On the generalized dimensions for the Fourier spectrum of the Thue-Morse sequence

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Abstract. We present explicit relations for the generalized dimensions D_q of the spectral measure of the Thue-Morse symbolic sequence for positive integer values of q. Each D_q is expressed through the eigenvalue of the corresponding $q \times q$ matrix. We also demonstrate how these dimensions can be recovered from the products of the values of the autocorrelation function.

1. Introduction

The Thue-Morse symbolic sequence $\{M_j\}, (j = 1, 2, \ldots)$ is defined through the rule $M_j = (-1)^{\Phi(j-1)}$ where $\Phi(m)$ denotes the sum of digits in the binary representation of the number m. There are several equivalent definitions which describe the same symbolic object. Thus, it can be obtained from the starting point $M_1 = 1$ by means of repetitive substitutions (inflations) when each symbol 1 is substituted by $1\ -1$ and each -1 is inflated into $-1\ 1$. Yet another definition (which we will exploit below) is based on concatenations. This goes as follows: given a symbolic string K_n of length 2^n we append to it the string $\overline{K_n}$ in which each symbol "1"of K_n is replaced by "–1" and vice versa; the recursion $K_{n+1} = K_n\overline{K_n}$ combined with the initial condition K_0=1 yields the Thue-Morse sequence.

During its nearly century-long history [1, 2, 3] the Thue-Morse sequence has found numerous applications in many domains of mathematics and mathematical physics. Being viewed as a deterministic dynamical system in which the index j plays the role of discrete time, it is neither regular (periodic or quasiperiodic) nor chaotic, but demonstrates a "marginal" behaviour. Respectively, its Fourier spectrum (also called "structure factor" in the context of solid-state physics) is neither discrete nor absolutely continuous with respect to Lebesgue measure, but purely singular continuous [3, 4]: the spectral measure is carried by the dense non-denumerable set with zero Lebesgue measure. The combination of self-similarity and weak disorder has made of the Thue-Morse sequence an indispensable case study for the studies of long-range effects in one-dimensional classical and quantum patterns. To provide an adequate list of quotations in this field is a formidable task by itself; we refer the readers to the recent review [5] and the representative list of references therein. Among the other applications we should mention the quantum rotator kicked by the force which obeys the Thue-Morse law: it serves as an example of a system driven by the external action which is neither regular nor random [6, 7]. Yet another application has been recently found in the field of autonomous dissipative dynamical systems where the onset of singular continuous (fractal) power spectra has been traced to the Thue-Morse symbolic coding of the respective attractor [8].

The fractal properties of the set which supports the spectral measure of the Thue-Morse sequence were numerically studied in the last decade. In their computer-assisted description of multfractality for various sequences generated by substitution rules, Godrèche and Luck [9] provided the numerical estimates for the singularity spectrum $f(\alpha)$ of the spectral measure. Later Liviotti [10] employed the wavelet technique for the same purpose.

Since the Thue-Morse sequence plays a prototypic role for weakly aperiodic systems, the exact knowledge of the characteristics of its spectral properties is of especial methodical interest. In our previous work we utilized the relation between the decay rate of the integrated autocorrelation function and the correlation dimension D_2 of the spectral measure [11, 12] to obtain the exact value of the correlation dimension D_2 for the spectral measure of the Thue-Morse sequence: $D_2 = 3 - \log(1+\sqrt{17})/\log 2 = 0.64298\ldots$ [13]. Below, we will use a different approach and express the values of the generalized dimensions D_q for integer q through the leading eigenvalues of the appertaining $q \times q$ matrices. Further, we generalize the results of [11, 12] by relating D_q to the growth rates of the higher products of the values of the autocorrelation function. The case of the information dimension D_1 does not conform to this scheme; we present the series which allows to calculate it to arbitrary precision.

