Lepton Magnetic Moments: what they tell us*

Fred Jegerlehner

Humboldt-Universität zu Berlin, Institut für Physik, Newtonstrasse 15, D-12489 Berlin, Germany

Recently, the exciting new Fermilab (FNAL) Muon g-2 measurement impressively confirmed the final Brookhaven (BNL) result from 2004 and, with a result four times more precise, has launched a new serious attack on the Standard Model (SM). On the theoretical side, ab initio lattice QCD (LQCD) calculations of hadronic vacuum polarization have made remarkable progress. They are now the new standard for studying the leading non-perturbative contributions, which have previously hindered matching with the precision required for full exploitation of the experimental results. The lattice results affected both leading hadronic contributions – the hadronic vacuum polarization (HVP) and the hadronic light-by-light (HLbL) contributions – by increasing the previously generally accepted $e^+e^- \to$ hadrons based dispersion relation results. The shifts reduced the discrepancy between theory and experiment to "nothing missing." One of the most prominent signs of "Beyond the Standard Model" (BSM) physics has disappeared: the SM appears validated more than ever, in agreement with what also searches at the Large Hadron Collider (LHC) at CERN tell us! A triumph of the SM, in spite of the fact that the SM cannot explain known cosmological puzzles like dark matter or baryogenesis and why neutrino masses are so tiny, the absence of strong CP violation for example. I also argue that the discrepancy between the data-driven dispersive result and the lattice QCD results for the hadronic vacuum polarization can be largely explained by correcting the e^+e^- data for $\rho^0 - \gamma$ mixing effects.

1. Introduction

This text is an update of an earlier article [1] and the more detailed book [2], which were written when the 2004 BNL measurement of $a_\mu = (g_\mu - 2)/2$ at 540 ppb [3] had set the precision limit before the Fermilab Muon g-2 experiment [4,5] provided new, more precise measurements of a_μ, which recently reached the milestone of 124 ppb [5]. With four times the precision and a statistical error roughly equal to the combined systematic error, the SM was subjected to an unprecedented precision test. While the Fermilab experiment confirmed the final BNL result,

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the theoretical prediction moved closer to the experimental result due to significant advances in ab initio lattice QCD calculations of the non-perturbative QCD contributions that had been an obstacle to the precision of the SM prediction of a_μ.

The story of the lepton magnetic moments is the story of “The closer you look the more there is to see” and highlights the SM to the deepest. The magnetic moments of leptons (g_ℓ factors) and the associated anomalies of the muon and electron play a decisive role in high-precision verification of the Standard Model. These quantities are not only among the most precisely measured in particle physics, but can also be predicted with high accuracy. At the same time, they offer promising insights into physics beyond the Standard Model. The key relationship for testing possible new heavy states is

$$\frac{\delta a_\ell}{a_\ell} \propto \frac{m_\ell^2}{M^2} \quad (M \gg m_\ell),$$

where m_ℓ is the mass of the lepton and M is the mass scale of new physics. This equation shows that the fractional change in a_ℓ is proportional to the square of the lepton mass over the square of the new physics scale, making the muon about $(m_\mu/m_e)^2 \sim 4\times 10^4$ times more sensitive to BSM physics than the electron. For the muon, theory and experiment now agree to 9 decimal places, so that $\delta a_\mu \sim \frac{\alpha}{\pi}\frac{m_\mu^2}{M_{\rm NP}^2}$ can effectively probe intermediate mass scales beyond the heaviest SM ingredient, the top quark mass M_t up to about 1 TeV, a region where new heavy states are essentially ruled out directly by the LHC.

In any case, the “Muon-g-2 drama” will have an essential impact on which SM extensions remain viable. Paradigms such as naturalness and the supposed related hierarchy problem are definitely not constructive, and new guiding principles must take their place. The most fruitful approach here is to search for emergent structures, as outlined in [6, 7], for example.

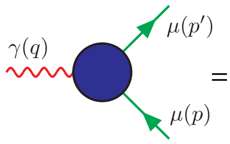
2. A new milestone in the experimental determination of the muon anomaly

Any particle with spin $\vec{s}$ has a magnetic moment $\vec{\mu}$ (internal current circulating)

$$\vec{\mu} = g_\mu \frac{e\hbar}{2m_\mu c}\,\vec{s}\ ; \quad g_\mu = 2\,(1 + a_\mu)$$

In non-relativistic quantum mechanics (QM) the g-factor would be $g = 1$ in relativistic QM Dirac 1928 predicted it to be $g = 2$ with leading radiative QED correction written conventionally as the lepton anomaly $a_\ell = (g_\ell - 2)/2 = \frac{\alpha}{2\pi}$ as first calculated by Schwinger in 1948. That $g_e = 2$ was found by Stern and Gerlach in 1922 quite some time before Dirac’s theory explanation. The anomaly $g_e = 2(1.00119 \pm 0.00005)$ was discovered much later in 1948 by Kusch and Foley 48 followed by Schwinger’s calculation later in 1948 given by $\frac{\alpha}{2\pi} \simeq 0.00116 \cdots$

The extreme precision one is able to reach in measuring and calculating the lepton anomalous magnetic moments is possible by taking the simplest object you can think of in the static limit which is provided with the electromagnetic vertex



$$= (-\mathrm{i}e)\, \bar{u}(p') \left[\gamma^\mu F_1(q^2) + \mathrm{i}\, \frac{\sigma^{\mu\nu} q_\nu}{2m_\mu} F_2(q^2) \right] u(p)$$

where

$$F_1(0) = 1 \quad ; \quad F_2(0) = a_\mu \tag{1}$$

The point is that a_μ is responsible for the Larmor (spin) precession frequency $\vec{\omega}_a$ which shows up when polarized muons are orbiting in a homogeneous magnetic field. The polarized muons are produced by shooting protons on a target which is producing pions which decay by parity (P) violating weak process $\pi^+ \to \mu^+ \nu_\mu$; $\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$. Here the maximal parity violation in both processes in production and decay are crucial as it allows one for a perfect tracking of the spin polarization information in the experiment.

The Larmor precession $\vec{\omega_a}$ of a beam polarized muons in a homogeneous magnetic field $\vec{B}$ is illustrated in Fig. 1. In a storage ring the muon beam must be focused which requires electric fields and if both electric and magnetic fields are present the Larmor precession frequency is given by the Bargmann-Michel-Telegdi

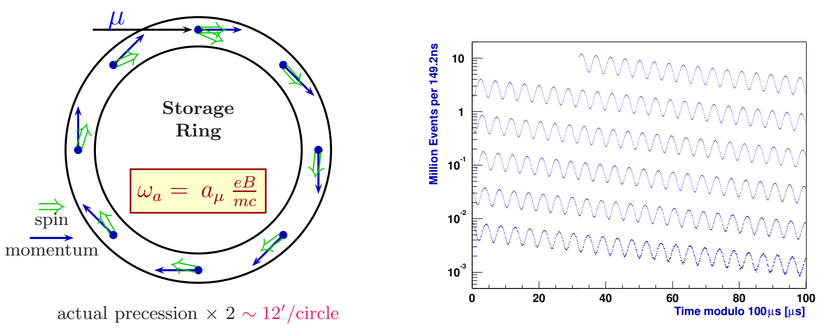


Fig. 1. left: At the Magic Energy (muon beam energy ~ 3.1 GeV), the angular frequency $\vec{\omega_a}$ is directly proportional to the magnetic field $\vec{B}$, so that the a_μ measurement is a frequency counting experiment. Right: Distribution of counts versus time for the 3.6 billion decays in the 2001 negative muon data–taking period of BNL [Courtesy of the E821 collaboration. Reprinted with permission from [3]. Copyright (2007) by the American Physical Society]

