Quantum Chromodynamics and strong interaction physics

F. Jegerlehner *
Humboldt-Universität zu Berlin, Newtonstrasse 15, D-12489 Berlin, Germany

*Lectures given at the University of Silesia, Katowice, Poland. * * *

Contents

1 Internal symmetries, unitary groups **1**

1.1 Finite dimensional representations of $SU(n)$ 4

1.2 Combining representations, reduction 5

1.3 Application to $SU(3)_{\text{flavor}}$ 8

1.4 The spectrum of low lying hadrons: 11

1.5 One particle states: pions and nucleons 14

1.5.1 Interpolating fields 14

1.5.2 Noether Theorem 15

1.6 Outlook 16

2 Spontaneous symmetry breaking and Nambu–Goldstone bosons **26**

2.1 The Goldstone theorem 26

2.2 Models of spontaneous symmetry breaking 30

2.2.1 A model with spontaneous breaking of a discrete symmetry 31

2.2.2 The Goldstone model 34

3 Chiral symmetry and quark flavor mixing **38**

3.1 Prolog: QCD as a part of the Standard Model of fundamental interactions 38

3.1.1 The matter fields 39

3.1.2 The gauge fields 42

3.2 Chiral transformations, chiral symmetry and the axial-vector anomaly 42

3.2.1 Chiral fields and the $U(1)$-axial current 42

3.2.2 The chiral group $U(n)_V \otimes U(n)_A$ 45

3.2.3 The Adler Bell, Jackiw triangle anomaly 47

3.3 Yukawa interaction of quarks and quark flavor mixing 51

3.3.1 Flavor mixing pattern 57

3.4 The Chiral Structure of Low Energy Effective QCD 61

4 Gauge principles and gauge invariance **68**

4.1 Abelian gauge theory: Quantum Electrodynamics 68

4.2 Non-Abelian gauge theories 76

4.2.1 Global non-Abelian symmetry 76

4.2.2 Local non-Abelian symmetry 77

4.2.3 Minimal couplings of the matter fields 78

4.2.4 Non-Abelian gauge field strength tensor, Yang-Mills action 80

4.2.5 Equations of motion and currents . . . 82

5 Quantization of gauge theories 87

5.1 The QCD Feynman rules . . . 88

6 Dimensional regularization 93

6.1 Tools for the Evaluation of Feynman Integrals . . . 96

6.1.1 $\epsilon = 4 - d$ Expansion, $\epsilon \to +0$ . . . 96

6.1.2 Bogolubov-Schwinger Parametrization . . . 97

6.1.3 Feynman Parametric Representation . . . 97

6.1.4 Euclidean Region, Wick–Rotations . . . 98

6.2 Scalar One–Loop Integrals . . . 100

6.3 Selected Properties of Scalar Integrals . . . 102

6.4 Tensor Integrals . . . 103

6.5 Special Integrals for massless propagators . . . 105

7 One–Loop Renormalization 111

7.1 The quark self–energy . . . 111

7.2 The gluon self–energy . . . 116

7.3 The Faddeev-Popov ghost self–energy . . . 125

7.4 The quark-gluon vertex . . . 127

8 The renormalization group, QCD and asymptotic freedom 138

8.1 Renormalization Group: the general solution . . . 138

8.2 Asymptotic Behavior . . . 143

8.2.1 Short distance or UV behavior . . . 143

8.2.2 Long distance or IR behavior . . . 144

8.3 Universality of the asymptotic behavior . . . 148

8.4 RG-invariance of physical observables . . . 149

8.5 The "running" parameters of QCD . . . 150

8.6 α_s in perturbation theory . . . 151

8.7 How does the RG work for asymptotically free theories? . . . 157

8.8 Decoupling of heavy states and the decoupling theorem . . . 159

8.9 MOM scheme, automatic decoupling . . . 161

8.10 N_f - flavor effective QCD and matching conditions . . . 164

8.11 Renormalization Scheme Dependence . . . 170

8.12 General $O(\alpha^2)$ framework for predicting $R(s)$ . . . 176

8.12.1 Current correlators and the Källen-Lehmann spectral representation 186
8.13 High energy hadron-production in e^+e^-–annihilation 188
8.14 Non–Perturbative Effects, Operator Product Expansion 195
8.14.1 Testing non–perturbative hadronic effects via the Adler function 197
8.15 Some considerations on hadron production at low energy 200
8.15.1 Pions and Scalar QED 201
8.15.2 The Vector Meson Dominance Model 203
8.15.3 Mass and Width of Vector Mesons 205
8.15.4 $\rho^0 - \gamma$–Mixing 207
8.15.5 Vector–Meson Production Cross–Sections 208
8.15.6 $\rho^0 - \omega$–Mixing 209
8.15.7 Gounaris-Sakurai Parametrization of $F_\pi(s)$ 210
8.15.8 The Theory of the Pion Form Factor in the Threshold Region 211
8.15.9 Beyond Chiral Perturbation Theory: Models with Massive Spin 1 Fields . . 217

9 Quarkonia **223**
9.1 Charmonium 224
9.2 Charmed Mesons 228
9.3 The Υ resonances 230
9.4 Bottom Mesons 232
9.5 Non-Relativistic Potential Models 233
9.5.1 More on the heavy quark potential 240
9.6 A digression: positronium in QED 245
9.6.1 e^+e^-–annihilation into photons: $e^+e^- \to 2\gamma,\ 3\gamma$ 247
9.7 Quarkonia Decays 251
9.8 Matrix elements of mesonic bound states 254
9.9 Applications 262
9.9.1 Determination of α_s 262

10 Jets **269**
10.1 QCD corrections $O(\alpha_s)$ to the 2–jet cross–section 276
10.1.1 Virtual corrections 277
10.1.2 Real soft gluons 278
10.1.3 Virtual plus real soft gluons 280
10.2 3–Jet Events 281
10.3 Summary and Discussion 283

A Solved Problems **289**

A.1 Exercises: Section **??** . . . 289

A.2 Exercises: Section **??** . . . 297

A.3 Exercises: Section **??** . . . 305

A.4 Exercises: Section **??** . . . 306

A.5 Exercises: Section **??** . . . 310

A.6 Exercises: Section **??** . . . 315

A.7 Exercises: Section **??** . . . 317

A.8 Exercises: Section **??** . . . 319

A.9 Exercises: Section **??** . . . 321

A.10 Exercises: Section **??** . . . 324

1 Internal symmetries, unitary groups

Empirically, physical states are known to often show up in multiplets of symmetry groups. Familiar examples are the flavor symmetries in strong interaction physics $SU(2)_{\text{flavor}}$ (= isospin), $SU(3)_{\text{flavor}}$ (= isospin plus hypercharge), etc. Internal symmetries are to be distinguished from the space-time symmetries, which are unified in the Poincaré group. The symmetry groups G of interest here are the groups $SU(n)$, defined by the set of complex $n \times n$-matrices U which are **unitary** ($U^{-1} = U^+$) and **unimodular** ($\det U = 1$) and with matrix multiplication as the group operation. The requirement of unitarity ensures that the transition probabilities between states are preserved:

$$|< \varphi \mid \psi >|^2 = |< \varphi' \mid \psi' >^2 = |< \varphi \mid U^+ U \mid \psi >|^2$$

of course $\mid \psi > \rightarrow \mid \psi' > = U \mid \psi >$ is a symmetry if and only if all group elements U commute with the total Hamiltonian H of the system:

$$[U, H] = 0 \ \ \forall \, U \ \in \ G.$$

Since any unitary matrix $\tilde{U}$ can be written as a product $\tilde{U} = U e^{i\varphi}$ of a matrix U with $\det U = 1$ and a phase factor $e^{i\varphi}$, a unitary group $U(n)$ is equivalent to a direct product $SU(n) \otimes U(1)$. Therefore we may restrict ourselves to a consideration of the simple groups $SU(n)$. Possible $U(1)$ factors may be discussed separately.

The groups $SU(n)$ have $r = n^2 - 1$ real continuous parameters $\omega_i \, (i = 1, \ldots, r)$. A complex $n \times n$ matrix has $2n^2$ real parameters, unitarity implies n^2 conditions and $\det U = 1$ yields one further condition. Therefore, $SU(n)$ is characterized by r infinitesimal **generators** T_i and a general $SU(n)$ transformation can be written as

$$U = U(\omega) = \exp \left(i \sum_{i=1}^{n^2-1} T_i \omega_i \right)$$

and r is called **order** of the group.

The generators are Hermitian $T_i = T_i^+$ (which guarantees that U is unitary), traceless $\text{Tr } T_i = 0$ (which implies $\det U = 1$) and may be normalized so that $\text{Tr } (T_i T_j) = \frac{1}{2} \delta_{ij}$.

A convenient (non unique) basis for the matrices T_i, written conventionally as $T_i = \lambda_i / 2$, can be constructed as follows. For the $n - 1$ possible diagonal traceless Hermitian λ_i choose

$$\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}, \ldots, \sqrt{\frac{2}{n(n-1)}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & -(n-1) \end{pmatrix}. \quad (1.1)$$

Then form the $\frac{n(n-1)}{2}$ off-diagonal matrices λ_i with 1 in a given off-diagonal position above the diagonal, 1 in the transposed position and zeros elsewhere. Also form the $\frac{n(n-1)}{2}$ off-diagonal matrices λ_i with a $-i$ in a given off-diagonal position above the diagonal, $+i$ in the transposed position and zeros elsewhere.

The **Lie-algebra** (commutation rules)

$$[T_i, T_k] = ic_{ikl}T_l$$

determines the structure constants c_{ikl}, which are real, totally antisymmetric and satisfy the **Jacobi identity**

$$c_{ikn}c_{nlm} + \text{ cyclic terms in}(ikl) \equiv 0.$$

A Lie group G and its structure constants c_{ikl} uniquely determine each other in a neighborhood of the identity element of G. In a Lie-algebra there is a maximum number l of simultaneously commuting (i.e. diagonal) elements. l is called the **rank** of the group. The $SU(n)$ groups have rank $l = n - 1$, which is obvious in the basis given above. The states belonging to a $SU(n)$ multiplet may be labeled, as usual, by the eigenvalues of the simultaneous eigenstates of the l diagonal matrices which we denote by $H_1, \ldots, H_l$. The structure of a multiplet is characterized by a **weight diagram** which displays the eigenvalues of the states on a $(H_1, \ldots, H_l)$ plot.

The remaining generators may be combined into pairs of **ladder operators** (a raising and a lowering operator) $E_{\pm\alpha}(\alpha = 1, \ldots, \frac{r-l}{2})$ which map the different eigenstates of a multiplet into each other. The $E_{\pm\alpha}$'s are non-Hermitian matrices with 1 in a given off-diagonal position and zeros elsewhere.