2. Spectral measure

A generic observable built from the Thue-Morse sequence should attain only two values. Therefore, the spectral properties do not depend on the choice of the observable, and we select the most convenient one: the value of $M_j = \pm 1$ itself. Let us take the value ω $(0 \le \omega \le 1)$ and consider the partial Fourier sums formed by the first $l_n = 2^n$ symbols of the sequence: $\sigma_n(\omega) = \sum_{j=1}^{2^n} M_j \mathrm{e}^{2\pi i j\omega}$, and the finite-length approximations to the power spectrum $S_n(\omega) = 2^{-n}|\sigma_n(\omega)|^2$. The concatenation rule through which the Thue-Morse sequence is built, implies $\sigma_{n+1}(\omega) = \sigma_n(\omega)(1-\mathrm{e}^{2^{n+1}\pi i\omega})$ and, respectively, $S_{n+1}(\omega) = S_n(\omega)(1-\cos 2^{n+1}\pi\omega)$. The evolution of $S_n(\omega)$ under the growing n allows to conclude both on the nature of the power spectrum at the given point ω and on the global distribution of the spectral measure μ:

$$\mu(\omega) = \lim_{n\to\infty} S_n(\omega) = \lim_{n\to\infty} \prod_{j=0}^{n-1} (1 - \cos 2^{j+1}\pi\omega) \tag{1}$$

The properties of $\mu(\omega)$ can be reconstructed from the asymptotic features of the infinite product (1) (also called the Riesz product [4]). It is straightforward to see that $S_n(\omega)$ vanishes at $\omega = 2^{-k}m$ for $n \geq k+1$ and arbitrary integer m, k [9]. For all the other rational values of ω, the ratio $\rho = S_{n+1}(\omega)/S_n(\omega)$ oscillates periodically with n. (These oscillations are preceded by a a transient whose length equals the multiplicity of the factor 2 in the factorization of the denominator of ω). When the geometric mean value $\langle\rho\rangle$ over the period of oscillations exceeds 1, the spectral sums S_n grow "on the average": $S_n \sim \langle\rho\rangle^n$ or, in terms of the length of the symbolic string, $S_n \sim l_n^\gamma$ where the growth rate γ equals $\log\langle\rho\rangle/\log 2$. However, no values of ω enable the delta peaks in the spectrum (or, in the diffraction jargon, the "Bragg peaks"): this would require $\gamma = 1$. This means that the discrete (atomic) component is absent in the spectrum. The fastest growth and, respectively, the largest γ is attained at $\omega = 2^{-k}m/3$ (of course, m should not be a multiple of 3); in this case $\gamma = \log 3/\log 2 - 1 = 0.584\ldots$[9]. Already this subset of ω-values is dense on the interval $[0,1]$; this fact alone is sufficient to ensure that the capacity (box-counting dimension) D_0 of the set which carries the spectral measure, equals 1. However, since the latter measure is not absolutely continuous with respect to the Lebesgue measure, the other generalized dimensions D_q can differ from 1.

To describe the multifractal properties of $\mu(\omega)$, an appropriate partition of the interval $[0,1]$ should be introduced. A standard way (see e.g. [9] where this formalism was applied to spectral measures) is to divide the unit interval into N small boxes of the length $\varepsilon = 1/N$, so that the probability to locate measure in the k-th subinterval is $p_k = \int_{(k-1)\varepsilon}^{k\varepsilon} \mu(\omega)d\omega$. The partition function is defined for any real number q as $Z(q,\varepsilon) = \sum_{k=1}^{N} p_k^q$. Assuming under the fixed q the scaling law $Z(q,\varepsilon) \sim \varepsilon^{\tau(q)}$, we arrive in the standard way [9] at the generalized (Rényi) dimensions: $D_q = \tau(q)/(q-1)$.

Since the normalization condition $\int_0^1 S_n(\omega)d\omega = 1$ holds for every n, we can use a sequence of $S_n(\omega)$ as approximations to the probability density. In its turn, this provides a sequence of approximations

$$\tilde{Z}(q,\varepsilon,n) = \sum_{k=1}^{1/\varepsilon} \left(\int_{(k-1)\varepsilon}^{k\varepsilon} S_n(\omega)d\omega \right)^q$$

to the partition function. As ε tends to zero, we can replace in these approximations summation by integration:

$$\tilde{Z}(q,\varepsilon,n) = \varepsilon^{q-1} \sum_{k=1}^{1/\varepsilon} \left(\frac{1}{\varepsilon} \int_{(k-1)\varepsilon}^{k\varepsilon} S_n(\omega)d\omega \right)^q \varepsilon \;\rightarrow\; \varepsilon^{q-1} \int_0^1 S_n^q(\omega)d\omega \tag{2}$$