equation (derived in 1959):

$$\vec{\omega}_a = \frac{e}{m}\left[a_\mu \vec{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\vec{\beta}\times\vec{E}\right]^{E\sim 3.1\text{GeV}}_{\text{at ``magic }\gamma\text{''}} \simeq \frac{e}{m}\left[a_\mu \vec{B}\right] \tag{2}$$

and one sees that there exist a Magic Energy at which $\vec{\omega}_a$ is directly proportional to the magnetic $\vec{B}$ field. This happens at ~ 3.1 GeV when the energy-dependent relativistic Lorentz boost factor is tuned such that the coefficient in front of the electric field term $\left(a_\mu - \frac{1}{\gamma^2-1}\right) = 0$ is zero. The crucial point is that for the ultra-relativistic beam energy 3.1 GeV the muons life-time in lab frame $\gamma\tau_\mu$ 29 times longer, such that muons circulate many times before they decay. This principle developed at CERN has been applied in later experiments at Brookhaven (BNL) and Fermilab (FNAL). The new Fermilab muon $g-2$ measurement applying sophisticated improved technology brightly confirmed the 2004 final BNL result

$$a_\mu[\text{BNL}] = 11659209.1(5.4)(3.3)[6.3]\times 10^{-10} \tag{3}$$

now improved to

$$a_\mu[\text{FNAL}] = 11659207.05(1.14)(0.95)[1.48]\times 10^{-10} \tag{4}$$

and potentially triggers a new serious attack on the SM. The combined result is

$$a_\mu^{\text{exp}} = (11\,659\,207.15 \pm 0.80 \pm 0.95)[1.24])\times 10^{-10} \quad (\text{FNAL/BNL})\,. \tag{5}$$

This is a remarkable achievement with the precision of 124 ppb today from 540 ppb at BNL, a factor 4 improvement with statistical error as small as the systematic one. There are good reasons why a very different experimental technique for mea-

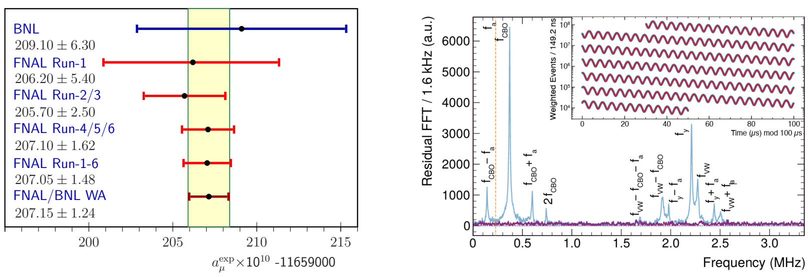


Fig. 2. Left: BNL vs. FNAL21/23/25 results. Right: $N(t) = N_0\,\mathrm{e}^{-t/\gamma\tau_\mu} \times \{1 + A\,\cos[\omega_a\,t - \varphi_0]\}$ modulo well understood corrections. N_0 is the normalization and A the P-violation asymmetry.

suring a_μ is much-needed. To come – novel complementary experiment at J-PARC in Japan [8–10] where a strict $\vec{E}=0$ cavity is utilized. In place of the ultra relativistic (CERN, BNL, Fermilab)) muons one then can work with ultra cold (J-PARC) muons such that one obtains a measurement with very different systematics.

3. Theory Today

The theoretical prediction of a_μ includes the entire SM, and the result one obtains reads:

$$\begin{aligned} a_\mu^{\rm SM} = F_2(0) &= a_\mu^{\rm (QED+EW+HVP_{LO}+HVP_{NLO}+HVP_{NNLO}+HLbL_{LO}+HLbL_{NLO})} \\ &= 0.00116591810(43) \quad \text{White Paper 2020 (WP20)}\,. \end{aligned} \tag{6}$$

This result, worked out by the Muon g-2 Theory Initiative [11], relies on the values $a_\mu^{\rm HVP,\,LO} = 693.1(4.0)\times 10^{-10}$ and $a_\mu^{\rm LbL,\,LO} = 90(17)\times 10^{-10}$ for the problematic leading hadronic contributions[1]. An interesting point concerns the inclusion of τ decay-spectra[2] in the calculation of $a_\mu^{\rm HVP,\,LO}$, which leads to the increased value $a_\mu^{\rm HVP,\,LO}[e^+e^- + \tau] = 705.3\pm 4.5\times 10^{-10}$ [26]. Similarly, calculations of $a_\mu^{\rm HVP,\,LO}$ based solely on τ data[3] by Roig et al. [32,33] yield $a_\mu^{\rm HVP,\,LO}[\tau] = 703.1^{+4.1}_{-4.0}\times 10^{-10}$. We learn that by including the τ data, both calculations yield values close to the known lattice QCD result from the Budapest-Marseille-Wuppertal (BMW) Collaboration [34]: $a_\mu^{\rm HVP,\,LO}[\rm LQCD : BMW] = 707.5 \pm 5.5\times 10^{-10}$. These results should be compared with the value $a_\mu^{\rm HVP,\,LO} = 717.0\pm 1.5\times 10^{-10}$, which would match the discrepancy between the SM prediction and the experimental result. Also, if we compare the result determined from τ data by Belle 2008: $a_\mu^{\pi\pi}[2m_\pi, 1.8{\rm GeV}] = (523.5\pm 3.9)\times 10^{-10}(\tau:\ \text{Belle})$, with the result obtained from the e^+e^- data $a_\mu^{\pi\pi}[2m_\pi, 1.8{\rm GeV}] = (504.6\pm 10.1, 1)\times 10^{-10}(e^+e^-:\ \text{CMD-2, SND})$, there is a difference of 18.9×10^{-10}. Adding this difference to the dispersive result of the e^+e^- data $a_\mu^{\rm HVP,\,LO}[e^+e^-] = (694.79\pm 4.18)\times 10^{-10}$, we obtain 713.7×10^{-10}. Recently, the Novosibirsk energy scanning experiment CMD-3 2023 reported an $e^+e^- \to \pi^+\pi^-$ cross-section measurement in the energy range from 0.327 to 1.2 GeV, which also yielded the significantly higher value $a_\mu^{\rm HVP,\,LO}(2\pi, \text{CMD-3}) = (526.0\pm 4.2)\times 10^{-10}$, which, compared to the corresponding WP20 estimate $(506.0\pm 3.4)\times 10^{-10}|_{\rm WP20}$, results in a difference of 20.0×10^{-10}. Adding this to

[1] My value from 2007 was $a_\mu^{\rm SM} = 0.00116591793(68)$ in [12, 13], i.e., the central value of WP20 changed only by $81.0-79.3 = 1.7$ in 10^{-10}, while the error could be reduced by 30% mainly due to more precise e^+e^- data in the calculation of the hadronic contributions. The WP20 result for $a_\mu^{\rm HVP,\,LO}$ was obtained by the e^+e^--data driven dispersion relation approach. The relevant e^+e^- data have been obtained form the energy scan data by CMD-2 [14], SND [15], SND20 [16] and CMD-3 [17] and the Initial State Radiation (ISR) data from KLOE [18–20], BaBar [21, 22], BESIII [23] and CLEO-c [24]

[2] This was pioneered by Davier et al. in [25].

[3] $\tau^- \to \nu_\tau\pi^0\pi^-$ spectra were recorded by ALEPH [27,28], OPAL [29], CLEO [30] and Belle [31].

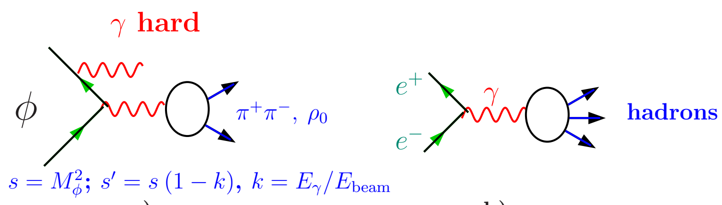


Fig. 3. a) Radiative return or Initial State Radiation (ISR), b) Standard energy scan measurement of the hadronic cross-sections.