For $SU(2)$ and $SU(3)$ we list some basic properties in the following.

a) $SU(2)$: Order $r = 3$, rank $l = 1$

Structure constants: $c_{ikl} = \epsilon_{ikl}$ the fully antisymmetric permutation tensor.

Generators: $T_i = \frac{\tau_i}{2}$; τ_i the Pauli matrices [1]

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Diagonal operators: $H_1 = \frac{\tau_3}{2} = I_3$: 3^{rd} component of isospin.

Eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Eigenvalues of I_3 : $\frac{1}{2}, -\frac{1}{2}$

Ladder operators: $E_{\pm 1} = \frac{1}{2}(\tau_1 \pm i\tau_2)$

$$E_{+1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

[1] Properties of the Pauli matrices:

$$\begin{aligned} [\tau_i, \tau_k] &= 2i\epsilon_{ikl}\tau_l \,, \quad \{\tau_i, \tau_k\} = 2\delta_{ik} \\ \tau_i^+ &= \tau_i \,, \quad \tau_i^2 = 1 \,, \quad Tr\, \tau_i = 0 \\ \tau_i\tau_k &= \frac{1}{2}\{\tau_i, \tau_k\} + \frac{1}{2}[\tau_i, \tau_k] = \delta_{ik} + i\epsilon_{ikl}\tau_l \end{aligned}$$

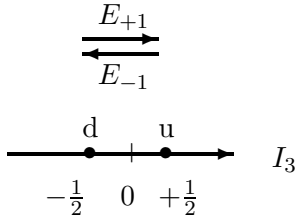


Figure 1.1: Weight diagram for the (u, d) quark doublet.

b) $SU(3)$: Order $r = 8$, rank $l = 2$

Structure constants : $c_{ikl} = f_{ikl}$, where the non-vanishing entries are permutations of the elements

$$f_{123} = 1$$
$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = 1/2$$
$$f_{458} = f_{678} = \sqrt{3}/2.$$

Generators: $T_i = \frac{\lambda_i}{2}$; λ_i the Gell-Mann matrices [2]

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Diagonal operators:

$$H_1 \;=\; \frac{\lambda_3}{2} = I_3 \quad 3^{rd} \text{ component of isospin}$$

[2] Properties of the Gell-Mann matrices:

$$\begin{aligned} [\lambda_i, \lambda_k] &= 2i\, f_{ikl} \lambda_l \;, \;\; \{\lambda_i, \lambda_k\} = \frac{4}{3}\delta_{ik} + 2 d_{ikl} \lambda_l \\ Tr\, \lambda_i &= 0 \;, \;\; Tr\, \lambda_i \lambda_k = 2\delta_{ik} \\ Tr\, \lambda_i\, [\lambda_k, \lambda_l] &= 4i\, f_{ikl} \;, \;\; Tr\, \lambda_i\, \{\lambda_k, \lambda_l\} = 4i\, d_{ikl} \end{aligned}$$

$$d_{118} = d_{228} = d_{338} = -d_{888} = 1/\sqrt{3} \;; \;\; d_{448} = d_{558} = d_{668} = d_{778} = -1/(2\sqrt{3}) \;;$$
$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = 1/2 \;.$$

$$H_2 \;=\; \frac{\lambda_8}{2} \doteq \frac{\sqrt{3}}{2} Y \;,\;\; Y \text{ hypercharge}$$

Eigenvectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eigenvalues of (I_3, Y): $\left(\frac{1}{2}, \frac{1}{3}\right)$, $\left(-\frac{1}{2}, \frac{1}{3}\right)$, $\left(0, -\frac{2}{3}\right)$

Ladder operators:

$$E_{\pm 1} \;=\; T_1 \pm i T_2 \;:\; E_{+1} \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; E_{-1} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{\pm 2} \;=\; T_4 \pm i T_5 \;:\; E_{+2} \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; E_{-2} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$E_{\pm 3} \;=\; T_6 \pm i T_7 \;:\; E_{+3} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \;,\; E_{-3} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

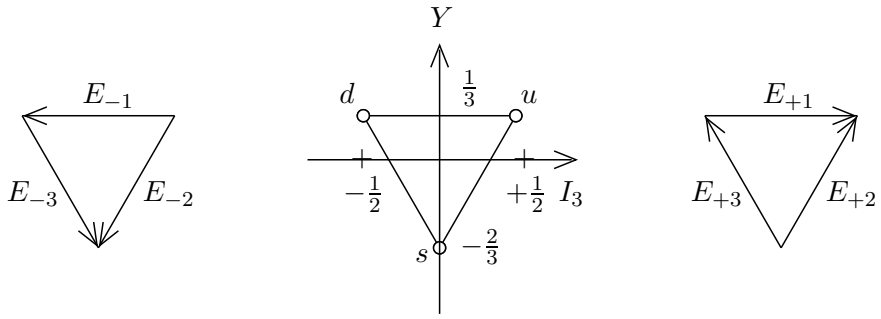


Figure 1.2: Weight diagram for the (u, d, s) quark triplet.

1.1 Finite dimensional representations of $SU(n)$

Given the structure constants c_{ikl} of $SU(n)$ any set of Hermitean traceless $N \times N$ matrices $\tilde{T}_i (i = 1, \ldots n^2 - 1)$ satisfying the Lie-algebra

$$\left[\tilde{T}_i, \tilde{T}_k\right] = i c_{ikl} \tilde{T}_l$$

is called a **representation** of the $SU(n)$ Lie-algebra. The unitary unimodular matrices $\tilde{U} = \exp(i \sum_{i=1}^{n^2-1} \tilde{T}_i \omega_i)$ then form a representation of $SU(n)$. The smallest non-trivial irreducible representation is the **fundamental representation** of dimension $N = n$. This is the representation which defines $SU(n)$. In gauge theories the fundamental spin 1/2 matter fields of quarks and leptons are in this representation. The Jacobi identity implies that there always exists the **adjoint representation** of dimension r with generators

$$(\tilde{T}_i)_{kl} = -i c_{ikl}$$

In gauge theories the spin 1 gauge fields in any case must be in this representation, as we shall see below.

The complex **conjugate** of a representation is also a representation since

$$(U_1 U_2)^* = U_1^* U_2^*.$$

Two representations are **equivalent** if we can transform one into the other by a change of basis:

$$S D_1(U) S^{-1} = D_2(U) \ \ \forall\, U \ \in \ G.$$

The conjugate representation n^* of the fundamental representation n is not a new representation if it is equivalent to n. In fact for the fundamental representation 2 of $SU(2)$ 2^* is equivalent to 2. In contrast, the conjugate representations 3^* of the fundamental representation 3 of $SU(3)$ is a new (inequivalent) representation. In QCD this crucial property of $SU(3)$ allows to distinguish color triplets of quarks (which transform according to the 3 representation) from color triplets of antiquarks (which transform according to the 3^* representation).

A representation is called **irreducible** if it cannot be transformed by a change of basis to block-diagonal form:

$$D(U) = \begin{pmatrix} D_1(U) & 0 \\ 0 & D_2(U) \end{pmatrix} = D_1(U) \oplus D_2(U) \ \ \forall\, U \ \in \ G.$$

If such a transformation exist, the representation is **reducible**. The irreducible representations are the basic building blocks of any representation. Particle multiplets are classified by the irreducible representations of a symmetry group.

The possible irreducible representations can be constructed by decomposing products of the fundamental representation into irreducible blocks. In the following we briefly discuss how this can be done.

1.2 Combining representations, reduction

Let $\psi_i (i = 1, \ldots n)$ be a **vector** transforming under the fundamental representation n of $SU(n)$. A **tensor product** $\psi_{i_1} \ldots \psi_{i_m}$ forms a tensor $\psi_{i_1 \ldots i_m}$ which transforms according to

$$\psi_{i_1 \ldots i_m} \to \psi'_{i_1 \ldots i_m} = U_{i_1 i'_1} \ldots U_{i_m i'_m} \psi_{i'_1 \ldots i'_m} \ .$$

For $m > 1$ this product representation, denoted by $n \otimes n \otimes \ldots \otimes n$ (m factors), is reducible.

One can decompose $\psi_{i_1 \ldots i_m}$ into a sum of tensors of different **symmetry class** with respect to **permutations** of the indices $i_1 \ldots i_m$ as follows:

Choose a set of positive integers $n_1, n_2 \ldots, n_k$ with $n_1 \geq n_2 \geq \ldots \geq n_k$ which form a partition of m : $n_1 + n_2 + \ldots + n_k = m$. Then group the indices $i_1 \ldots i_m$ into k classes $(i_{11} \ldots i_{1n_1})$, $(i_{21} \ldots i_{2n_2}), \ldots, (i_{k1} \ldots i_{kn_k})$ and write them in form of a tableau of k stacked rows where the first row has n_1 boxes containing the indices $i_{11} \ldots i_{1n_1}$, the second row has n_2 boxes containing the indices $i_{21} \ldots i_{2n_2}$ and so on. The tableau obtained is called a **Young tableau** (often called

Young diagram).

1. symmetrize in rows →

2. antisymmetrize in columns ↓

i_{11}	i_{12}	·	·	·	·	...	i_{1n_1}
i_{21}	i_{22}	·	·	...	i_{2n_2}		
⋮							
i_{k1}	i_{k2}	...	i_{kn_k}				

With a Young tableau we associate a tensor of a given symmetry class by the following construction. By **convention** first symmetrize $\psi_{i_1 \ldots i_m}$ in each group of indices appearing in the rows. Afterwards anti-symmetrize in the indices appearing in each column. By this a tensor of a given symmetry class is defined. According to the convention (first symmetrize in rows then anti-symmetrize in columns) tableaus with indices permuted in columns represent the same tensor. Tableaus with indices permuted in rows represent the same tensor if and only if the indices are not anti-symmetrized with indices in a different row.

For $SU(n)$ a tensor index can take the values $i = 1, \ldots, n$ only. Hence, there cannot be more than n rows for anti-symmetrization.

One easily verifies that group transformations do not mix tensors from different symmetry classes. The following central theorem holds:

a) Tensors in a given symmetry class form an invariant irreducible subspace. The group representation induced (by projection to the invariant subspace) in this subspace by the fundamental representation is irreducible.

b) The irreducible representations generated through all possible symmetry classes are exhaustive (i.e. there are no irreducible representations which cannot be obtained this way).