Apparently, infinite decrease of ε under fixed finite n makes no sense: in this way, one would approach the smooth $S_n(\omega)$ and obtain trivial scaling $\tau = q - 1$. The decrease of ε should be combined with the simultaneous increase of n; this refinement of the partition allows to explore the asymptotical fine structure of $\mu(\omega)$. According to Eq.(1), the resolution in frequency domain for $S_{n+1}(\omega)$ is twice as good as that for $S_n(\omega)$. Respectively, seeking for the asymptotic scaling properties, one should compare $\tilde{Z}(q,\varepsilon,n)$ with $\tilde{Z}(q,\frac{\varepsilon}{2},n+1)$. Therefore $\tau(q)$ under fixed n and ε is evaluated as

$$-(\log 2)^{-1} \log \frac{\tilde{Z}(q,\frac{\varepsilon}{2},n+1)}{\tilde{Z}(q,\varepsilon,n)} \tag{3}$$

which, taking into account the asymptotics (2), yields finally

$$D_q = \frac{\tau(q)}{q-1} = 1 - \frac{\log\lambda(q)}{(q-1)\log 2} \tag{4}$$

where the growth rate $\lambda(q)$ is given by

$$\lambda(q) = \lim_{n\to\infty} \frac{\int_0^1 S_{n+1}^q(\omega)d\omega}{\int_0^1 S_n^q(\omega)d\omega} = \lim_{n\to\infty} \frac{\int_0^1 \left(\prod_{k=0}^{n}(1-\cos(2^{k+1}\pi\omega)\right)^q d\omega}{\int_0^1 \left(\prod_{k=0}^{n-1}(1-\cos(2^{k+1}\pi\omega)\right)^q d\omega} \tag{5}$$

In this way, computation of the generalized dimensions D_q has been reduced to the evaluation of $\lambda(q)$.

3. Generalized dimensions D_q for integer values of q

In general, the only way to find the value of $\lambda(q)$ seems to be the direct numerical integration of numerator and denominator in Eq.(5) with subsequent extrapolation to the limit $n \to \infty$. However, the case of integer values of $q > 1$ admits a simplification. The 2^n terms in the expansion of $S_n(\omega)$ into $\cos 2\pi k\omega$ range from $k = 0$ to $k = 2^n - 1$. Respectively, for integer $q > 0$ the cosine-expansion of $S_n^q(\omega)$ contains the terms until $\cos 2\pi(2^n - 1)q\omega$: $S_n^q(\omega) = \sum_{k=1}^{q(2^n-1)} b_k^{(n)} \cos 2\pi k\omega$.

Let us pick out from the set of the coefficients $\{b_k^{(n)}\}$ the subset which corresponds to the multiples of 2^n: $a_j^{(n)} \equiv b_{j\times 2^n}^{(n)}$, $j = 0, 1, \ldots, q-1$. Obviously, of the whole expansion only the ω-independent term $a_0^{(n)}$ contributes to the integrals in (5), and $\lambda(q)$ equals $\lim_{n\to\infty} a_0^{(n+1)}/a_0^{(n)}$. When we proceed from S_n to S_{n+1}, the coefficients $a_j^{(n+1)}$ at $\cos 2\pi j 2^{n+1}\omega$ come into the consideration. By combinatorial arguments it can be easily shown that each of $a_j^{(n+1)}$ is a linear combination of $a_k^{(n)}$: thus, for $q = 2$ we have

$$\begin{aligned} a_0^{(n+1)} &= \frac{3}{2}a_0^{(n)} - a_1^{(n)} \\ a_1^{(n+1)} &= \frac{1}{2}a_0^{(n)} - a_1^{(n)} \end{aligned} \tag{6}$$

for $q = 3$ the recursion relations are

$$\begin{aligned} a_0^{(n+1)} &= \frac{5}{2}a_0^{(n)} - \frac{15}{8}a_1^{(n)} + \frac{3}{4}a_2^{(n)} \\ a_1^{(n+1)} &= \frac{3}{2}a_0^{(n)} - 2a_1^{(n)} + \frac{5}{2}a_2^{(n)} \\ a_2^{(n+1)} &= -\frac{1}{8}a_1^{(n)} + \frac{3}{4}a_2^{(n)} \end{aligned} \tag{7}$$

and so on. In case of S_q one has to do with the $q \times q$ matrix, whose elements are combinations of binomial coefficients with the coefficients of expansion of $\cos^n x$ into $\cos jx$. Although the general expressions are rather complicated, the computation of the matrix elements for not too large q is straightforward.