$a_\mu^{\text{HVP, LO}}[e^+e^-] = (693.1 \pm 4.0) \times 10^{-10}|_{\text{WP}}$ yields $(713.1 \pm 4.7) \times 10^{-10}$, a surprising result since the second experiment with the SND detector at the same storage ring yielded a result of $a_\mu^{\text{HVP, LO}}(2\pi,\ \text{SND20}) = (508.3 \pm 4.2) \times 10^{-10}$, which roughly corresponds to the other e^+e^--data-based analyses. The large difference between the CMD-3 value and the SND20 result (both e^+e^- scan results from the e^+e^- collider facility in Novosibirsk) is another puzzle[4]!

While the $e^+e^- \to \pi^+\pi^-$ $R(s)$-ratio measurements (see Fig. 3) show significant discrepancies between the ISR experiments KLOE and BaBar, as well as between the scan experiments CMD-3 and SND20, for which we still have no explanations, we learn that taking the τ data into account yields results that systematically increase $a_\mu^{\text{HVP, LO}}$, making them compatible with the results of lattice QCD, in particular with the BMW value.

The unexpected shift in theory has resulted from recent advances in lattice calculations of the leading non-perturbative hadronic contributions, the vacuum polarization $a_\mu^{\text{HVP, LO}}$ and the light-light scattering $a_\mu^{\text{LbL, LO}}$. In particular, the significantly higher $a_\mu^{\text{HVP, LO}}$ result from BMW 2019 [34, 36] and subsequent validations by the Mainz/CLS [37–39] and RBC/UKQCD 2024 [40, 41] collaborations have opened a new window for the $a_\mu^{\text{HVP, LO}}$ contributions. For the significant shifts implied by the lattice QCD results, the Muon g-2 Theory Initiative (WP25) estimated

$$a_\mu^{\text{HVP, LO}} = 693.1(4.0) \to 713.2(6.1) \ \text{ and } \ a_\mu^{\text{LbL, LO}} = 90(17) \to 112.6(9.6) \qquad (7)$$

as necessary corrections. The resulting discrepancy between theory and experiment has consequently decreased to

$$\delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 38 \pm 63 \times 10^{-11} = \Delta a_\mu^{\text{BSM}}??? \qquad (8)$$

and this deviation must be confronted with subdominant SM effects such as the contribution of the weak interactions 12.9 σ, the HLbL effect 9.6 σ, and the higher

[4] CMD-3 seems to apply different radiative corrections (sQED vs RLA?) from other experiments; see [17, 35].

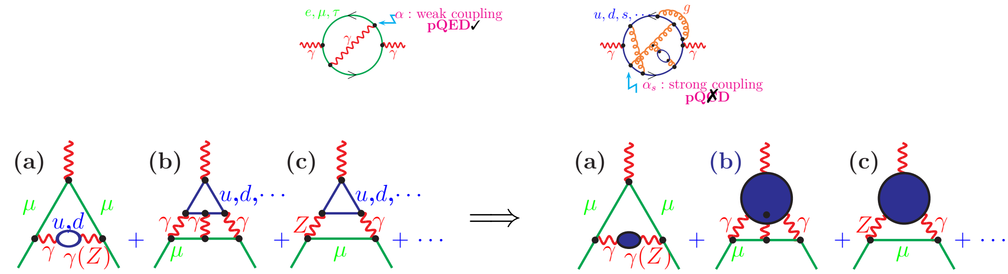


Fig. 4. The three classes of hadronic contributions in which light quark loops appear as hadronic "blobs": (a) Hadronic vacuum polarization (HVP) of order $O(\alpha^2), O(\alpha^3)$; (b) Hadronic light-light scattering (HLbL) of order $O(\alpha^3)$; (c) Hadronic effects in 2-loop electroweak (EW) radiative corrections of order $O(\alpha G_F m_\mu^2)$.

order (HO) HVP effect -8.2 σ. It is noteworthy that if one were to stick with the WP20 prediction, an apparently significant BSM effect would be 1.6 times larger than the phenomenologically well-established weak contribution. Furthermore, what we know from the intensive LHC searches makes it very unlikely that a gap of 5.6 σ could be real. We now have the unbelievably precise agreement to 9 decimal places in the muon g factor.

For details, I refer to my 2017 book [2] (for a short overview see [12] or [42]) or the recent WP25 report and the references contained therein. The updated QED prediction of a_μ is (see [43–46])

$$\begin{aligned} a_\mu^{\rm QED} &= \frac{\alpha}{2\pi} + 0.765,857,423(16)\left(\frac{\alpha}{\pi}\right)^2 + 24.050,509,82(28)\left(\frac{\alpha}{\pi}\right)^3 \\ &\quad +130.8734(60)\left(\frac{\alpha}{\pi}\right)^4 + 750.010(872)\left(\frac{\alpha}{\pi}\right)^5 \\ &= 116584718.8(2)\times 10^{-11} \end{aligned} \tag{9}$$

For the weak contributions, we have [47–50]

$$a_\mu^{\rm EW} = \underset{\rm LO}{194.79(1)}\times 10^{-11} - \underset{\rm NLO}{40.38(36)}\times 10^{-11} = 154.4(4)\times 10^{-11}\ . \tag{10}$$

4. Hadronic radiation corrections: the challenge for theory

Contributions from hadrons (quark loops) at low energy scales are a general problem in electroweak precision physics. About the muon anomaly, the relevant diagrams are displayed in Fig. 4. The calculation of non-perturbative effects is possible using hadron production data in conjunction with dispersion relations (DR), effective low-energy modeling within a resonance Lagrangian approach (RLA) such as Hidden Local Symmetry (HLS) or the Extended Nambu-Jona-Lasinio (ENJL) model, and ab initio lattice QCD calculations. In particular, for class (a), the HVP contribution can be calculated using a dispersion integral over $e^+e^- \to$ hadrons data. Here we have an independent amplitude determined by a specific data channel, a task that seems quite simple but, as we know, has its pitfalls. HLS global fits can help better understand the consistency of results of different experiments, which, in many cases, do not agree satisfactorily. Currently, ab initio lattice QCD seems to be the best option; in class (b), the HLbL contribution can be evaluated via the resonance Lagrangian approach (RLA) (CHPT extended by VDM in accordance with the chiral structure of QCD) as in the references [51–55]. A more model-independent approach is the $\gamma\gamma \to$ hadrons data-driven dispersive approach by Colangelo et al. [56, 57], in which 28 independent amplitudes are to be determined from an equal number of independent data sets[5]. In fact, data are available for only a few channels; nevertheless, the approach allows a reliable estimate of the HLbL contribution, since the leading piece is given by single-particle exchanges, similar to the RLA estimates, which use the same data to determine the amplitudes. Once again, the most promising are ab initio lattice QCD calculations [59, 60], in particular the position space approach [61–65], are in progress and have delivered first valuable results; finally, we have class (c), which contributes to the EW part. It receives leading contributions from quark and lepton triangle diagrams that have an $f\bar{f}Z$ vertex (f being a lepton or a quark) consisting of a vector (V) and an axial vector (A) term, while the other two vertices are of the vector type. Since the VVV part vanishes according to Furry's theorem, only the VVA part contributes, which is given by the well-known Adler-Bardeen-Jackiw anomaly (ABJ), which is both perturbative and non-perturbative due to the non-renormalization theorem, i.e., the leading effects can be calculated reliably, also because anomaly cancellation takes effect [53, 66].