Symmetrization and anti-symmetrization obviously reduces the number of independent components of a tensor. The number of independent components of a tensor of a given symmetry class is equal to the **dimension** of the irreducible representation.

The irreducible representation of highest dimension is represented by the totally symmetric tensor.

i_1	i_2	...	i_m

There is only **one** such representation in $n \otimes \ldots \otimes n$ (m factors)

$$\psi_{(i_1 \ldots i_m)} = \frac{1}{m!} \sum_{\text{permutations } p} \psi_{ip(1)} \cdots_{ip(m)} \ .$$

A column with n boxes represents a tensor of rank zero i.e. a singlet and corresponds to a 1-dimensional trivial representation:

$$\varphi = \epsilon^{i_1 \ldots i_n} \psi_{i_1} \ldots \psi_{i_n}$$

Therefore, if a column with n boxes is part of a larger Young tableau it can be omitted!

A $n-1$ fold antisymmetric product of ψ_i's transforms as a n^* (complex conjugate of the fundamental representation n):

$$\chi^i = \epsilon^{ii_1 \ldots i_{n-1}} \psi_{i_1} \ldots \psi_{i_{n-1}}$$

since

$$\psi_i \chi^i = \varphi = \epsilon^{ii_1 \ldots i_{n-1}} \psi_i \psi_{i_1} \ldots \psi_{n-1}$$

is a singlet.

We now present some specific properties of $SU(2)$ and $SU(3)$:

a) $SU(2)$:
The fundamental representation is 2 . 2^* is equivalent to 2 . Indices have 2 possible values $i = 1, 2$.

A tableau is a singlet. as a part of a larger tableau can be omitted i.e. $\equiv$ etc. All nontrivial representations are characterized by a row:

Tableau: , , , etc.

Dimension: 2 3 4

Product representations and their reduction follow by combining corresponding tableaus in all possible ways.

Examples: $SU(2)$ interpreted as spin

1.

$$2 \otimes 2 \quad = \quad \square \times \square = \begin{array}{|c|}\hline \\ \hline \end{array} + \square\square = 1 \oplus 3$$

i.e. two spin 1/2 particles can group into a singlet of spin 0 and a triplet of spin1

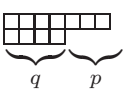
2.

$$\begin{aligned} 2 \otimes 2 \otimes 2 = (\begin{array}{|c|}\hline \\ \hline \end{array} + \square\square) \times \square \quad &= \quad \begin{array}{|c|c|}\hline \\ \hline \end{array} + \begin{array}{|c|c|}\hline \\ \hline \end{array} + \square\square\square \\ &= \quad \square + \square + \square\square\square \\ &= \quad 2 \oplus 2 \oplus 4 \end{aligned}$$

i.e. three spin 1/2 particles can group into two doublets of spin 1/2 and a quartet of spin 3/2.

b) $SU(3)$:
The fundamental representation is 3 . 3^* is inequivalent to 3. Indices have 3 possible values $i = 1, 2, 3$.

A tableau is a singlet. as part of a larger diagram can be omitted i.e. $\equiv$. etc.
All non-trivial representations are characterized by tableaus with one or two columns:



Each corresponds to a n^* i.e. an irreducible representation is characterized by two indices (p, q) and transforms as a tensor

$$\psi^{j_1 \ldots j_q}_{i_1 \ldots i_p} \text{ symmetrized in } (i_1 \ldots i_p) \text{ and } (j_1 \ldots j_q)$$

where $i_1 \ldots i_p$ transform under 3 and $j_1 \ldots j_q$ under 3^*.

We may write $\psi^{j_1\cdots}_{i_1\ldots}$ in product form

$$\psi^{j_1\ldots j_q}_{i_1\ldots i_p} = \chi^{j_1}\ldots\chi^{j_q}\psi_{i_1}\ldots\psi_{i_q}$$

with $\chi^i = \epsilon^{ikl}\psi_k\psi_l$. Together with the symmetrization it can be shown that the trace condition

$$\sum_{j=1}^{3}\psi^{jj_2\ldots j_q}_{ji_2\ldots i_p} = 0$$

must hold. This restricts the number of independent components of the tensor, which equals the dimension of the irreducible representation $D(p,q)$: One finds

$$D(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2).$$

The generators $\tilde{T}_i$ of a given irreducible representation can be worked out from the transformation law

$$\psi'^{j_1\ldots j_q}_{i_1\ldots i_p} = U^*_{j_1j'_1}\ldots U^*_{j_qj'_q}\,U_{i_1i'_1}\ldots U_{j_pj'_p}\psi^{j'_1\ldots j'_q}_{i'_1\ldots i'_p}$$

for infinitesimal transformations.

The simplest irreducible representation are given in the following table:

(p,q)		$D(p,q)$	*tableau*	*tensor*	
$(0,0)$		1		1	singlet
$(1,0)$		3		ψ_i	triplet
$(0,1)$		3^*		ψ^i	antitriplet
$(2,0)$		6		ψ_{ik}	sextet
$(0,2)$		6^*		ψ^{ik}	antisextet
$(1,1)$	adjoint	$8 = 8^*$		ψ^i_k	octet
$(3,0)$		10		ψ_{ikl}	decaplet
$(0,3)$		10^*		ψ^{ikl}	antidecaplet

1.3 Application to $SU(3)_{\text{flavor}}$

Low lying hadronic states may be classified in $SU(3)_{\text{flavor}}$ multiplets. The relevant quantum numbers are the baryon number B, isospin I and strangeness S. We can achieve that multiplets are centered on the origin if we replace strangeness S by hypercharge Y

$$Y = B + S.$$

Empirically, the electric charge of a hadron is given by

$$Q = I_3 + \frac{Y}{2}\ .$$

In the quark model of hadrons mesons ($B = 0$) are quark – antiquark states

$$M = (q\bar{q})$$

baryons ($B = 1$) are three quark states

$$B = (qqq)$$

where $q = u, d, s$. The quarks (u, d, s) are in the fundamental representation 3, the antiquarks $(\bar{u}, \bar{d}, \bar{s})$ in the representation 3^*.

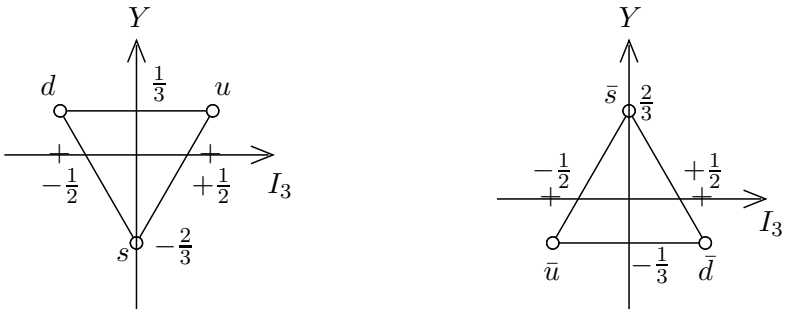


Figure 1.3: Basic building blocks of the $SU(3)$ quark model.

Direct products of representations may be reduced (decomposed) into irreducible blocks by combining boxes of the corresponding Young tableaux in all possible ways with the restriction that antisymmetric pairs must be preserved. The latter condition is nontrivial but may be satisfied by the following construction:

In order to append to the first tableau the second one in all admissible ways which respect the (anti -) symmetrization, we place in each box of the second tableau letters (in lexicographic order) with identical letters in each given row (symmetrized). Thus we insert a's in the first row, b's in the second row, etc. All boxes of the second tableau are now appended to the right-hand ends of the rows of the first one (which represents the upper left-hand corner of the new diagram) in all possible ways. Thus we first append all a's to the first tableau (in all admissible ways) with no more than one a per column (anti-symmetrized). To the such obtained enlarged tableaux append all b's (in all admissible ways) with no more than one b per column, etc.

Some of the tableaux such obtained are not admissible because they do not take into account properly the (anti -) symmetrization of the original boxes and have to be thrown away (also in order to avoid double counting).

Here we need a definition: a sequence of letters a, b, c, $\cdots$ is *admissible* if at any point of the sequence at least as many a's have occurred as b's, at least as many b's have occurred as c's etc.

Examples: a) admissible: abcd, aabcb, ...

b) not admissible: abb, acb, ...

Now consider for each tableau constructed above the full sequence of letters formed by reading from *right to left* in the first row, then in the second row etc. The tableaux which we have to throw away are those which lead to sequences of letters which are not admissible.

The properties of the composed new tableaux may be summarized as follows:

1. Each tableau must be a Young tableau.

2. The number of boxes in the new tableau must be equal to the sum of the number of boxes in the original two tableaux.

3. If dealing with $SU(n)$, no tableau has more than n rows.

4. Making a journey through the tableau starting with the top row and entering each row from the right, at any point the number of b's encountered in any of the attached boxes must not exceed the number of previously encountered a's and the number of c's encountered in any of the attached boxes must not exceed the number of previously encountered b's, etc.

5. The letters must be in anti-lexicographical order when reading across a row from left to right.

6. The letters must differ and be in lexicographic order when reading a column from top to bottom.

The first three rules should be obvious. The purposes of the three rules 4) to 6) are to assure that states which were previously symmetrized are not anti-symmetrized in the product and vice versa, and to avoid double counting states.

Examples:

1. $3 \otimes 3 = \square \times \square = \begin{array}{|c|}\hline\\\hline\end{array} + \square\square = 3^* \oplus 6$
2. $3 \otimes 3^* = \square \times \begin{array}{|c|}\hline\\\hline\end{array} = \begin{array}{|c|c|}\hline\\\hline\end{array} + \begin{array}{|cc|}\hline\\\hline\end{array} = 1 \oplus 8$
3. $3 \otimes 3 \otimes 3 = (\begin{array}{|c|}\hline\\\hline\end{array} + \square\square) \times \square = \begin{array}{|c|c|}\hline\\\hline\end{array} + \begin{array}{|cc|}\hline\\\hline\end{array} + \begin{array}{|cc|}\hline\\\hline\end{array} + \square\square\square$
   $= 1 \oplus 8 \oplus 8 \oplus 10$

More than two tableaux may be combined by first combining the first two, then combining the result with the third one and so on.

Exercise: Show that $8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 10 \oplus 27$.

