-1 bifurcation (line h in Fig.1). To get a better understanding of the bifurcation events accompanying the destruction of this contour, we have derived the discrete return mapping for the trajectories, visiting the vicinities of the saddle-points; this mapping may be reduced to a one-dimensional recursion relation of the form:

$$x_{i+1} = f(x_i);\;\; f(x) = -\mu + \mid x \mid^{\nu_1} (1 + \varepsilon x \mid x \mid^{\nu_3})^{\nu_2} \quad (2)$$

Here ε is a factor of the order unity; the increase of the parameter μ describes the growth of Ra ($\mu = 0$ corresponds to the formation of the basic heteroclinic connection). The non-integer exponents $\nu_{1,2,3}$ are expressed through the ratios of the eigenvalues of the saddle points: $\nu_1 = \gamma_2\lambda_2/(\gamma_1\lambda_1)$, $\nu_2 = -\gamma_2/\gamma_1$ and $\nu_3 = (\lambda_3 - 2\lambda_2)/\lambda_2$, where $\lambda_{1,2,3}(\lambda_3 \approx \lambda_2 < 0 < \lambda_1)$ and $\gamma_{1,2}(\gamma_2 < 0 < \gamma_1)$ are the closest to zero eigenvalues of the flow, linearized near the equilibrium and near the roll solutions, respectively. The function $f(x)$ has minima at $x = 0$ (of the order ν_1) and at $x = -e^{-1/(1+\nu_3)}$ (of the order v_2). The stability of the bifurcating orbits and the general character of the evolution is determined by the values of $\nu_{1,2}$, which can be calculated directly from Eq.(1).

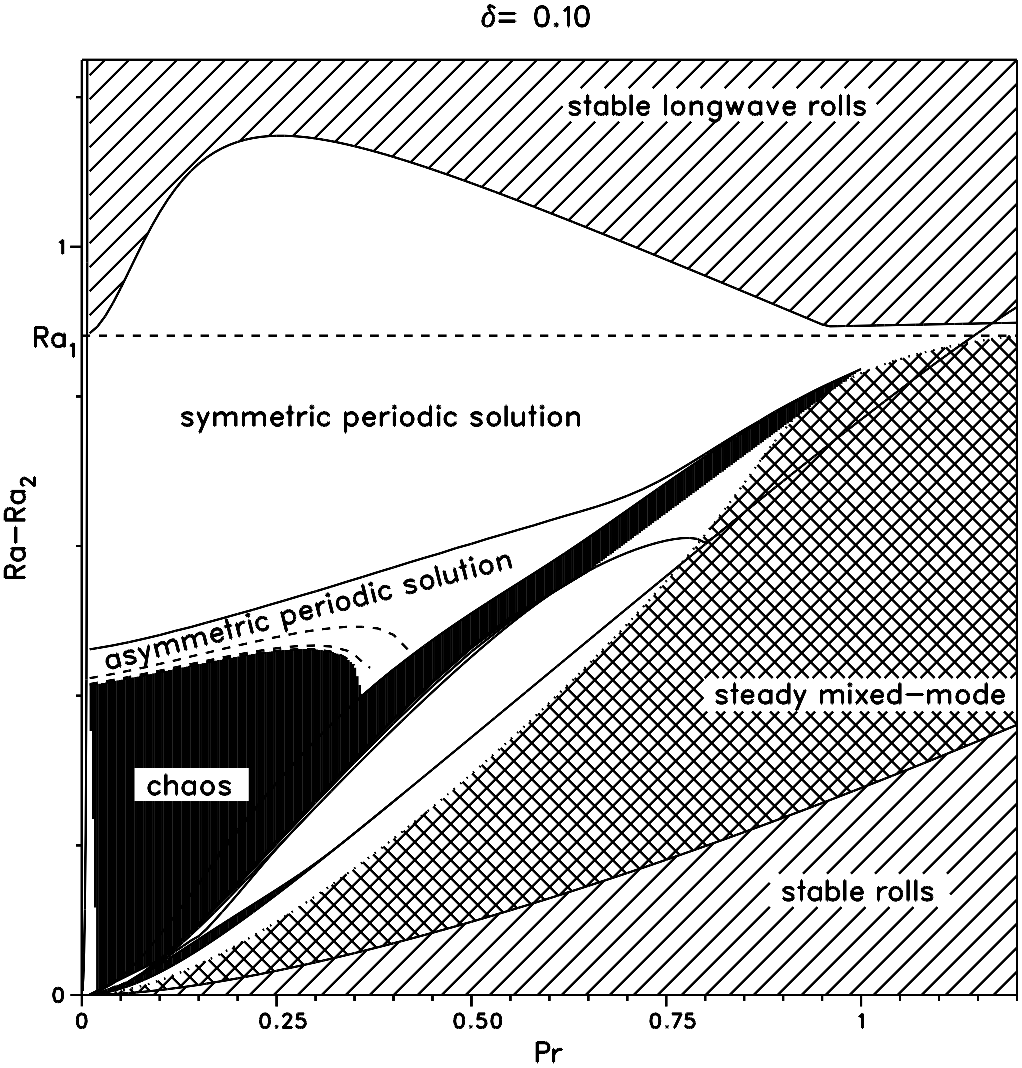

Figure 1. Bifurcation diagram for the equations (1).