5. Hadronic Vacuum Polarization (HVP) – Data & Status

The leading non-perturbative hadronic contributions $a_\mu^{\text{HVP, LO}}$ (see diagram (a) in Fig. 4) can be calculated using

$$R_\gamma(s) \equiv \sigma^{(0)}(e^+e^- \to \gamma^* \to \text{hadrons})/\frac{4\pi\alpha^2}{3s}$$

[5] This has been commented in [58]: the authors point out "in the absence of experimental data, reconstruction of light-by-light scattering amplitudes from their absorptive parts is ambiguous and requires additional theoretical input.

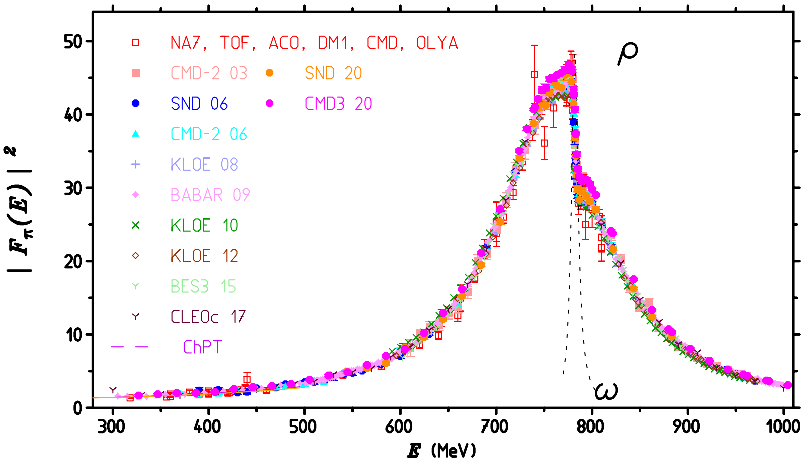


Fig. 5. The prominent ρ^0 resonance in the process $e^+e^- \to \pi^+\pi^-$ has been investigated in a large number of experiments, which unfortunately are not in satisfactory agreement.

data and the dispersion relation (DR)

$$a_\mu^{\text{HVP, LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left(\int_{4m_\pi^2}^{E_{\text{cut}}^2} \mathrm{d}s \, \frac{R_\gamma^{\text{data}}(s)\,\hat{K}(s)}{s^2} + \int_{E_{\text{cut}}^2}^{\infty} \mathrm{d}s \, \frac{R_\gamma^{\text{pQCD}}(s)\,\hat{K}(s)}{s^2}\right) \tag{11}$$

where $\hat{K}(s)$ is an analytically known bounded function. The amplification factor $1/E^4$ gives the low-energy data a very high weight. Therefore, the ρ^0 resonance in the $\pi^+\pi^-$ channel makes the main contribution to $a_\mu^{\text{HVP, LO}}$. As a result, $\sim 75\%$ come from the range $4m_\pi^2 < m_{\pi\pi}^2 < M_\phi^2$. In this data-driven approach, the experimental error from the data implies a theoretical uncertainty that poses a real obstacle to reliable, precise results. The R-data stem from the experiments CMD-2, SND, KLOE, BaBar, BESIII, CLEOc, SND20, CMD3, and are depicted together with older results in Fig. 5 The the balance of the contributions from different energy sections and the error related square errors are shown in Fig. 6. My e^+e^-–data based LO HVP evaluation reads

$$a_\mu^{\text{had}(1)} = (697.17 \pm 4.18)\, 10^{-10} \,. \tag{12}$$

The higher order hadronic correction of Fig. 7 can be calculated by appropriate DRs with corresponding kernel functions and up to three hadronic blob insertions. Summary of the hadronic and weak contributions (based on dispersive approach) where the LO-HVP and LO-HLbL results have to be updated by LQCD results! The uncertainties of the subleading NLO and NNLO and weak contributions are below one σ, i.e., they are DRA save. A selection of the HVP results is shown in Fig. 9. Similarly, Fig. 10 shows a series of HLbL results.

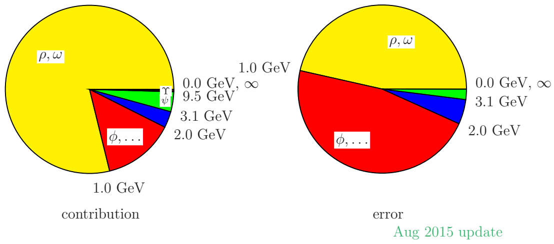


Fig. 6. Left: Contributions from various energy ranges. Right: Distribution of the error squares from corresponding ranges.

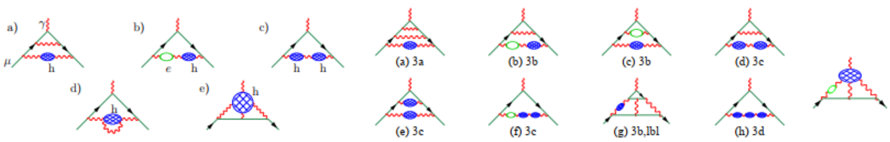


Fig. 7. Hadronic higher order contributions: involving LO, NLO and NNLO vacuum polarization, and light-by-light scattering insertions.

5.1. The Role of τ Decay spectra

The lesson we learn from Fig. 9 is that including the charged channel $\tau \to \pi\pi^0\nu_\tau$ spectra brings the data-driven dispersive results close to the lattice QCD results from the Budapest-Marseille-Wuppertal (BMW). The isovecttor (I=1) τ spectra are much simpler since there is no $\gamma, \rho, \omega, \phi$ mixing, in particular no $\rho^\pm$ mixing with the photon. While τ data have to be supplemented by the $\rho - \omega$ mixing contribution the e^+e^- data have to be corrected for $\rho^0 - \gamma$ mixing, which

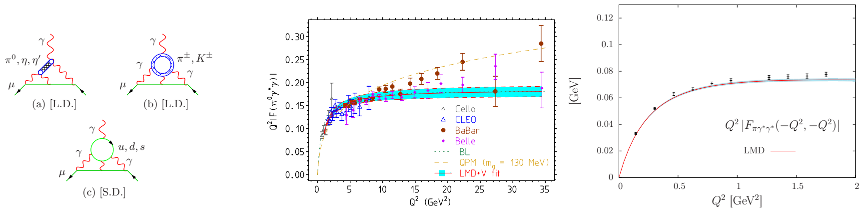


Fig. 8. Left: HLbL diagrams in meson effective theory (extended ChPT) for the long distance (L.D.) and quark loops for the short distance (S.D.) contributions. Right: $e^+e^- \to e^+e^-\gamma\gamma^* \to e^+e^-\pi^0$ data extracted $\pi^0\gamma\gamma$ form factor. CELLO and CLEO measurement of the π^0 form factor $\mathcal{F}_{\pi^0\gamma^*\gamma}(m_\pi^2, -Q^2, 0)$ at high space–like Q^2. Outdated by BABAR? Belle conforms with theory expectations!. First measurements of $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2; -Q^2, -Q^2)$ in lattice QCD.

Table 1. HVP, HLbL and weak contributions to a_μ where the HVP contributions rely on e^+e^- data. The HLbL is estimated mainly using transition form factors like $\gamma\gamma *^{\rightarrow} \pi^0$ and $\gamma\gamma^* \rightarrow \pi^+\pi^-, \pi^-\pi^0$.

$$
\begin{array}{lcrl}
a_\mu^{\mathrm{had}(1)} & = & (697.17 \pm 4.18) \times 10^{-10} & \text{(LO)} \\
a_\mu^{\mathrm{had}(2)} & = & (-10.89 \pm 0.067) \times 10^{-10} & \text{(NLO)} \\
a_\mu^{\mathrm{had}(3)} & = & (1.242 \pm 0.010) \times 10^{-10} & \text{(NNLO)} \\
a_\mu^{\mathrm{had,LbL}} & = & (10.34 \pm 2.88) \times 10^{-10} & \text{(HLbL)} \\
a_\mu^{\mathrm{weak}} & = & (15.4 \pm 0.4) \times 10^{-10} & \text{(LO+NLO)}\ .
\end{array}
$$

make e^+e^- data look like τ data concerning this QED-QCD mixup inherent in the neutral channel only.