Since the relations (6),(7) etc. are linear, λ is simply the largest eigenvalue of the corresponding matrix and can be found from the respective characteristic equation. For $q = 2$ the equation is

$$\lambda^2 - \frac{\lambda}{2} - 1 = 0 \tag{8}$$

which yields $\lambda = (1 + \sqrt{17})/4 = 1.28077064\ldots$ and $D_2 = 1 - \log\lambda/\log 2 = 0.642981\ldots$.

For $q = 3$ we have

$$\lambda^3 - \frac{5}{4}\lambda^2 - \frac{3}{2}\lambda - 1 = 0 \tag{9}$$

and, respectively, $\lambda = 1.777389781\ldots$ and $D_3 = 0.58511995\ldots$; for $q = 4$ the characteristic equation is

$$\lambda^4 - \frac{13}{8}\lambda^3 - \frac{55}{16}\lambda^2 + \frac{17}{8}\lambda + 1 = 0 \tag{10}$$

with $\lambda = 2.579911342\ldots$ and $D_4 = 0.5442261703\ldots$, etc.

4. Generalized dimensions through the autocorrelation function

The same results can be expressed in terms of the autocorrelation functions. For an observable ξ_j and integer t ($-\infty < t < \infty$) the autocorrelation function is defined as $C(t) = C(-t) = (\langle\xi_k\xi_{k+t}\rangle - \langle\xi_k\rangle^2)/(\langle\xi_k^2\rangle - \langle\xi_k\rangle^2)$ where the angular brackets denote averaging with respect to the position k. In our case, $\langle M_k\rangle$=0 (this follows e.g. from the concatenation building rule), and the above definition is reduced to $C(t) = \langle M_k M_{k+t}\rangle$. Being merely the Fourier transform of the power spectrum, $C(t)$ can be easily recovered from the spectral sums. In this sense, casting the Riesz product into the form of the trigonometric sum immediately yields the cosine-transform of the power spectrum and is especially convenient: let $c_j^{(n)} = 2\int_0^1 S_n(\omega)\cos(2\pi j\omega)d\omega$ denote the coefficient at $\cos 2\pi j\omega$ in S_n, then

$$C(j) = \frac{1}{2 - \delta_{0j}} \lim_{n\to\infty} c_j^{(n)} \tag{11}$$

(the factor $\frac{1}{2}$ at $j \neq 0$ enters the expression because S_n includes the contributions of both $C(j)$ and $C(-j)$).

This presentation allows to reformulate the above analysis of the growth of the coefficient $a_0^{(n)}$ in the expansions of $S_n^q(\omega)$. In the q-th power of the cosine-transform of $S_n(\omega)$, the ω-independent part is formed by the product terms with the vanishing net sum of trigonometrical arguments. Respectively, in the time domain the integral $\int_0^1 S_n^q(\omega)d\omega$ is represented by the sum of the products of q values of $C(j)$:

$$U(q,T) = \sum_{|j_1|+|j_2|+\ldots+|j_{q-1}|<T} C(j_1)C(j_2)\ldots C(j_{q-1})C(-j_1-j_2-\ldots-j_{q-1}) \tag{12}$$

where T is the length of the segment of the symbolic string, contributing to S_n. This sum grows with T according to the power law: $U(q,T) \sim T^{\kappa(q)}$ where $\kappa(q) = \log\lambda(q)/\log 2$. Respectively,

$$D_q = 1 - \frac{\kappa(q)}{q-1} \tag{13}$$

The expression (13) relates the Rényi dimensions of the spectral measure D_q for integer q with the growth law for the sums of $q-$products of the values of the autocorrelation function. For the case $q = 2$ it is equivalent to the formula derived in [11].