The quantum numbers of quarks are given by:

Quark	spin	B	Q	I_3	S	Y
u	1/2	1/3	2/3	1/2	0	1/3
d	1/2	1/3	−1/3	−1/2	0	1/3
s	1/2	1/3	−1/3	0	−1	−2/3

Exercise: Use the Young tableaux to construct the meson states in

$$3 \otimes 3^*$$

and the baryon states in

$$3 \otimes 3 \otimes 3 \, .$$

Notice that the indices of the tensors $\psi^{j_1 \ldots j_q}_{i_1 \ldots i_p}$, which characterize a irreducible representation (p, q), in the quark model have two different interpretations. Each upper index has associated either

an antiquark or an anti-symmetrized pair of quarks. For lower indices antiquarks and quarks are interchanged: i.e.

$$\text{Upper index}: \quad \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (ds) \\ (su) \\ (ud) \end{pmatrix}$$
$$B = -1/3 \qquad B = 2/3$$

$$\text{Lower index}: \quad \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (\bar{d}\bar{s}) \\ (\bar{s}\bar{u}) \\ (\bar{u}\bar{d}) \end{pmatrix}$$
$$B = 1/3 \qquad B = -2/3$$

where (ud) etc. denote anti-symmetrized pairs. Which interpretation is to be used is uniquely fixed if we specify the baryon number B of the state.

1.4 The spectrum of low lying hadrons:

Mesons: $\bar{q}q'$ bound states
A $\bar{q}q'$ with orbital angular momentum L has Parity $P = (-1)^{L+1}$. For $q' = q$ we have a $\bar{q}q$ bound state which is also an eigenstate of charge conjugation C with $C = (-1)^{L+S}$, where S is the spin 0 or 1. The $L = 0$ states are the pseudoscalar mesons, $J^P = 0^-$, and the vectors mesons, $J^P = 1^-$.

In the limit of exact $SU(3)$ the pure states would read

$$\begin{aligned} \pi^0 &= (\bar{\mathbf{u}}\mathbf{u} - \bar{\mathbf{d}}\mathbf{d})/\sqrt{\mathbf{2}} \\ \eta_1 &= (\bar{\mathbf{u}}\mathbf{u} + \bar{\mathbf{d}}\mathbf{d} + \bar{\mathbf{s}}\mathbf{s})/\sqrt{\mathbf{3}} \\ \eta_8 &= (\bar{\mathbf{u}}\mathbf{u} + \bar{\mathbf{d}}\mathbf{d} - \mathbf{2}\bar{\mathbf{s}}\mathbf{s})/\sqrt{\mathbf{6}} \end{aligned}$$

$$\begin{aligned} \rho^0 &= (\bar{\mathbf{u}}\mathbf{u} - \bar{\mathbf{d}}\mathbf{d})/\sqrt{\mathbf{2}} \\ \omega_1 &= (\bar{\mathbf{u}}\mathbf{u} + \bar{\mathbf{d}}\mathbf{d} + \bar{\mathbf{s}}\mathbf{s})/\sqrt{\mathbf{3}} \\ \omega_8 &= (\bar{\mathbf{u}}\mathbf{u} + \bar{\mathbf{d}}\mathbf{d} - \mathbf{2}\bar{\mathbf{s}}\mathbf{s})/\sqrt{\mathbf{6}} \end{aligned}$$

In fact $SU(2)_{\text{flavor}}$ breaking by the quark mass difference $m_d - m_u$ leads to $\rho - \omega$–mixing [mixing angle $\sim^\circ$] (Glashow 1961) [4]:

$$\begin{aligned} \rho^0 &= \cos\theta\,\rho' + \sin\theta\,\omega' \\ \omega &= -\sin\theta\,\rho' + \cos\theta\,\omega' \end{aligned}$$

Similarly, the substantially larger $SU(3)_{\text{flavor}}$ breaking by the quark masses, leads to large $\omega - \phi$–mixing [mixing angle $\sim 36^\circ$ close to so called ideal mixing where $\phi \sim$ is a pure $\bar{s}s$ state] (Okubo 1963) [5]:

$$\begin{aligned} \phi &= \cos\theta\,\omega_8 + \sin\theta\,\omega_1 \\ \omega &= -\sin\theta\,\omega_8 + \cos\theta\,\omega_1 \end{aligned}$$

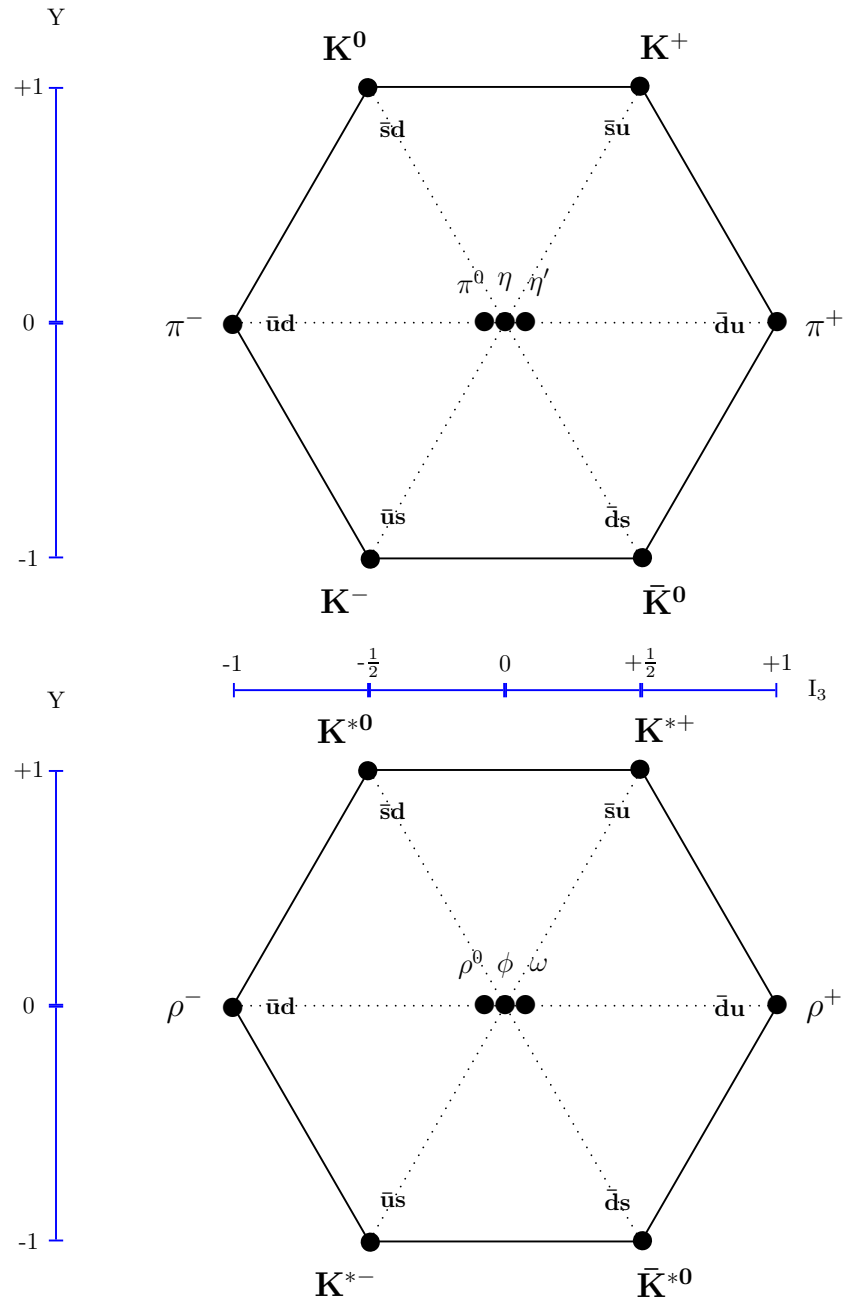


Figure 1.4: The pseudoscalar-meson and vector-meson octets plus the singlets (nonets). States with the same conserved quantum numbers mix, like $\eta_8, \eta_1 \to \eta, \eta'$ and $\rho - \omega$ and $\omega - \phi$.

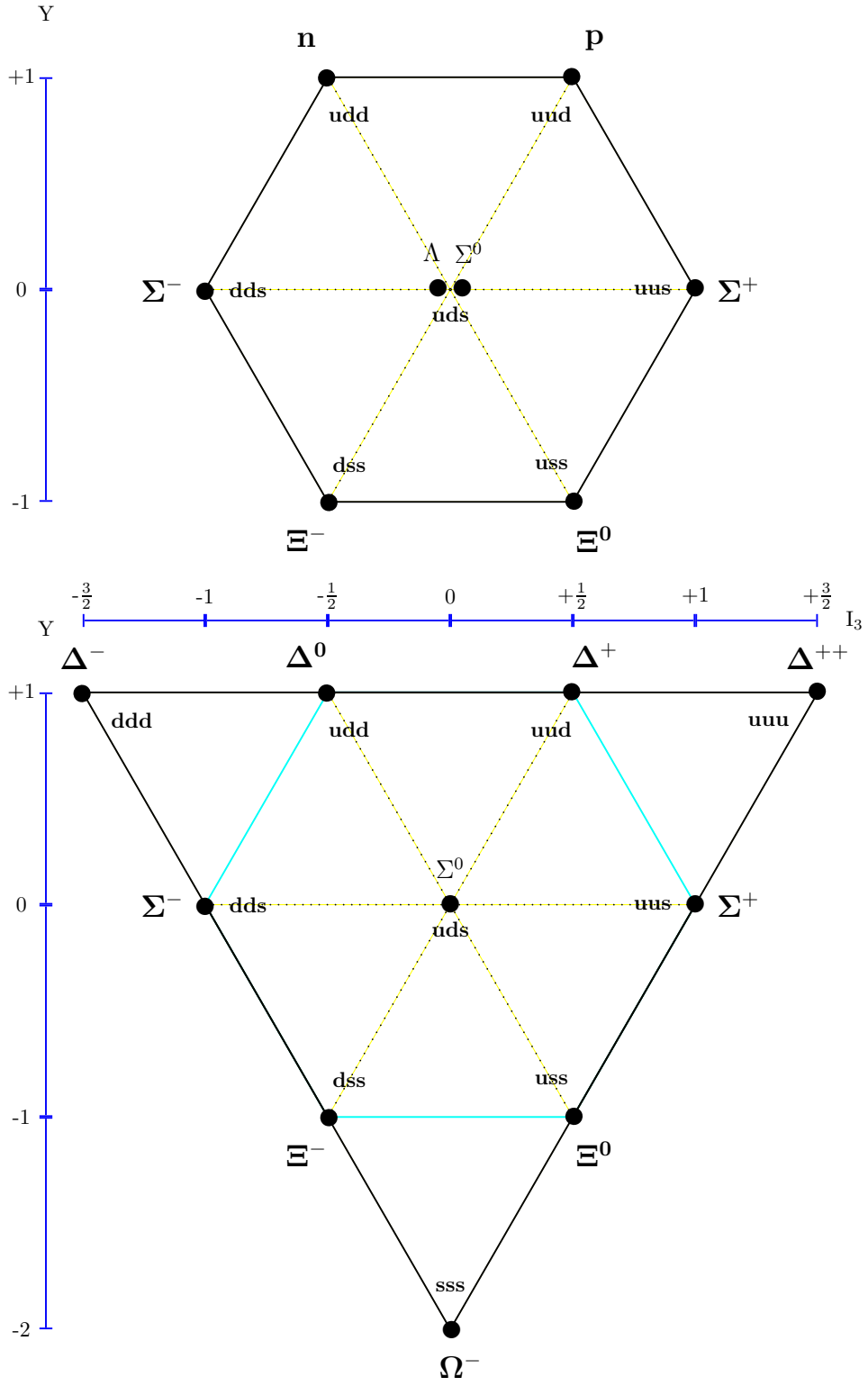


Figure 1.5: The baryon octet and decaplet

1.5 One particle states: pions and nucleons

Here we briefly present some notation for one particle states and their interpolating fields. The normalization of the states is given by

$$\langle\alpha(p)|\beta(p')\rangle = (2\pi)^3\, 2\,\omega(p)\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,\delta_{\alpha\beta}\ ,$$

with $\omega(p)=\sqrt{\vec{p}^{\,2}+m^2}$, m the mass of the state, $\vec{p}$ its momentum.