6. Ab initio method: HVP by lattice QCD

The primary object for HVP in LQCD is the electromagnetic quark current correlator in Euclidean configuration space

$$\langle J_\mu(\vec{x}, t)\, J_\nu(\vec{0}, 0)\rangle\,, \quad \text{where} \ \ , J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \cdots \tag{13}$$

A Fourier transform yields the bare vacuum polarization function $\Pi(Q^2)$

$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4 x \mathrm{e}^{\mathrm{i}\, Qx} \langle J_\mu(x)\, J_\nu(0)\rangle = \left(Q_\mu Q_\nu - \delta_{\mu\nu}\, Q^2\right) \Pi(Q^2) \tag{14}$$

which is needed for the calculation

$$a_\mu^{\mathrm{HVP}} = 4\alpha^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2) \left\{\Pi(Q^2) - \Pi(0)\right\} \tag{15}$$

with $f(Q^2)$ as the known kernel function. The restrictions here concern the discretization of the configuration space in a finite volume (finite number of degrees of freedom): We have a lattice with lattice spacing a in a finite box with volume $V = L^4$. This requires extrapolations to the continuum and to infinite volume, as shown in Fig. 11. Computationally, CPU power is a significant limitation due to the large number of degrees of freedom that must be integrated over the QCD path integral. The light u, d, and s quarks are particularly costly, which is usually expressed in terms of the pion mass, with the CPU requirement being approximately proportional to $(1/m_\pi)^3$. Only recently have simulations with the physical pion mass become possible, illustrating the significant progress made by LQCD. Lattice QCD (LQCD) is a competitive method for calculating low-energy hadron physics matrix elements. It is a straightforward approach based on first principles. Note that for lattice approximations in a finite box, the momenta are quantized:

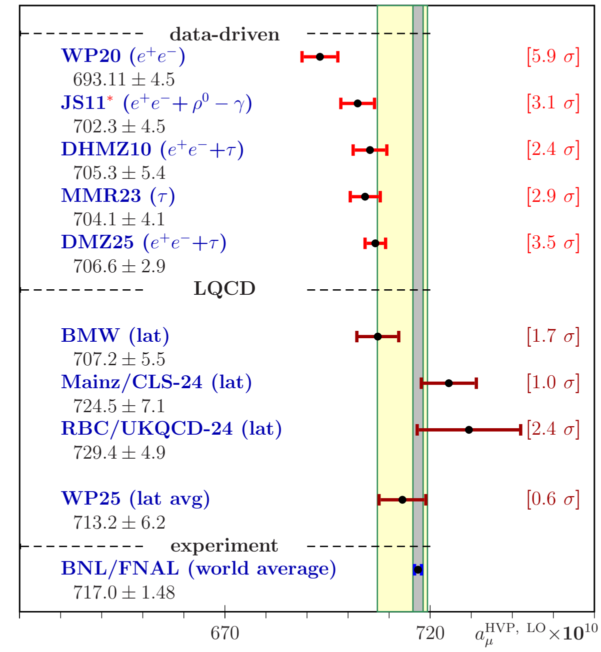


Fig. 9. A selection of $a_\mu^{\text{HVP, LO}}$ results obtained using different approaches. WP20 [11] is the dispersive result based exclusively on $e^+e^- \to$ hadron data. JS11 refers to an e^+e^--data-based result corrected for $\rho^0 - \gamma$ interference, which convincingly explains the puzzle surrounding the e^+e^-- versus τ-spectra relation [67]. This correction is necessary to separate the irreducible QCD component from the normally ignored QED-QCD mixing inherent in the experimental $\pi^+\pi^-$ production data. DHMZ10 [26] and the update DMZ25 [68] have included the isospin breaking (IB)-corrected τ data in addition to the e^+e^- data. MMR23 is an analysis based exclusively on τ data [69]. BMW, Mainz/CLS, and RBC/UKQCD are the latest lattice QCD results, which currently provide the most reliable hadronic contributions. The last point is the update of WP25 [70] (wheat-colored band), the result of the consensus theory compared to the experimental result from BNL/FNAL (gray band), represented by the fictitious $a_\mu^{\text{HVP, LO}}$ term required to close the gap between experiment and the theory prediction when dropping $a_\mu^{\text{HVP, LO}}$.

$Q_{\text{min}} = 2\pi/L$, where L is the length of the lattice box. The limit $Q_{\text{min}} = 2\pi/L \to 0$ is equivalent to the infinite volume limit $L \to \infty$. Lattice data are available for $Q^2 > (2\pi/L)^2$. Extrapolation to $Q^2 = 0$ can be done using Padè approximants or experimental HVP data. To achieve the required accuracy, LQCD data down to $Q^2_{\text{min}} \approx 0.1$ GeV^2 are needed. Indeed, current simulations reach $m_\pi a L \gtrsim 4$ so that for $m_\pi \sim 200$MeV or $Q_{\text{min}} \sim 314$MeV. Currently, about 44% of the contribution

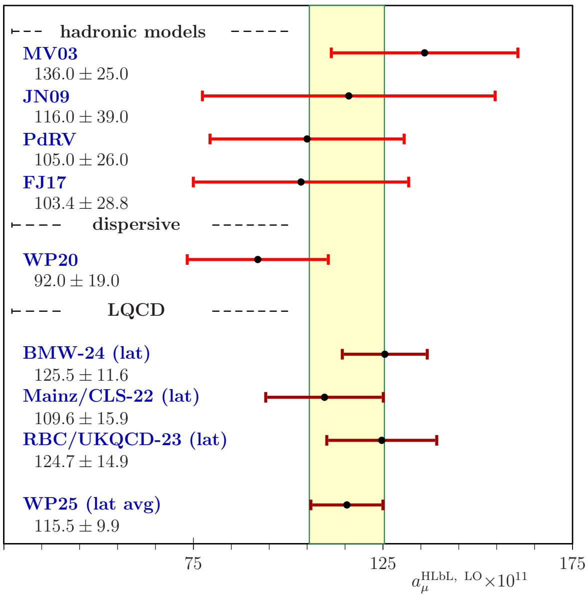


Fig. 10. A selected history of $a_\mu^{\rm LbL,\ LO}$ results. Pioneered by Hayakawa, Kinoshita and Sanda [51], Beijnens, Pallante and Prades [52] and Knecht and Nyffeler [53, 54]. Data point shown are MV03 [55], JN09 [42], the consensus PdRV [71], my book FJ17 [2] and the result based on the dispersive approach by Colangelo et al. of the White Paper WP20. The lattice QCD results are BMW-24 [72], Mainz/CLS-22 [64, 65] and RBC/UKQCD-23 [59, 60]. WP25 represents the current consensus of the Mon g-2 theory initiative [70] (White Paper update).

