On the correlation dimension of the spectral measure for the Thue-Morse sequence

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Abstract

Using the relationship between the decay rate of autocorrelation and the characteristics of singular Fourier spectra, we show that the correlation dimension of the spectral measure for the infinite Thue-Morse symbolic sequence equals $3-\log(1+\sqrt{17})/\log 2 = 0.64298\ldots$.

KEY WORDS: singular continuous spectrum; multifractality; autocorrelation function.

The Thue-Morse infinite binary sequence $\{M_j\}, (j = 1, 2, \ldots)$ is formed by two symbols (in our notation, 1 and -1) and can be obtained from the starting point $M_1 = 1$ by means of repetitive substitutions (inflations) according to the rule [1, 2]:

$$\begin{array}{rcrr} 1 & \rightarrow & 1 & -1 \\ -1 & \rightarrow & -1 & 1 \end{array} \tag{1}$$

Several equivalent definitions are possible, for instance, $M_j = (-1)^{k+1}$ where k is the sum of digits in the binary representation of j. With respect to a shift along the symbolic chain, the Thue-Morse sequence supports a unique translation-invariant ergodic probability measure [3]. Although organized by a simple deterministic rule, the Thue-Morse sequence is not periodic. The Fourier spectrum (in physical applications the term "structure factor" is frequently used) of the infinite chain is long known to be purely singular continuous [4, 5]: the spectral measure is neither atomic (discrete) nor absolutely continuous with respect to the Lebesgue measure, but is concentrated on a Cantor set. This convenient combination of computational simplicity and intrinsic complexity turned the Thue-Morse sequence into a standard tool, widely exploited in the context of long-range ordering and disordering in one-dimensional patterns. To mention only a few recent applications, it was prescribed as a governing law for the external force in a kicked quantum rotator [6, 7] and served as a ground state configuration for the classical lattice gas model [8] and Ising spin system with short interaction[9]. In the description of the critical phenomena for the aperiodic quantum Ising lattices [10, 11, 12] it dictated the constants of exchange coupling. Further, the Thue-Morse sequence played a role of a one-dimensional potential for the discretized Schrödinger equation in the study of the gaps distribution in the energy spectrum [13], and its singular spectra have been recovered in the diffraction patterns of the specially prepared GaAs-AlAs superlattice heterostructures [14, 15]; the latest advances in this direction are thoroughly discussed in [16] (where also the exhaustive list of appertaining references can be found). Finally, the Thue-Morse symbolic code of the attractor at the accumulation point of the sequence of "homoclinic doublings" in symmetrical flows with saddle equilibria [17, 18] was recently understood as one of the mechanisms responsible for the generation of singular continuous spectra in continuous dissipative systems [19].

The multifractal properties of the Fourier spectrum for the Thue-Morse sequence have been investigated by Godrèche and Luck [20], who determined numerically the generalized dimensions D_q and the singularity spectrum $f(\alpha)$ for the spectral measure (see also a recent application of the wavelet technique in [21]). Apparently, the box-counting dimension D_0 equals 1 (the Cantor set which carries the spectral measure, is dense); estimation of D_q for the general case $q \neq 0$ requires extensive numerical computations. In this note we explicitly calculate the value of the correlation dimension D_2 for the spectral measure. In doing this, we utilize the exact relation derived in [22, 23, 24] which binds the properties of the autocorrelations to those of the singular spectral measure, and can be formulated as follows. Let the autocorrelation function for an observable x_j ($j = 1, 2, \dots$) be introduced as $C(t) = (\langle x_j x_{j+t}\rangle - \langle x_j\rangle^2)/(\langle x_j^2\rangle - \langle x_j\rangle^2)$ and the integrated autocorrelation function as $C_{\rm int}(T) = \frac{1}{T}\sum_{t=0}^{T} |C(t)|^2$, respectively. Then $C_{\rm int}(T) \sim T^{-D_2}$ where D_2 is the correlation dimension of the spectral measure. Thus the direct evaluation of D_2 from the Fourier spectral data can be replaced by the estimation of the decay rate for the integrated autocorrelation; as shown below, the self-similarity of the Thue-Morse sequence allows to determine this rate analytically (note that autocorrelation function generically does not depend on the choice of observable, so we simply put $x_j = M_j$).

By construction the Thue-Morse sequence is invariant under the substitution (1) and under the inverse operation of "binary decimation": crossing out each even symbol. This is reflected in the recurrence properties of the correlation function: $\sum_j M_j M_{j+t} = \sum_j M_j M_{j+2t}$ and, respectively,

$$C(2t) = C(t). \tag{2}$$

This ensures repetitive non-decaying peaks for arbitrary large values of t (and thus prohibits the spectrum to be absolutely continuous with respect to the Lebesgue measure: the necessary condition of such continuity is the decay of correlations for $t \to \infty$ [25, 26]). The analogous recurrence relation for the odd argument values follows from the invariance of the infinite sum $\sum_j M_j M_{j+2t+1}$ under the binary decimation. After decimation, the odd symbols remain unchanged whereas the even ones are replaced by their counterparts (1 by -1 and vice versa), hence the pairwise products acquire the opposite sign. Consequently,

$$C(2t+1) = -\frac{C(t) + C(t+1)}{2}. \tag{3}$$

The relations (2, 3) combined with "initial condition" $C(0) = 1$ allow one to obtain the exact values of $C(t)$ for any value of t. They ensure that the inequality $|C(t)| \leq \frac{1}{3}$ holds for all t (naturally, except for $C(0)$=1).