<u>π–mesons:</u>
$|\pi^+(p)\rangle,\ |\pi^0(p)\rangle,\ |\pi^-(p)\rangle$ are one pion states of momentum p. Some times it is convenient to use a cartesian basis for the π's: $|\pi^a(p)\rangle$ $a=1,2,3$. The charge eigenstates are then given by

$$\begin{aligned}
|\pi^+(p)\rangle &= \frac{1}{\sqrt{2}}\left\{|\pi^1(p)\rangle + \mathrm{i}|\pi^2(p)\rangle\right\}\ ,\\
|\pi^0(p)\rangle &= |\pi^3(p)\rangle\ ,\\
|\pi^-(p)\rangle &= \frac{1}{\sqrt{2}}\left\{|\pi^1(p)\rangle - \mathrm{i}|\pi^2(p)\rangle\right\}\ ,
\end{aligned}$$

such that

$$T_\pm|\pi^0\rangle = \mp\sqrt{2}\,|\pi^\pm\rangle\ .$$

The pseudoscalar mesons may be written as a $SU(2)$ isospin triplet $|t_3;p\rangle_\pi$ with $t_3=1,0,-1$ for π^+,π^0,π^-.

<u>Nucleons:</u>
the nucleon states $|p(p,s)\rangle,\ |n(p,s)\rangle$ are in a $SU(2)$ isospin doublet $|t_3;p,s\rangle_N$ with $t_3=\frac{1}{2}$ the proton and $t_3=-\frac{1}{2}$ the neutron. s denotes the spin $[=\pm\frac{1}{2}]$. Similar for the other hadrons.

1.5.1 Interpolating fields

For the pion in the Cartesian basis the pseudoscalar fields are real

$$\varphi^\pi_a(x) = \{\varphi^\pi_a(x)\}^+$$

and create or annihilate a bare π^a and are normalized by (free field = plane wave solution)

$$\langle 0|\varphi^\pi_a(x)|\pi^b(p)\rangle = \mathrm{e}^{-\mathrm{i}\,px}\,\delta_a{}^b\ .$$

As a $SU(2)$ representation one may write the multiplet field as a triplet field

$$\phi(x) = \begin{pmatrix} \varphi^+ \\ \varphi^0 \\ \varphi^- \end{pmatrix}$$

or as 2×2 matrix field

$$\Phi(x) = \frac{1}{\sqrt{2}}\sum_{a=1}^{3} T_a\varphi_a(x) = \begin{pmatrix} \varphi^0/\sqrt{2} & \varphi^+ \\ \varphi^- & -\varphi^0/\sqrt{2} \end{pmatrix}$$

in the Cartesian and charged basis, respectively. In the latter representation the field is an element of the $SU(2)$ Lie algebra.

Similarly, for the nucleons we have an interpolating field $\psi_p(x)$ and $\psi_n(x)$, which we may write as an isodoublet field

$$\Psi_N(x) = \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix} .$$

As usual proton field $\psi_p(x)$ creates a bare $\bar{p}$ at x and/or annihilates a bare p at x and similar for the neutron. With $(\Psi_N)_{t_3=+\frac{1}{2}} = \psi_p$ and $(\Psi_N)_{t_3=-\frac{1}{2}} = \psi_n$ the normalization in this case reads

$$\langle 0| \left(\Psi_N\right)_{t_3}(x) \, |t_3'; p, s\rangle_N = \mathrm{e}^{-\mathrm{i}\, px} \, \delta_{t_3,t_3'} \ .$$

1.5.2 Noether Theorem

Symmetries of physical systems usually correspond to an invariance under a change of frames and mathematically are described by Lie Groups of transformations like the unitary groups discussed above. The key insight about symmetries we owe to Emmy Noether 1918: the Noether theorem. It says that if a system (e.g. as defined by a certain Lagrangian) is invariant under a set of symmetry transformations there esists a corresponding set of conserved four–currents $j_i^\mu(x) = (\rho_i(t,\vec{x}), \vec{j}_i(t,\vec{x}))$ satisfying the continuity equation $\partial_\mu j_i^\mu(x) = 0$. The trsnsformations are then generated by a corresponding set of generalized charges [which represent the generators of the symmetry group] which are given by space - integrals

$$Q_i = \int \mathrm{d}^3x \, j_i^0(t,\vec{x})$$

of the j_i^0 component of a conserved current. Current conservation implies the time independence of the "charges" Q_i: $\frac{\mathrm{d}Q_i}{\mathrm{d}t} = 0$ ["charge" conservation].

In hadron physics the flavor currents related to the flavor symmetries $SU(2)$ (isospin), $SU(3)$ (isospin+strangeness), $\cdots$ play an important role. This will be discussed in Sects. 3.2and 4.2.1.

1.6 Outlook

Nature plays all kind of games with symmetries. There are global and local space-time symmetries and global and local internal symmetries. The latter determine strong, weak and electromagnetic interactions by the local gauge group

$$G_{\rm loc} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y .$$

$SU(3)_c$ is the color gauge group of the strong interactions (QCD). $SU(2)_L \otimes U(1)_Y$ is the electroweak gauge group ($SU(2)_L$ the left-handed weak isospin and $U(1)_Y$ the weak hypercharge) which is broken to the Abelian electromagnetic $U(1)_{\rm em}$ of QED. Local space-time symmetry (general coordinate transformation invariance, general relativity) leads to classical gravity theory. Gravitational interactions break Poincaré invariance, which "in practice" appears as an absolute global symmetry due to the extreme weakness of the gravitational force. Global charge-like symmetries remain the most mysterious symmetries. The only known exact global symmetries are Abelian symmetries with the associated quantum numbers being **quantized**.

$$\begin{array}{lll} Q & \text{electric charge} & U(1)_Q \\ B & \text{baryon number} & U(1)_B . \end{array}$$

The quantization of these charges is barely understood, notice that $U(1)$ invariance only implies the conservation of the corresponding charge not its quantization[3].

Baryon number B is an additive quantum number like the electric charge Q. It derives from a global $U(1)_B$ invariance of all standard model interactions. Empirically it is tested most accurately by the stability of the proton. Proton lifetime limits are

$$\begin{array}{ll} \tau_p^{\rm geo} > & 1.6 \times 10^{25} \text{ years (geochemical estimate independent on decay modes)} \\ \tau_p^{\rm lab} > & 10^{31} - 3 \times 10^{32} \text{ years (absence of "main" decay modes) .} \end{array}$$

Possible decay modes which where searched for are:

$$\begin{array}{lll} p & \to & e^+\gamma,\ e^+\pi^0,\ e^+\rho^0,\ e^+\omega,\ e^+K^0, \ldots , \\ & & \bar\nu_e\pi^+,\ \bar\nu_e\rho^+,\ \bar\nu_e K^+, \ldots , (e \to \mu) \\ p & \to & \pi^+ + \pi^0 \end{array}$$

By convention $B(p) = 1$, $B(e^-) = 0$. All observations support the assignments $B(B) = 1$ and $B(\bar B) = -1$ for baryons B and antibaryons $\bar B$, respectively. All other particles have $B = 0$.

□

Excursion on the *Baryon Asymmetry Of The Universe*: While we have no direct evidence that baryon number is not strictly conserved we know that at some level, presumably not far from current experimental limits, baryon number must be violated. Otherwise the observed baryon asymmetry, the asymmetry between matter and antimatter observed in our universe (galaxies, stars, planets,... but no anti–galaxies,anti–stars, anti–planets,...) could not be

[3]Note that also the lepton numbers $L_\ell (\ell = e, \mu, \tau)$, related to approximate $U(1)_{L_\ell}$ symmetries, are known to be conserved with high accuracy. They are broken at the level of the tiny neutrino masses, which must be non-zero and non-degenerate in order to allow for the observed neutrino oscillations.

explained from properties of the fundamental interactions of nature. It would (and could) be just an accidental asymmetry in the initial condition at the moment of the creation of the universe at the big–bang.

In the early very hot universe matter and antimatter was created by the highly energetic photon collisions at (almost) equal rate. In fact with a tiny asymmetry: per unit density $\rho_{\bar{B}}$=1.000000000 of antimatter a portion ρ_B=1.000000001 of matter must have been created. The expansion of the universe cooled down the radiation and matter and antimatter annihilated almost completely into photons with the relict $\rho_B - \rho_{\bar{B}}$=0.000000001 of matter. Thus the baryon number in units of the number of photons in the universe is $N_B/N_\gamma \sim 10^{-9}$.

A theory which is able of explaining the origin of the baryon asymmetry must satisfy three conditions (Sakharov 1967):

- It must violate B,
- it must violate CP and
- the universe must be out of thermal equilibrium.

The latter condition is satisfied since we know that the universe is expanding and not in a stationary equilibrium state.

Within the SM B is conserved to all orders in perturbation theory and only violated by extremely tiny non–perturbative effects. The latter are due to the existence of non–trivial classical solutions of the $SU(2)$ Yang–Mills equations, the so called instantons (Belavin et al. 1975). Quantum effects due to these four-dimensional pseudoparticles lead to symmetry breaking via Adler-Bell-Jackiw anomalies ('t Hooft 1976).