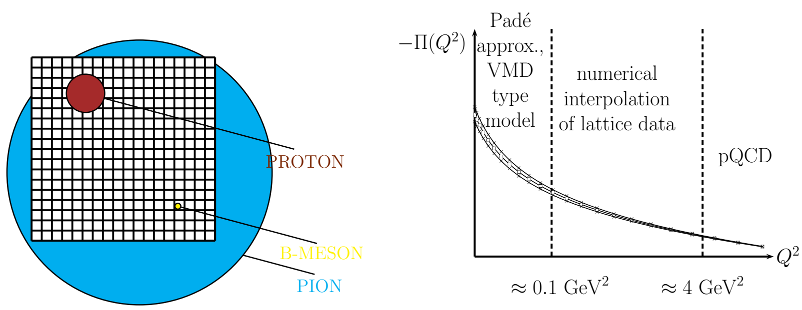


Fig. 11.

of low x to $a_\mu^{\rm HVP,\ LO}$ is not yet covered by lattice data directly. Data from BMW,

Collaboration	$a_\mu^{\text{HVP, LO}}$	$a_\mu^{\text{HVP, LO}}\lvert\text{corr}$
BMW-20	707.5(5.5)	707.2(5.5)
Mainz/CLS-24	724.5(7.1)	724.5(7.1)
RBC/UKQCD-24	734.5(13.0)	729.4(13.0)

Mainz/CLS-24, RBC/UKQCD, combined and averaged by the WP25 group, yield:

$$\underset{9.6\%}{a_\mu^{\text{SD}}(\text{iso}) = 69.06(22) \times 10^{-10}}, \underset{33.1\%}{\text{a}_\mu^{\text{W}}(\text{iso}) = 236.16(42) \times 10^{-10}},$$
$$\underset{57.2\%}{a_\mu^{\text{LD}}(\text{iso}) = 407.9(5.0) \times 10^{-10}} \; . \tag{16}$$

so that

$$a_\mu^{\text{HVP, LO}}(\text{iso})\,\big|\text{Avg.1}^{\text{lat}} = 713.1(5.0) \times 10^{-10} \tag{17}$$

and, including IB corrections,

$$a\mu^{\text{HVP, LO}}(\text{cor})\,\Big|_{\text{Avg.1}}^{\text{lat}} = 713.2(6.1) \times 10^{-10} \tag{18}$$

The individual flavor contributions of the light quarks (u, d) are about 90%, those of the strange quarks about 8%, and those of the charm quarks about 2%.

Previously, in 2017, LQCD results for the leading order a_μ^{HVP} in units of 10^{-10} were obtained by the Brookhaven, Zeuthen, Mainz, Edinburgh, and other groups. A first indication of a larger HVP value was obtained by the BMW collaboration (see Fig. 12). The recent LQCD results: have driven the latest WP25 study, and

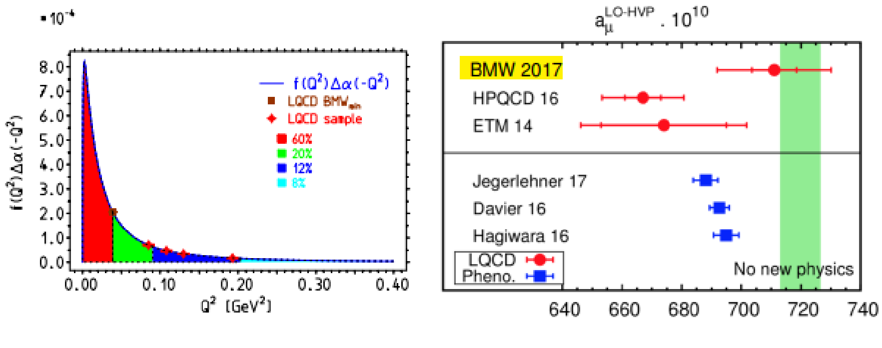


Fig. 12. Left: Percentage contributions from regions as predicted by the kernel function $f(Q^2)$. Right: Representation from Budapest-Marseille-Wuppertal 2017.

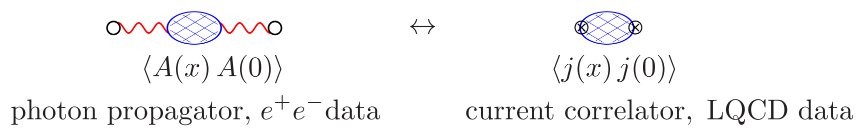
the new consensus value was ascertained to be

$$a\mu^{\text{HVP, LO}}\Big|_{\text{WP25}}^{\text{LQCD}} = 713.2(6.1) \times 10^{-10} \; . \tag{19}$$

7. What is the problem?

Here we have to remind the role of τ decay spectra for the $a_\mu^{\text{HVP, LO}}$ evaluation. The lesson we learn from Fig. 9 is that including the charged channel $\tau \to \nu_\tau \pi\pi^0$ spectra brings the data-driven dispersive results close to the lattice QCD results from the Budapest-Marseille-Wuppertal (BMW) Collaboration. The problem with the DRA-HVP evaluation must therefore be related to the experimental e^+e^- data.

Despite determined efforts to calculate and model QED corrections for hadronic processes(see [35, 73–80]), open questions remain. What can explain the large difference between experimental e^+e^--HVP data and lattice QCD-HVP data? In the process $e^+e^- \to$ hadrons, experiments measure the photon propagator, i.e., a mixture of QCD and QED effects that is difficult to disentangle. This requires the removal of external QED corrections, while the radiation of hadrons beyond sQED is not well understood, although sQED is a good approximation for low-energy meson production. In contrast, lattice QCD measures the hadronic blob exclusively, as a correlator of two hadronic currents:



It is interesting to compare the dipion spectra of $e^+e^- \to \pi^+\pi^-$ and $\tau^- \to \nu_\tau \pi^-\pi^0$ taking into account the Isospin Breaking (IB) effects (see Fig. 13). In many respects, the τ spectrum is much simpler than that of e^+e^-, since the charged $\rho^\pm$ does not mix with other states in the dominant low-energy range below about 1.05 GeV (above ϕ, below ρ'). The τ decays are mediated by the very heavy W boson, in stark contrast to e^+e^- annihilation, which is mediated far below the Z boson by the massless photon, which mixes with ρ^0, ω, and φ. τ spectra provide isovector $I = 1$ dipion spectra which should agree with the $I = 1$ part of the e^+e^- annihilation upon appropriate isospin breaking corrections. In contrast to the charged current (CC) data, understanding the neutral current (NC) data requires more complex QCD-driven hadron phenomenology modeling: Chiral perturbation theory (CHPT) has been extended to include spin-1 resonances, which are well described by the vector meson dominance model (VMD) or, in perfect form, by the Resonance Lagrangian Approach (RLA)) [81, 82], such as the Hidden Local Symmetry (HLS) (massive Yang-Mills) or the Extended Nambu-Jona-Lasinio (ENJL). These models predict the dynamic widths and dynamic mixing of $\gamma, \rho^0, \omega, \phi$ rather accurately.

A key effect that is often overlooked is the mixing of ρ^0, ω, and φ with the photon, especially the $\rho^0 - \gamma$ mixing, which directly impacts the relationship between the photon propagator and the One-Particle-Irreducible (1PI) HVP blob. This mixing appears in the NC measurements $e^+e^- \to \pi^+\pi^-$, but not in the CC

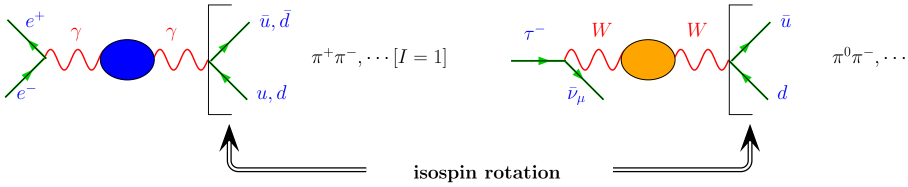


Fig. 13. τ spectra provide isovector $I = 1$ dipion spectra which should agree with the $I = 1$ part of the e^+e^- annihilation upon appropriate isospin breaking corrections.

data $\tau^- \to \nu_\tau \pi^- \pi^0$. This issue was examined in [67] to clarify the known discrepancy between the dipion spectra from τ decays $v_\pm(s)$ and the e^+e^- data $v_0(s)$, which persisted despite applying the commonly accepted isospin-breaking correction (IB) $R_{\mathrm{IB}}(s)$ to the τ data.