Consider now the recurrences between the sums $S_n \equiv \sum_{n+1}^{2n} C^2(t)$ for $n = m, 2m, 4m, \dots$, m - arbitrary positive integer. On applying (2) and (3) to the even and odd terms in S_{4m} respectively, we obtain

$$S_{4m} = S_{2m} + \frac{S_{2m}}{2} + \frac{1}{2}\sum_{2m}^{4m-1} C(t)C(t+1) \tag{4}$$

Denoting the last term in the right hand size of (4) by Π_{2m} and transforming it with the help of (2, 3), we arrive at

$$\Pi_{2m} = -\sum_{m}^{2m-1}\left(C^2(t) + C(t)C(t+1)\right). \tag{5}$$

Thus we are left with the two coupled linear recurrences

$$\begin{aligned} S_{2j} &= \frac{3}{2}S_j + \frac{1}{2}\Pi_j \\ \Pi_{2j} &= -S_j - \Pi_j \end{aligned} \tag{6}$$

Introducing further the variables $\kappa_j \equiv S_{2j}/S_j$ and $\xi_j \equiv \Pi_{2j}/S_j$ and noticing that the equality $\kappa_j(\xi_j+2)=2$ is independent of j, we end up with a simple one-dimensional mapping

$$\kappa_{2j} = \frac{1}{\kappa_j} + \frac{1}{2} \quad . \tag{7}$$

which in the domain $\kappa > 0$ (only this is of relevance, since S_j >0) has the globally attracting fixed point $\kappa_* = \frac{1+\sqrt{17}}{4}$=1.280776406.... Therefore, for large j holds: $S_{m2^j} \propto (\kappa_*)^j S_m$. Let us turn now to the integrated autocorrelation $C_{\rm int}(T)$. Since this function is by definition a ratio of two monotonous functions of T, it is easy to see that, irrespectively of T, the inequality

$$\frac{C_{\rm int}(2T)}{2} < C_{\rm int}(t) < 2C_{\rm int}(T) \tag{8}$$

holds for any t from the interval $(T, 2T)$. Obviously, for the values $T=2^n m$ under large n

$$C_{\rm int}(T) = \frac{1+\sum_{j=1}^{n} S_{m2^j}}{m2^n} \propto T^{\frac{\log\kappa_*}{\log 2}-1} \quad . \tag{9}$$

In combination with (8), this ensures the power-law decay of $C_{\rm int}(T)$. According to Wiener [25], vanishing of $C_{\rm int}(\infty)$ implies the absence of the discrete component in the spectrum. Consequently, the spectral measure for the Thue-Morse sequence is purely singular continuous, and its correlation dimension equals:

$$D_2 = 1 - \frac{\log\kappa_*}{\log 2} = 3 - \frac{\log(1+\sqrt{17})}{\log 2} = 0.64298136\ldots \tag{10}$$

Cf. the values of other generalized dimensions: the capacity D_0=1 and the information dimension D_1=0.730 (numerical value).

The fact that the relation (9) is independent of m, implies that the prefactor before T^{1-D_2} on the right hand size of (9) must tend to a periodic function of the logarithmic variable $\log T/\log 2$. Our numerical estimates, however, show the influence of this time-dependence to be rather weak: the prefactor oscillates log-periodically between 0.75 and 0.77.

The same procedure can be implemented for other symmetrical substitution sequences, for instance, for the m-tuplings: $A \to AB^{m-1}$, $B \to BA^{m-1}$. Obviously, the case m=2 yields the Thue-Morse sequence. For m=3 the autocorrelation function obeys the recurrences

$$C(3t) = C(t),\;\; C(3t+1) = -\frac{C(t+1)}{3},\;\; C(3t+2) = -\frac{C(t)}{3}$$

and the correlation dimension of the spectral measure equals $3-\frac{\log 11}{\log 3}$=0.8173.... For m=4 one has:

$$C(4t) = C(t),\;\; C(4t+1) = -C(4t+3) = \frac{C(t)-C(t+1)}{4},\;\; C(4t+2) = 0$$

and $D_2 = \frac{5}{2} - \frac{\log(3+\sqrt{41})}{\log 4}$=0.8834.... Higher values of m can be treated in a similar way.

We expect that the knowledge of recurrent properties for the autocorrelation function of substitution symbolic sequences (the obvious (2) is commonly used, but the less evident relations like (3) seem to be overlooked) and that of the exact values like (10) can be helpful in answering many subtle questions which arise in the context of one-dimensional "quasicrystals" in substitutional systems with various bond-length ratios [27, 28], extended electronic states in one-dimensional lattices [29], and kicked quantum systems with corresponding potentials [6, 7].

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