While B is practically conserved in the SM, CP violation is naturally incorporated in the SM with three (or more) families of quarks and leptons (Kobayashi and Maskawa 1973). In fact three families are known to exist, the last member of the third family the top quark with a mass of about 175 GeV has been found some time ago (CDF and D0 at Fermilab 1995). More recently CP violation has been established (Babar at SLAC and Belle at KEK 2001) in the B–meson system to be a large effect in accord with the SM prediction given the CP violation in the $K^0 - \bar{K}^0$ system, which has been discovered as a small effect $\varepsilon \simeq 2.3 \times 10^{-3}$ long time ago (Christenson, Cronin, Fitch and Turlay 1964).

The SM is very unlikely able to predict the correct size of the baryon asymmetry. The latter thus is a clear indication that the SM is only part of the full story.

□

The lepton numbers L_ℓ ($\ell = e, \mu, \tau$) are other additive quantum numbers which seem to be strictly conserved at first sight. By convention $L_\ell(\ell^-) = 1$. That L_μ is separately conserved follows from the non-observations of the decays

$$\begin{array}{cc}
\mu^+ \to e^+ + \gamma & \Gamma(\mu \to e\gamma)/\Gamma(\mu \to \text{all}) < 1.2 \times 10^{-11} \\
\mu^+ \to e^+ + e^- + e^+ & \Gamma(\mu \to 3e)/\Gamma(\mu \to \text{all}) < 1.0 \times 10^{-12} \\
K_L \to e + \mu & \Gamma(K_L \to e\mu)/\Gamma(K_L \to \text{all}) < 4.7 \times 10^{-12} \\
K^+ \to \pi^+ + e + \mu & \Gamma(K^+ \to \pi^+ e\mu)/\Gamma(K^+ \to \text{all}) < 2.1 \times 10^{-10} \\
\mu^- + (Z, A) \to e^- + (Z, A) & \Gamma(\mu^-\text{Ti} \to e^-\text{Ti})/\Gamma(\mu^-\text{Ti} \to \text{all}) < 4.0 \times 10^{-12} \\
\mu^- + (Z, A) \to e^+ + (Z-2, A) & \Gamma(\mu^-\text{Ti} \to e^+\text{Ca})/\Gamma(\mu^-\text{Ti} \to \text{all}) < 3.6 \times 10^{-11} \quad .
\end{array}$$

Tests of the separate conservation of L_τ are much less stringent: The best limits are:

$$\Gamma(\tau \to e\gamma)/\Gamma(\tau \to \text{all}) < 2.7 \times 10^{-6} \quad \text{and} \quad \Gamma(\tau \to \mu\gamma)/\Gamma(\tau \to \text{all}) < 1.1 \times 10^{-6} \quad .$$

Within the experimentally well established electroweak standard model strict lepton number conservation is only possible if the neutrinos are strictly massless. Non-vanishing neutrino masses lead to **neutrino–oscillations**. Neutrino mixing searches (ν-oscillations $\nu_\ell \leftrightarrow \nu_{\ell'}$) have confirmed the effect recently which implies the existence of non–vanishing neutrino masses. Present direct upper limits on the neutrino masses are:

$$
\begin{aligned}
m_{\nu_e} < \quad & 3.0 \text{ eV} \quad && (\text{from } {}^3H \to {}^3He\, e^-\, \bar{\nu}_e) \\
m_{\nu_\mu} < \quad & 190 \text{ keV} \quad && (\text{from } \pi \to \mu\, \nu_\mu) \\
m_{\nu_\tau} < \quad & 18.2 \text{ MeV} \quad && (\text{from } \tau^- \to 3\pi\, \nu_\tau)
\end{aligned}
$$

Lower bounds are not yet so easy to establish at present but observed neutrino mixing phenomena indicate values of about two to three orders of magnitude lower than the above direct upper limits. In any case this implies corresponding lepton numbers L_ℓ ($\ell = e, \mu, \tau$)–violations.

Another important limit is the absence of $\Delta L_e = 2$ transitions. The limit from neutrino-less double beta decay $(Z, A) \to (Z + 2, A) + e^+ + e^-$ is $t_{1/2} > 1.6 \times 10^{25}$ years for ^{76}Ge . The observation of such reactions would imply that the electron-neutrino is a massive **Majorana neutrino**, a self-conjugate fermion which is its own antiparticle.

Global non-Abelian symmetries are approximate (broken) only and correlated with the hierarchy of the fundamental interactions. The weaker the interaction the less symmetries it respects. Strong interaction symmetries are:

$$
\left.\begin{array}{l}
\left.\begin{array}{ll}
\text{Isospin (charge independence of nucleon forces) } I & SU(2)_I \\
\text{Strangeness } S & \\
\text{Charm } C & \\
\quad \vdots &
\end{array}\right\} SU(3)_{\text{flavor}}
\end{array}\right\} SU(4)_{\text{flavor}}
$$

The larger the symmetry group the stronger it is broken by growing mass differences of the states in the multiplets. These symmetries are furthermore broken by electromagnetic and weak interactions. The latter in addition breaks parity P maximally and CP in accordance with the Cabibbo-Kobayashi-Maskawa (CKM) mixing scheme of the three quark–lepton family electroweak SM. CP violation was observed for the first time in K^0 decays in 1964 as a small effect at the 3 ppm level. The corresponding effect in B^0 decays has been established 2001 at dedicated B–factories.

Let us elaborate more on *Quantum Chromodynamics* and the *chiral symmetry group*: The modern theory of strong interaction is QCD. It views hadrons like the pions, the nucleons etc. as composite objects made out of quarks. Mesons are quark antiquark bound states, nucleons are three quark bound states etc. The quarks not only carry flavor quantum numbers like isospin, strangeness, charm, etc. but an additional one called *color*. More precisely quarks are color triplets antiquarks are color antitriplets of $SU(3)_c$. Color is a local symmetry (much like the local $U(1)_{\text{em}}$ gauge invariance in QED) and requires the existence of eight colored gauge bosons the *gluons* which glue together the quarks in the hadrons. QCD is a unbroken non-Abelian gauge theory, which has the property of *asymptotic freedom*, the strength of the interaction becomes weaker and weaker as we look at shorter and shorter distances, i.e., inside the hadrons. Complementary it becomes stronger and stronger as we go to larger and larger distances. This means that we cannot separate the quarks in the hadrons to become free particles. If we try to separate a $q\bar{q}$–pair the color field between them get squeezed into flux tubes which at the end form strings which execute a linearly rising force. In this way quarks and gluons get in fact permanently confined inside the hadrons. This phenomenon is called *confinement*. Only objects not carrying net color can become free, these are the hadrons. They have typical sizes of about 1 fermi (=10^{-13} cm) and the color forces are screened at distances beyond the size of a hadron. The remnant forces are what we observe as nuclear forces in atomic nuclei or in low energy hadron scattering. Thus in spite of the fact

the force carriers, the gluons, are massless the strong interaction forces are rather short ranged. Thus, interestingly, the spectrum of possible states of QCD are not the fields in the Lagrangian, the quarks and gluons, but the hadrons, and those must be color neutral which means they must be color singlets.

Thus from the point of view of the color $SU(3)_c$ the spectrum can be found by determining all possible singlets which we may form from quarks in 3 or 3^* and gluons in 8:

$$\begin{aligned} 3 \otimes 3^* &= 1 \oplus 8 \\ 3 \otimes 3 \otimes 3 &= 1 \oplus 8 \oplus 8 \oplus 10 \\ 3^* \otimes 3^* \otimes 3^* &= 1 \oplus 8 \oplus 8 \oplus 10^* \\ 8 \otimes 8 &= 1 \oplus 8 \oplus 8 \oplus 10 \oplus 10^* \oplus 27 \end{aligned}$$

Only the singlets play a role here. They correspond to the mesons $\bar{\psi}_{i_c\alpha}(\Gamma)_{\alpha\beta}\psi_{i_c\beta}$ baryons, antibaryons and the so called *glueballs*, respectively. The glueballs, the singlet in $8 \otimes 8$ of gluons, do not contain any valence quarks. They are expected to show up as broad resonances between 1 and 2 GeV and have not yet been established by experiments.

A completely different kind of application concern the flavor symmetries, which are approximate global symmetries of the strong interactions and hence, in modern terminology, of the QCD Lagrangian. Since some of the quarks are rather light one may look at the approximation where quark masses are switched off. The QCD Lagrangian then has a huge global flavor symmetry, namely, the **chiral group**

$$\mathcal{G}_F = U(N_F)_V \otimes U(N_F)_A \simeq SU(N_F)_V \otimes SU(N_F)_A \otimes U(1)_V \otimes U(1)_A \tag{1.2}$$

with N_F the number of quark flavors. In the context of the SM, as we shall see later, this corresponds to the symmetric phase of the electroweak– or flavor– sector of the SM, i.e., before the local gauge symmetry $SU(2)_L \otimes U(1)_Y$ is broken by the Higgs mechanism to the residual exact electromagnetic $U(1)_{\rm em}$ local gauge symmetry. The unbroken phase may be understood as an *asymptotic symmetry* which is approached asymptotically at very high energies, when all masses which are small relative to a given energy scale are negligible. Since as we increase the number of flavors from $N_F = 2$ to 6, the above symmetry is broken more and more by increasingly heavy quark masses[4]

quark flavor	u	d	s	c	b	t
mass (MeV)	$\sim$5	$\sim$9	190	1650	4750	172600

the chiral symmetry is good only for the light flavors: for $N_F = 2$ we have with very good accuracy the isospin $SU(2)$ for $N_F = 3$ the slightly more broken $SU(3)$ of isospin plus strangeness, symmetries which manifest themselves in the hadronic spectrum. One might ask what is the precise sense of a *broken symmetry*, a symmetry which is not truly a symmetry? Associated with a symmetry there are currents and generalized charges, the generators of the symmetry transformations (see next Sec.) of particle multiplets. The point is that in spite of the fact that the symmetry is not perfect the states may be classified or labeled in terms of corresponding

[4]Note the in each doublet the quarks with the larger charge magnitude like c and t have also larger mass than s and b, respectively. The lightest two quarks are an exception, the u quark is lighter than the d quark which is of existential importance as it makes the proton to be lighter than the neutron and the neutron to decay into protons and not vice versa. Thus the inversion is crucial for the stability of the proton and hence for all structure in the universe.

quantum numbers like isospin and stangeness which satisfy the appropriate Lie algebra. As the symmetry is approximate only, the corresponding charges are not strictly conserved and thus are time-dependent to some extent.