5. Cases of $q \to 0$ and large q

In case of small q the growth rate λ can be computed explicitly:

$$\lambda(q) = \lim_{n\to\infty} \frac{1 + q\int_0^1 \log S_{n+1}(\omega)d\omega}{1 + q\int_0^1 \log S_n(\omega)d\omega} + O(q^2) = 1 + q\int_0^1 \log\frac{S_{n+1}(\omega)}{S_n(\omega)}d\omega + O(q^2) = 1 - q\log 2 + O(q^2) \tag{14}$$

For the generalized dimension this yields:

$$D_q = 1 - \frac{\log\lambda}{(q-1)\log 2} = 1 - q + O(q^2) \tag{15}$$

In the opposite limit $q \to \infty$ the dominating contribution into $\int_0^1 S_n^q(\omega)d\omega$ is made by the values of ω which enable the fastest growth of the local finite-length approximations $S_n(\omega)$. The peaks in S_{n+1} which belong to the most rapidly growing family are $3/2$ times higher than the peaks at the same places in S_n; at the same time, due to the improvement of the spectral resolution, the width of these peaks is halved. Therefore, $\lambda \cong \frac{1}{2}(3/2)^q$. Respectively,

$$D_q \cong \frac{q}{q-1}\left(2 - \frac{\log 3}{\log 2}\right) \tag{16}$$

As q grows, D_q tends to $D_\infty = 2 - \log 3/\log 2 = 0.415037\ldots$.

In fact, already moderate values of q are "large" enough: thus, the exact value of D_4 quoted above differs from the estimate (16) by less than 0.01; in case of $q = 6$ this difference is less than 0.0007, and for $q = 8$ it is even less than 0.00007.

6. Information dimension D_1

The case $q = 1$ should be treated separately. Differentiation of numerator and denominator in (4) provides the expression for the value of the information dimension:

$$D_1 = 1 - \frac{1}{\log 2}\lim_{n\to\infty}\int_0^1 \left(S_{n+1}(\omega)\log S_{n+1}(\omega) - S_n(\omega)\log S_n(\omega)\right)d\omega \tag{17}$$

Let us transform the integral in this expression:

$$\begin{aligned}
&\int_0^1 \left(\prod_{j=1}^{n+1}(1-\cos 2^j\pi\omega)\sum_{j=1}^{n+1}\log(1-\cos 2^j\pi\omega) - \prod_{j=1}^{n}(1-\cos 2^j\pi\omega)\sum_{j=1}^{n}\log(1-\cos 2^j\pi\omega)\right)d\omega = \\
&\int_0^1 S_n(\omega)\log(1-\cos 2\pi\omega)d\omega - \int_0^1 \cos 2\pi\omega\sum_{j=2}^{n}\log(1-\cos 2^j\pi\omega)d\omega \; + \\
&\int_0^1 \left(\prod_{j=2}^{n+1}(1-\cos 2^j\pi\omega)\sum_{j=2}^{n+1}\log(1-\cos 2^j\pi\omega) - \prod_{j=1}^{n}(1-\cos 2^j\pi\omega)\sum_{j=1}^{n}\log(1-\cos 2^j\pi\omega)\right)d\omega
\end{aligned} \tag{18}$$

Of the three integrals, the last one vanishes, since the contributions of its first and second parts mutually balance each other. Similarly, the second integral vanishes because $\sum_{j=2}^{n}\log(1-\cos 2^j\pi\omega)$ does not contain terms proportional to $\cos 2\pi\omega$: its cosine-expansion starts with $\cos 4\pi\omega$. Thus, the only remaining part is the first term

$$\int_0^1 S_n(\omega)\log(1-\cos 2\pi\omega)d\omega = -\log 2 - \sum_{j=1}^{2^n-1}\frac{c_j^{(n)}}{j} \tag{19}$$

where $c_j^{(n)}$, as above, is the coefficient at $\cos 2\pi j\omega$ in S_n. Taking into account Eq.(11) which relates $c_j^{(n)}$ to the values of the autocorrelation function $C(j)$, we get

$$D_1 = 1 - \frac{-\log 2 - \lim_{n\to\infty}\sum_{j=1}^{2^n-1}\frac{c_j^{(n)}}{j}}{\log 2} = 2 + \frac{2}{\log 2}\sum_{j=1}^{\infty}\frac{C(j)}{j} \tag{20}$$

Thus the problem is reduced to the estimation of the value $\Lambda \equiv \sum_{j=1}^{\infty} C(j)/j$.