For sufficiently low values of $Pr(Pr < 0.35)$ we observe $\nu_2 < 1$. This property means that in the vicinity of the unstable roll solutions expansion prevails over contraction. The corresponding extremum of the mapping (2) has the shape of a cusp, which results in the strong local instability. The mechanism of transition to chaos, which is observed in this parameter domain, is reminiscent of the Lorenz equations: growth of parameter leads to the heteroclinic "explosion", which creates a countable set of unstable periodic orbits. This set represents a "metastable chaos", since at this stage the chaotic set as a whole is unstable, and after a few irregular bursts the solutions exhibit periodic oscillations. An increase of Ra leads to the disappearance of the stable periodic orbit through a tangent bifurcation, and to the emergence of a chaotic attractor. For the observer the chaotic state seems almost ordered: nearly regular oscillations of the poloidal component of the flow are accompanied by a random "spiking" of the mean flow component. The relatively narrow range of Ra values in which this chaotic attractor exists, is limited from above by another tangent bifurcation, which generates a stable periodic motion.

Further increase of Ra for $Pr < 1$ leads to another sequence of bifurcations, which eventually produces a temporally chaotic pattern. These heteroclinic bifurcations involve the central saddle point which corresponds to the static equilibrium. Strong contraction in the vicinity of the latter counterbalances the expansion near the roll solutions, and the inequality $\nu_1 > 1$ for the overall saddle index ν_1 ensures the stability of the orbits born from the destruction of the heteroclinic contour. Analysis of the mapping

(2), corroborated by the results of direct numerical integration of equations (1), shows that the bifurcation scenario may be decomposed into elemental "steps". Each step begins with the destruction of a couple of two mutually symmetric heteroclinic contours (in each of them the trajectory makes 2^n turns around either of the mixed-mode steady solutions), followed by the birth of two stable periodic orbits. With increasing Ra these orbits undergo the symmetry-breaking bifurcation, and a couple of new periodic orbits bifurcates from them. A further increase of Ra leads to the restoration of symmetry: the periodic orbits are "glued" pairwise, forming a pair of new heteroclinic contours, with already 2^{n+1} turns in each of them. This process of the creation of the attracting orbit with progressively doubling number of turns ends with the chaotic attractor; the sequence of the bifurcation values of Ra converges at a geometric rate. As shown by the renormalization-group analysis, this rate as well as the quantitative scaling properties of the attractor, are entirely determined by the value of ν_1.

In the chaotic phase the spatio-temporal pattern typically exhibits relaxation like irregular oscillations of the roll planform. The rolls are slowly skewed and distorted until a fast burst-like deformation tears them off. The resulting disordered structure rapidly relaxes to a new roll pattern, which in turn displays a similar disintegration.

It should be mentioned, that the transition to chaos follows very closely the onset of convection; thus, for $\delta = 0.1$ the typical value of $\varepsilon = (Ra - Ra_c)/Ra_c$ does not exceed 10^{-3}. As mentioned above, with the decrease of δ the respective differences of Ra scale as δ^{-2}; this means that the distance between the onset of convection and the onset of chaos for the roll pattern is inversely proportional to the square of the aspect ratio.

The domain of chaos on the parameter plane is bounded from above by an inverse sequence of period-doubling bifurcations (for small values of Pr, see lines p_4, p_2 in Fig.1) or heteroclinic bifurcations (for moderate Pr), which eventually lead again to relatively simple regular oscillations. Finally, with a further increase of Ra the closed orbit corresponding to this time-periodic pattern disappears as a result of a tangent bifurcation. The system is attracted to the other steady pattern of finite-amplitude rolls with slightly greater wavelength corresponding to the mode with amplitude a_1. This initially unstable pattern bifurcating from the quiescent state at $Ra = Ra_1$, acquires stability as a result of the Hopf bifurcation (curve r in Fig.1), which is found to be supercritical for $Pr < 0.15$ and subcritical for higher values of Pr. Analysis of further possible destabilizations of this pattern at higher values of Ra requires the consideration of other components of both poloidal and toroidal parts of the velocity field with smaller wavenumbers, which are not represented in Eq.(1).

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