Surprisingly the chiral symmetry group (1.2) is not just $SU(N_F)$ which is what we observe in the hadron spectrum. The chiral group is doubled by the axial part, in fact in the massless limit left–handed and right–handed fields satisfy an $SU(N_F)$ independently and thus we obtain $SU(N_F)_L \otimes SU(N_F)_R$ which is equivalent to vector times axial-vector $SU(N_F)_V \otimes SU(N_F)_A$ (see Sec. 3.2 for more details). The only explanation for the absence of the parity doublers in the spectrum is that in fact the $SU(N_F)_A$ is spontaneously broken, i.e., the symmetry is manifest in the dynamics, represented by the massless QCD–Lagrangian, but absent in the ground state (vacuum) and hence in the space of states built up about the non–symmetric vacuum (see Sec. 2). In turn this implies the existence of a set of Nambu–Goldstone bosons, which must be massless: for $SU(2)$ we must have 3 Goldstone bosons the three pions $\pi^{\pm}, \pi^0$, for $SU(3)$ we must have 8 Goldstone bosons the pions plus $K^{\pm}$, K^0, $\bar{K}^0$ and η. Since in reality quark masses are non-vanishing the "would be Goldstone bosons" aquire a mass and one calls them *pseudo Goldstone bosons*. This mechanism explains why the pseudoscalar mesons are the lightest hadrons and why they have masses substantially lower than the other hadrons.

We finally mention that the $U(1)_V$ factor in (1.2) in fact corresponds to the baryon number conservation, while $U(1)_A$ is not a symmetry at all. The latter is broken by quantum corrections, the famous Adler–Bell–Jackiw anomaly.

For a more detailed discussion we refer to Sec. 3.2 for the chiral group and to Sec. 2 for the spontaneous symmetry breaking and the Goldstone phenomenon.

We have seen that we are often dealing with imperfect symmetries in nature. The various possibilities we have encountered may be classified as follows:

- Symmetries broken by weaker interactions: What are the enhanced symmetries we get if we switch of gravity, the weak forces and the electromagnetic interactions?
- Symmetries broken by mass terms or other terms in the Lagrangian of dimension less than four. In general such breakings disappear if we go to higher energies only if they are generated by
- spontaneous symmetry breaking.
- What is barely considered in the literature: symmetries may show up at low energies because one does not see the short distance details, which means that at high enough energies when probing short distances symmetries observed at low energy may be violated. Typically, interaction terms of dimension larger than four (non-renormalizable terms which naturally arise in low energy effective theories but which are suppressed if the high-energy cut-off is large enough) could violate symmetries we see at low energies. Low energy here means present accelerator energies as we expect the Planck scale $M_{\text{Planck}} \sim 10^{19}$ GeV to be the fundamental reference scale. Such non-renormalizable terms are absent in the SM and possibly only show up when we go much closer to the Planck scale M_{Planck}. Effects at the 1 ppm level we may expect only at energies $E/M_{\text{Planck}} \sim 10^{-3}$, i.e., $E \sim M_{\text{GUT}} \sim 10^{16}$GeV.

Appendix to Section 1:

Some useful formulae for matrix transformations

1.

$$\left\| \begin{aligned} e^{iA}Be^{-iA} &= B + i[A,B] + \tfrac{i^2}{2!}[A,[A,B]] + \ldots \\ &\quad + \tfrac{i^n}{n!}[A,[A,\ldots,[A,[A,B\underbrace{]]\ldots]}_{n}] + \ldots \end{aligned} \right.$$

holds for any two operators or matrices A and B .

Proof:

Replace A by λA and perform a Taylor expansion in λ

$$F(\lambda) = e^{i\lambda A}Be^{-i\lambda A} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\frac{\partial^n F}{\partial \lambda^n} \right) |_{\lambda=0}$$

and evaluate the Taylor coefficients. Using that A commutes with $e^{i\lambda A}$ we get

$$\frac{\partial F}{\partial \lambda} = e^{iA}\, i\, [A,B] e^{-i\lambda A}$$

and by repeated differentiation

$$\frac{\partial^n F}{\partial \lambda^n} = e^{i\lambda A}\, i^n\, [A,A,\ldots [A,B\underbrace{]]\ldots]}_{n} e^{-i\lambda A}.$$

For $\lambda = 1$ the result follows.

2.

$$\left\| \begin{aligned} &e^{i\sum_l T_l\omega_l}\, T_i\, e^{-i\sum_l T_l\omega_l} = T_k \left(e^{i\sum_l t_l\omega_l}\right)_{ki} \\ &\quad \text{i.e. } T_i \text{ transforms as a vector under the adjoint representation } (t_l)_{ik} = \\ &\quad -i\, c_{lik} \text{ or } (t_l)_{ki} = i\, c_{lik}. \end{aligned} \right.$$

Proof:

$$\begin{aligned} & e^{i\sum_l T_l\omega_l}\, T_i\, e^{-i\sum_l T_l\omega_l} \\ = \;& T_i + i\,[T_l\omega_l, T_i] + \frac{i^2}{2!}\,[T_{l_2}\omega_{l_2}\,[T_{l_1}\omega_{l_1}, T_i]] + \ldots \\ & + \frac{i^n}{n!}[T_{l_n}\omega_{l_n}, [T_{l_{n-1}}\omega_{l_{n-1}}, \ldots, [T_{l_1}\omega_{l_1}, T_i\underbrace{]\ldots]]}_{n} + \ldots \\ = \;& T_k\delta_{ki} + T_k\, i\, \left(t_l\omega_l\right)_{ki} + T_k \frac{i^2}{2!}\left(t_l\omega_l\right)^2_{ki} + \ldots + T_k\frac{i^n}{n!}\left(t_l\omega_l\right)^n_{ki} + \ldots \\ = \;& T_k \left(e^{i\sum_l t_l\omega_l}\right)_{ki} \end{aligned}$$

We have used 1. and $[T_l, T_i] = T_k\, i\, c_{lik} = T_k (t_l)_{ki}$ such that

$$\begin{aligned} [T_l\omega_l, T_i] &= T_k (t_l\omega_l)_{ki} \\ [T_{l_2}\omega_{l_2}, [T_{l_1}\omega_{l_1}, T_i]] &= [T_{l_2}\omega_{l_2}, T_{k_1}]\,(t_{l_1}\omega_{l_1})_{k_1 i} \\ &= T_k\,(t_{l_2}\omega_{l_2})_{kk_1}\,(t_l, \omega_{l_1})_{k_1 i} \\ &= T_k (t\omega)^2_{ki} \quad etc. \end{aligned}$$

where repeated indices have to be summed over.

3.

$$e^{-i\sum_l T_l\omega_l}\partial_\mu\left(e^{i\sum_l T_l\omega_l}\right) = T_k\left(\frac{1-e^{-i\sum_l t_l\omega_l}}{\sum_l t_l\omega_l}\right)_{ki}\partial_\mu\omega_i$$

is a linear combination of the generators and thus an element of the Lie-algebra

Proof:

$$\begin{aligned}
& e^{-i\sum_l T_l\omega_l}\partial_\mu\left(e^{i\sum_l T_l\omega_l}\right) \\
= \; & -i\,[T_l\omega_l,\partial_\mu] + \frac{i^2}{2!}\,[T_{l_2}\omega_{l_2},[T_{l_1}\omega_{l_1},\partial_\mu]] + \ldots \\
& + \frac{(-i)^n}{n!}\,[T\omega,[T\omega,\ldots,[T\omega,\partial_\mu]\ldots]] + \ldots \\
= \; & -i\,T_k\,\delta_{ki}\,(-\partial_\mu\omega_i) + \frac{i^2}{2!}\,[T_l\omega_l,T_i]\,(-\partial_\mu\omega_i) + \ldots \\
& + \frac{(-i)^n}{n!}\,[T\omega,[T\omega,\ldots,[T\omega,\underbrace{T_i]\ldots]}_{n-1}](-\partial_\mu\omega_i) + \ldots \\
= \; & -T_k\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\,(t\omega)^{n-1}_{ki}\partial_\mu\omega_i \\
= \; & T_k\left(\frac{1-e^{-i\sum_l t_l\omega_l}}{\sum_l t_l\omega_l}\right)_{ki}\partial_\mu\omega_i = T_k\Lambda_{ki}(\omega)\;\partial_\mu\omega_i
\end{aligned}$$

We have used 1. and $[T_l\omega_l,\partial_\mu] = T_l\omega_l\partial_\mu - \partial_\mu T_l\omega_l = -T_l(\partial_\mu\omega_l)$ and then proceed as in the proof of 2.

Exercises: Section 1

① Show that $8\otimes 8 = 1\oplus 8\oplus 8\oplus 10\oplus 10^*\oplus 27$.

② Isospin symmetry was introduced by Heisenberg in 1932 right after the discovery of the neutron (Chadwick 1932). It is obvious thar isospin symmetry is violated by elctromagnetism: the isodoublet members (p,n) have different charge $(1,0)$. The nucleon system is described by a isospinor Dirac field

$$\Psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} .$$

Write down the charges Q, B and T_i $(i=1,2,3)$ in terms of the nucleon field operator and show that B commutes with isospin, but Q does not. Write down the $T=1$ and $T=0$ two nucleon states.

③ Discuss the isospin properties of the triplet of pions (π^+,π^0,π^-) .

The isospin symmetry of the scattering operator S not only leads to relations between matrix elements but also to selection rules: Suppose

(a) T is a generator of a symmetry transformation such that $[T,S]=0$,

(b) $\mid\alpha>$ and $\mid\beta>$ are eigenstates of T i.e. $T\mid\alpha>=t_\alpha\mid\alpha>, T\mid\beta>=t_\beta\mid\beta>$

What does this imply for the S-matrix elements

$$S_{\beta\alpha} =< \beta \mid S \mid \alpha > \ ?$$

Find a few examples.

④ Use the Young tableaux to construct the meson states in

$$3 \otimes 3^* $$

and the baryon states in

$$3 \otimes 3 \otimes 3 \ .$$

The states in the pseudoscalar meson octet of flavor $SU(3)$ are characterized by the 3^{rd} component of isospin and by hypercharge $Y = B + S$ (B baryon number $B = 0$ for mesons, S strangeness $S = 0$ for pions). Display the weight diagram ($I_3 - Y$ plot) of the meson states. How are they composed of u, d and s quarks in the $SU(3)_{\text{flavor}}$ quark model ?

⑤ The structure constants c_{ikl} of a Lie-algebra $[T_i, T_k] = ic_{ikl}T_l$ satisfy the Jacobi identity.

$$c_{ikn}c_{nlm} + \text{ terms cyclic in } (ikl) = 0$$

Use this to show that $(\tilde{T}_i)_{kl} = -ic_{ikl}$ also satisfies the Lie-algebra (adjoint representation).