The invariance of the infinite Thue-Morse sequence with respect to the inflation and the inverse operation of binary decimation imposes the recurrent relations on the values of the autocorrelation function [3, 13]:

$$C(2j) = C(j), \quad C(2j+1) = -\frac{C(j)+C(j+1)}{2} \tag{21}$$

which, combined with the normalization condition $C(0) = 1$, allow to compute $C(j)$ for every value of j: $C(2^n) = -C(3\times 2^n) = -1/3$, $C(5\times 2^n) = C(7\times 2^n) = 0$, $C(9\times 2^n) = -C(11\times 2^n) = 1/6$, and so on ($n = 0, 1, \ldots$). This information alone does not allow to obtain Λ in a closed form. The series in (20) converges slowly ($\sim j^{-1}$); however, the recursions (21) allow to implement the "internal summation" which leads to the noticeable (in principle, indefinite) acceleration of the convergence. Let Ξ be a sum

$$\Xi = a + \sum_{j=1}^{\infty} f(j)C(j) \tag{22}$$

where a is a constant and $f(j)$ is some function of j. Transforming this expression with the help of (21), we obtain

$$\Xi = a - \frac{f(1)}{2} + \sum_{j=1}^{\infty}\frac{2f(2j)-f(2j-1)-f(2j+1)}{2}C(j) \tag{23}$$

This provides the iteration scheme:

$$a_{m+1} = a_m - \frac{f_m(1)}{2} \tag{24}$$

$$f_{m+1}(j) = \frac{2f_m(2j)-f_m(2j-1)-f_m(2j+1)}{2} \tag{25}$$

where, in order to compute Λ, we should start with $a_0 = 0$ and $f_0 = 1/j$.

Since the expression on the right of Eq.(25) is the (rescaled and shifted) finite-difference approximation for the second derivative of $f_m(j)$, the result of k iterations of Eq.(24,25) provides a series in $C(j)$ whose coefficients $f_k(j)$ converge as j^{-1-2k}. Successive transformations yield

$$\Lambda = -\frac{1}{2} - \sum_{j=1}^{\infty}\frac{C(j)}{2j(4j^2-1)} = -\frac{5}{12} + \sum_{j=1}^{\infty}\frac{(96j^2-9)C(j)}{4j(4j^2-1)(16j^2-1)(16j^2-9)} = \ldots \tag{26}$$

Although the further expressions are too cumbersome to quote them explicitly, their derivation is straightforward and can be easily performed with every program for symbolic computations. Already after the 7-th iteration the coefficients at $C(2)$, $C(3)$ and $C(4)$ have an order of, respectively, $10^{-15}, 10^{-18}$ and 10^{-20}, and it is enough to take the first two terms in the series (that is, $a^{(7)}$ and the term at $C(1)$), in order to produce Λ with 14 correct digits: $\Lambda = -0.43995518283629$ and, respectively, $D_1 = 0.73055767901739\ldots$

7. Discussion

The approach which we have proposed above, is not restricted to the particularities of the Thue-Morse sequence. In the general case, as soon as the building rules of the symbolic sequence allow to explicitly interrelate the consecutive approximations $S_n(\omega)$ and $S_{n+1}(\omega)$ to the spectral measure $\mu(\omega)$, the same technique can help to extract the generalized dimensions from the corresponding characteristic equations.

Similarly, the expression (13) which relates the generalized dimensions to the growth rate of the products of the autocorrelation function values, remains valid for a broad class of problems and can be applied not only to binary symbolic sequences but to general (stationary) datasets of computational or experimental origin. In case of processes with singular continuous or mixed Fourier spectra the direct evaluation of spectral sums (and thereby of the set which supports the spectral measure) is sensitive to the numerical details like frequency resolution, etc. Compared to this, the estimation of the autocorrelation function is a robust procedure. Thus the relationship (13) allows one in principle to recover the generalized dimensions from observational and numerical data. However, the rapid growth of the number of terms in the sums $U(q,T)$ (this number is proportional to T^{q-1}) makes its application for $q > 4$ hardly practical.

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