In fact, taking into account the $\rho - \gamma$ interference solves the mystery of τ (charged channel) vs e^+e^- (neutral channel). The $\rho - \gamma$ interference (which does not occur in the charged channel) is often mimicked by large shifts in the mass M_ρ and width Γ_ρ, but such large shifts are not consistent with known calculations of $M_{\rho^\pm} - M_{\rho^0}$, for example according to the Cottingham formula. The missing element in the standard derivations of the Gounaris-Sakurai formula (based on the VMD-I model, which breaks electromagnetic gauge invariance) is the omission of the $\gamma - \rho^0$ mixing propagator

$-\mathrm{i}\,\Pi^{\mu\nu\,(\pi)}_{\gamma\rho}(q) =$  ,

which we calculated using the gauge-invariant VMD-II approach together with scalar QED (sQED), which should be valid below 1 GeV (as can be learned from the $\gamma\gamma \to \pi^+\pi^-$ vs $\pi^0\pi^0$ data). We thus are confronted with a non-diagonal (γ, ρ^0) 2×2 matrix propagator, which we must diagonalize. We do not obtain the correct diagonalization by omitting the non-diagonal mixing term. The $\gamma - \rho^0$ mixing effect, which is proportional to eg in terms of couplings, where e is the electric charge and g is the effective coupling $\rho\pi\pi$, undergoes a resonance amplification, resulting in unexpectedly large distortions of the I=1 resonance peak[6], as shown in Fig. 14.

In [67], we **wrongly** corrected the τ spectral function[7] $v_-(s)$ as follows

$$v_0(s) = r_{\rho\gamma}(s)\, R_{\mathrm{IB}}(s)\, v_-(s)\,.$$

[6] This correction is robust, as it involves no new parameters that do not already appear in the GS formula. The effect only depends on the well-known leptonic ρ width $\Gamma_{\rho\to ee}$.

[7] That the correction $r_{\rho\gamma}(s)$ has to be applied to the e^+e^- data was already stated and discussed in Sect. 21.3 of [83].

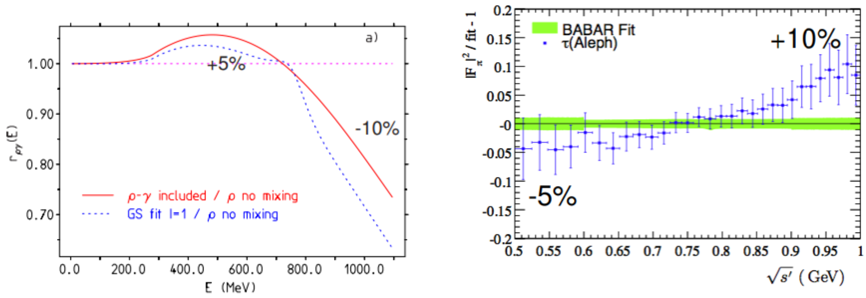


Fig. 14. Left: The ratio of the isospin $I = 1$ pion form factor $|F_\pi(E)|^2$ taking mixing into account, normalized to the case without mixing. Also shown is the same ratio of the $I = 1$ part of the e^+e^- data to the τ data GS fits, mimicked by fictitious parameter shifts in mass and width. Right: Best "proof" for our $\gamma - \rho^0$ mixing profile is the ratio of the ALEPH τ decay spectrum versus the BaBar e^+e^- spectrum [reproduced as part of Fig. 55 in `arXiv:1205.2228` by J. P. Lees et al.] [22]] (also see [26]).

But as we just argued, it is not the τ spectra that require correction for the absence of $\rho-\gamma$ mixing. Instead, this QED-QCD interference effect must be removed from the e^+e^- data in order to get the purely hadronic part of the 1PI self-energy. Thus the proper correction must read:

$$v_0(s)|_{\rm corr} = v_0(s)/r_{\rho\gamma}(s) = R_{\rm IB}(s)\, v_-(s)\ . \qquad (20)$$

In fact, as in the τ-channel, the $\gamma - \rho^0$ mixing is missing in the LQCD data, since this data are obtained by simulating the "QCD-only" path integral[8].

For the ρ^0 resonance range [0.63,0.96] GeV, the $\rho - \gamma$ interference leads to a shift [67][9].

$$\delta a_\mu^{\rm HVP,\,LO}[\rho\gamma] \simeq (5.1 \pm 0.5) \times 10^{-10}\ . \qquad (21)$$

This correction must be added to the standard e^+e^--based $a_\mu^{\rm HVP,\,LO}$.

$$a_\mu^{\rm HVP,\,LO}|_{\rm corr} = a_\mu^{\rm HVP,\,LO} + \delta a_\mu^{\rm HVP,\,LO}[\rho\gamma] \simeq (702.3 \pm 4.2) \times 10^{-10}\ . \qquad (22)$$

This result fits well with $a_\mu^{\rm HVP,\,LO}[ee+\tau] = (705.3 \pm 4.5) \times 10^{-10}$ from Davier et al. [26], where τ decay spectra are taken into account.

In summary, removing the $\rho^0 - \gamma$ mixing from the e^+e^- data is essential, as it brings the dispersive result into better agreement with the BMW lattice result, $a_\mu^{\rm HVP,\,LO} = (707.5 \pm 5.5) \times 10^{-10}$.

[8] Intrinsic QED effects relevant for HVP, such as final state radiation (FSR), must be included separately.

[9] It is important to apply the correction only to the $I = 1$ contribution. Due to the narrow widths of ω and φ, the mixing for these states is much smaller.

With respect to the value $a_\mu^{\text{HVP, LO}} = 713 \times 10^{-10}$ accepted by WP25, we conclude that at least part of the missing shift is explained. If the BMW results were to prevail over the WP25 value, the $\rho^0 - \gamma$ mixing would largely explain the discrepancy between the dispersive result based on e^+e^- data and the lattice and τ-based results.

The key point: QED admixtures are absent in lattice QCD[10], since QED is not included in the simulation, unlike in real measurements. This suggests that dispersive approach issues stem from an imprecise treatment of photon radiation by hadrons, such as pions (sQED vs. RLA; see [17, 35]).

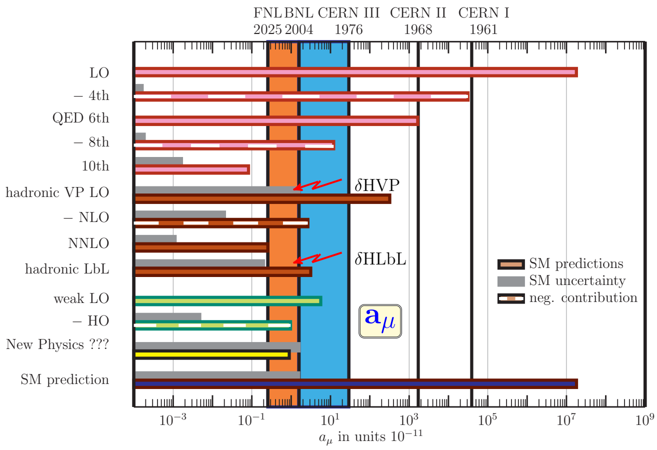


Fig. 15. Present muon $g - 2$ experiments testing various contributions. New Physics? = deviation $(a_\mu^{\text{exp}} - a_\mu^{\text{the}})/a_\mu^{\text{exp}}$. Limiting theory precision: hadronic vacuum polarization (HVP) and hadronic light-by-light (HLbL).

8. Electron g-2 Status and its Future

Here is a brief note on the status of the electron magnetic moment a_e, which is a test of QED that is highly sensitive to α electromagnetic as determined by atomic interferometry. In 2018, with $\alpha^{-1}(\text{Cs18}) = 137.035999046(27)$ one had $a_e^{\text{exp}} - a_e^{\text{the}} = (-84 \pm 36) \times 10^{-14}$, a deviation of $-\,2.3\ \sigma$ between theory and experiment. This presented a difficult situation for developers of BSM models, as the sign of δa_e differed from that of δa_μ, which was difficult to explain. In 2020, however, a more precise value of $\alpha^{-1}(\text{Rb20}) = 137.035999206(11)$ was obtained [84], which meant that $a_e^{\text{exp}} - a_e^{\text{the}} = (51 \pm 30) \times 10^{-14}$ changed sign and now predicted a gap of $+\,1.7\ \sigma$. Note that α, the most fundamental parameter in physics, changed by $5.4\ \sigma$ due to the switch from Cesium (Cs) to Rubidium

[10] QED effects relevant for HVP, such as final state radiation (FSR), must be included separately.

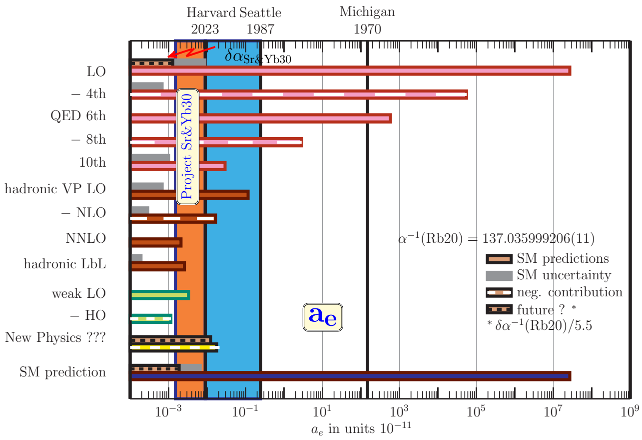


Fig. 16. Status and sensitivity of the a_e experiments testing various contributions. The error is dominated by the uncertainty of α(Rb20) from atomic interferometry. "New Physics" ? = deviation $(a_e^{\rm exp} - a_e^{\rm the})/a_e^{\rm exp}$, i.e., essentially absent presently. The blue band illustrates the improvement by the Harvard/Northwestern U. experiment. The orange band shows the possible progress by the Sr&Yb30 atomic interferometry project AION.

(Rb) atoms! Soon after, in 2022, a more accurate measurement improved the value of $a_e^{\rm exp} = 0.00115965218073(28)$ to $a_e^{\rm exp} = 0.00115965218059(13)$ [85]. But the QED prediction was also improved. In 2024, the universal 5-loop coefficient changed to $A_1^{(10)} = 5.873(128)$ [45] (crosschecked in [46]), which then leads to the prediction[11]

$$a_e^{\rm the} = 0.00115965218023(9) \;\; \text{and} \;\; a_e^{\rm exp} - a_e^{\rm the} = (36 \pm 16) \times 10^{-14}\;. \qquad (23)$$

Only a deviation of + 2.3 σ, which still agrees quite well with the SM prediction. In future atomic interferometry experiments (as part of the AION project [86]) based on Strontium and Yttrium, are expected to reduce the uncertainty of the electromagnetic fine structure constant such that an a_e prediction with an accuracy of 2×10^{-14} would be possible.

9. Conclusions

One of the most famous oracles that promised new physics beyond the SM has dissolved, thereby consolidating the SM up to the TEV scale, which is consistent with the LHC searches! However, this does not mean that the search for BSM

[11] In this result, also $a_e^{\rm LO-HVP}$ had to be slightly modified to take into account the change in $a_\mu^{\rm LO-HVP}$ (lattice vs dispersive result), i.e., $a_e^{\rm LO-HVP} = 1.871(11)\times 10^{-12}$ changes to $a_e^{\rm LO-HVP} = 1.923(09)\times 10^{-12}$.

physics as the main goal of particle physics is losing importance. What do a_μ and a_e tell us? High-precision physics may prove more difficult than expected to reach the limits of the SM, and this applies to both $a_\mu^{\rm exp}$ and $a_\mu^{\rm the}$. For the electron, the limitation arises from the need for extremely precise atomic interferometry to determine the electromagnetic fine structure constant in the Thomson limit. As far as $a_\mu^{\rm exp}$ is concerned, all experiments to date (CERN, BNL, FNAL) have used the magic γ trick, and an alternative type of experiment is urgently needed. Hopefully, the J-PARC project [8–10], which is promoting an experiment with ultra-cold muons in an $\vec{E} = 0$ field cavity, will be realized in the not-too-distant future.

Given the failure to provide a clear answer based on data-driven DRA, the MUonE determination (elastic μe scattering) of the HVP at CERN (directly measuring what LQCD calculates) would be a mandatory experiment that would need to be carried out [87–90]. Lattice QCD has proven to be an indispensable tool for predicting non-perturbative hadronic effects in electroweak precision observables. However, since a significant fraction of the results still have to be estimated by extrapolation to small lattice spacings and large volumes, further progress is required to reduce the extrapolation uncertainties.

As for an improved α determination as input for predicting a_e, the AION project (using Strontium and Yttrium ions) [86] promises significant progress.

Although lattice QCD already provides satisfactory results for the hadronic vacuum polarization, it remains an urgent task to understand the inconsistencies in the experimental data required in the dispersive approach: KLOE vs BaBar, SND20 vs CMD-3, $e^+e^- \to \pi^+\pi^-$ vs $\tau^- \to \pi^0\pi^-\nu_\tau$ data (NC vs CC); A better determination of the HLbL contribution requires more $\gamma\gamma \to$ hadrons data to improve the precision of known channels and explore new channels that have not been experimentally accessible so far.

Beyond BSM science fiction: Scrutinizing SM predictions and making progress in determining SM parameters such as M_t and M_H (which requires a high-precision Higgs/top quark pair factory) is a major challenge that is also important for a better understanding of early cosmology. But that would be another topic!

Outlook: We note that we have remarkable agreement between the following results:

$$
\begin{array}{lcll}
a_\mu^{\rm HVP-LO}[ee+\gamma\rho] & = & 702.3 \pm 4.2 \times 10^{-10} & \text{Eq. (22)} \\
a_\mu^{\rm HVP-LO}[ee+\tau] & = & 705.3 \pm 4.5 \times 10^{-10} & [26] \\
a_\mu^{\rm HVP-LO}[\tau] & = & 704.1 \pm 4.1 \times 10^{-10} & [91] \\
a_\mu^{\rm HVP-LO}[\rm LQCD : BMW] & = & 707.5 \pm 5.5 \times 10^{-10} & [34]
\end{array}
$$

Since Mainz/CLS and RB/UKQCD achieved slightly larger lattice results (compared to BMW), the 2025 white paper combined the lattice results to give the larger value $a_\mu^{\rm HVP-LO} = 713.2 \pm 6.2 \times 10^{-10}$. If the progress in the lattice results confirmed the slightly smaller BMW value, the mystery surrounding the discrepancy between the dispersive approach and the lattice results would be largely solved if the e^+e^- data were corrected for the $\rho^0 - \gamma$ interference. Then, the difference between the theoretical prediction and the experimental measurement at a level of 0.9 σ would still provide strong confirmation of the SM. In fact, the hybrid evaluation, which combines the "best" of the lattice and dispersive approaches: [92, 93] [BMW/DMZ-24], highlighted the result as "confirmation of the SM up to a level of 0.37 ppm."

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