⑥ Discuss the electromagnetic decays of π^0 and η^0 into photons. Use the fact that the C parity of the photon is $\eta_C^\gamma = -1$. What are the observed decays? What do they imply for the C parities of the neutral pseudoscalar mesons π^0 and η^0 ?. Which decays are strictly forbidden? Which Feynman diagram is responsible for these decays?

⑦ Which experiments (processes) allow us to determine the intrinsic parity (η_P) of the pions?

⑧ Lepton number L_e is another additive quantum number which is almost strictly conserved. $L_e(e^-) = 1$ by convention. Determine L_e for the other particles from the observed reactions:

1. $L_e(e^+) = -1,\ L_e(\gamma) = 0$:

$$\begin{aligned} p + e &\rightarrow p + e + \gamma \\ \gamma^* &\rightarrow e^+ + e^- \end{aligned}$$

2. $L_e(\pi^0) = L_e(\pi^\pm) = 0$:

$$\begin{aligned} \pi^0 &\rightarrow 2\gamma,\ \gamma + e^+ + e^- \\ p + \pi^- &\rightarrow n + \pi^0 \\ p + \pi^0 &\rightarrow n + \pi^+ \end{aligned}$$

3. $L_e(\bar{\nu}_e) = -1,\ L_e(\nu_e) = 1$:

$$\begin{aligned} \pi^- &\rightarrow e^- + \bar{\nu}_e \\ \pi^+ &\rightarrow e^+ + \nu_e \end{aligned}$$

From the last two reactions we learn the important result $\nu_e \neq \bar{\nu}_e$!

⑨ Baryon number conservation is responsible for the stability of the proton. By convention $B(p) = 1$, $B(e^-) = 0$. Determine the baryon numbers of particles from the observation of the following reactions:

a.) Baryons and mesons:

1. $B(\pi^0) = 0$:

$$p + p \rightarrow p + p + \pi^0$$

2. $B(n) = B(p)$, $B(\pi^\pm) = B(\pi^0) = 0$:

$$\begin{aligned} p + p &\rightarrow p + n + \pi^+ \\ \pi^- + p &\rightarrow n + \pi^0 \end{aligned}$$

3. $B(K^\pm) = B(K^0) = 0$:

$$\begin{aligned} K^\pm &\rightarrow \pi^\pm + \pi^0 \\ K^0 &\rightarrow \pi^+ + \pi^-,\ \pi^+ + \pi^- + \pi^0 \end{aligned}$$

4. $B(\Lambda)$, $B(\Sigma) = 1$:

$$\begin{aligned} \pi^- + p &\rightarrow \Lambda^0 + K^0,\ \Sigma^- + K^+ \\ \pi^+ + p &\rightarrow \Sigma^+ + K^+,\ \Sigma^0 + \Lambda^0 \end{aligned}$$

5. $B(\Xi)$, $B(\Omega^-) = 1$:

$$K^- + p \rightarrow \Xi^- + K^+,\ \Xi^0 + K^0,\ \Omega^- + K^+ + K^0$$

b.) Antibaryons:

6. $B(\bar{p}) = -1$:

$$p + p \rightarrow p + p + p + \bar{p}$$

7. $B(\bar{B}) = -1$:

$$p + \bar{p} \rightarrow \bar{n} + n,\ \bar{\Lambda}^0 + \Lambda^0,\ \bar{\Sigma}^0 + \Sigma^0,\ \bar{\Sigma}^\pm + \Sigma^\mp,\ \bar{\Xi}^+ + \Xi^-$$

c.) Photon:

8. $B(\gamma) = 0$:

$$p \rightarrow p + \gamma$$

d.) Leptons: All leptons are produced in pairs, $B(e^-) = 0$ by convention.

9. $B(e) = B(\mu) = 0$:

$$\gamma^* \rightarrow e^+ + e^-,\ \mu^+ + \mu^-$$

10. $B(\nu_e) = B(\nu_\mu) = 0$:

$$\begin{aligned} n &\rightarrow p + e^- + \bar{\nu}_e \\ \mu^- &\rightarrow e^- + \bar{\nu}_e + \nu_\mu \\ \mu^+ &\rightarrow e^+ + \nu_e + \nu_\mu^- \\ \pi^- &\rightarrow \mu^- + \bar{\nu}_\mu \\ \pi^+ &\rightarrow \mu^+ + \nu_\mu \end{aligned}$$

References

[1] F. Jegerlehner, *Electroweak Theory and LEP Physics*,
Lectures given in the "Troisième Cycle de la Physique en Suisse Romande" at the Ecole Polytechnique Fédérale Lausanne (see `http://www-com.physik.hu-berlin.de/~fjeger/books.html`)

[2] F. Jegerlehner, *The Anomalous Magnetic Moment of the Muon*, Springer Tracts in Modern Physics, Vol. 226, (Springer, Berlin, 2008)

[3] M. Gell-Mann, A. Pais, Phys. Rev. **97** (1955) 1387

[4] S.L. Glashow, Phys. Rev. Lett. **7** (1961) 469

[5] S. Okubo, Phys. Lett. **5** (1963) 165

2 Spontaneous symmetry breaking and Nambu–Goldstone bosons

In many cases symmetries are not ideally realized in nature. As an example, isospin invariance of strong interactions is not exact. It is broken by small mass-splittings among the members of isospin multiplets. For the nucleon doublet one has $(m_n - m_p)/m_p \simeq 1.29/938.27$, for the pion triplet $(m_{\pi\pm} - m_{\pi 0})/m_{\pi 0} \simeq 4.59/134.97$ and so on. In addition, electromagnetic and weak interactions violate the isospin symmetry of the strong interactions. Isospin is not conserved, for example, in decays like $\pi^0 \to \gamma\gamma$ (electromagnetic) or $n \to p + e^- + \bar{\nu}_e$ (weak).

Besides such **explicit** breaking one distinguishes **spontaneous** breaking of symmetries. A symmetry is said to be spontaneously broken if the ground state exhibits less symmetry than the Lagrangian (or Hamiltonian) of the system. In fact it may happen that symmetric equations of motion have non-symmetric solutions representing states of lower energy than the symmetric ones. Though the ground state exhibits non-trivial structure, in the spontaneously broken case, we still call it vacuum.

The phenomenon of spontaneously broken symmetries is well known from condensed matter physics. Let us consider the **Heisenberg model** of a ferromagnet, as an example. The model assumes nearest neighbor spin-spin interactions of spins $\vec{S}_{\vec{r}} = \left(S^x_{\vec{r}}, S^y_{\vec{r}}, S^z_{\vec{r}}\right)$ attached to the sites $\vec{r}$ of a cubic lattice. The corresponding interaction Hamiltonian

$$H = -J \sum_{<\vec{r},\vec{r}\,'>} \vec{S}_{\vec{r}} \cdot \vec{S}_{\vec{r}\,'}$$

is rotationally symmetric. Since parallel spins are favored energetically, below the critical temperature, the system exhibits "spontaneous" magnetization. We may choose it to point in the direction of the z-axis. The magnetization is the ground state expectation value of the spin variable $\vec{S}$

$$< S^z_{\vec{r}} > = < S^z_{\vec{0}} > = M \neq 0 \quad \text{or} \quad < \vec{S}_{\vec{0}} > = (0, 0, M) \neq 0$$

Generally, a local variable which has a non-vanishing ground state expectation value is called **order parameter**. In a state with non-zero magnetization the symmetry of the system is reduced to rotations around the magnetization axis. The original symmetry is "spontaneously broken". Of course, the magnetization may point in any direction, which means that there are infinitely many physically equivalent ground states. The different possible ground states are related to each other by rotations. Once we have chosen a particular ground state to describe the system the symmetry is spontaneously broken.

2.1 The Goldstone theorem

In abstract terms, we may characterize the situation as follows: Let the Lagrangian of a system be symmetric with respect to a group G of transformations. If a unique vacuum $\mid 0 >$ exists, then it must be invariant under G, meaning that $\mid 0 >$ is a singlet, and the symmetry is exact. In general, however, the ground state may be degenerate such that there exists more than one state of lowest energy. The set of ground states then must transform as a multiplet of G. For a continuous symmetry group a continuous "orbit" of vacua is obtained. Each ad hoc choice of one of the ground states as the "physical vacuum" of the system breaks the symmetry spontaneously. Typically, there exists a local field which transforms non-trivially under G and which has a non-vanishing vacuum expectation value (order parameter):

$$< 0 \mid \varphi_a(x) \mid 0 > = F_a \neq 0.$$

Since the symmetry is spontaneously broken, there is no symmetry principle which forces a field (or any operator) which has the quantum numbers of the vacuum to have vanishing vacuum expectation value. Of course we require the vacuum to be symmetric under translations and Lorentz transformations. Therefore, the field $\varphi_a(x)$ must be scalar and F_a must be a constant vector in the internal symmetry space. Let us suppose that φ_a transforms according to the fundamental representation of $G = SU(n)$. By a **global** transformation

$$\varphi_a \to \varphi'_a = e^{i\sum_i Q_i\omega_i}\varphi_a e^{-i\sum_i Q_i\omega_i} = \left(e^{i\sum_i T_i\omega_i}\right)_{ab}\varphi_b$$

we may arrange things such that only the real part of the nth component has non-vanishing vacuum expectation value:

$$< 0 \mid \varphi_a(x) \mid 0 >= 0\ ,\ a = 1, \ldots, n-1\ ;\ \ < 0 \mid \varphi_n(x) \mid 0 >\neq 0\ .$$

Since we assume the Lagrangian to have **global** $SU(n)$ symmetry, we have $n^2 - 1$ conserved Hermitian currents $j_i^\mu(x)$:

$$\partial_\mu j^\mu(x) = 0\ ;\ \ j_\mu^+ = j_\mu$$

and the generators of the group are represented by the charge operators

$$Q_i = \int d^3x\, j^0(\vec{x}, t)\ ,\quad \frac{dQ_i}{dt} = 0.$$

For infinitesimal transformations the transformation law for the field, given above, takes the form of a set of commutation relations

$$[Q_i, \varphi_a] = (T_i)_{ab}\,\varphi_b.$$

The T_i's are the generators in the $n \times n$ fundamental matrix representation given in Sec. 1. If we take the vacuum expectation value we obtain

$$< 0 \mid [Q_i, \varphi_a] \mid 0 >= (T_i)_{ab} < 0 \mid \varphi_b(x) \mid 0 >= (T_i)_{an}\, v.$$

If the symmetry is **not** broken spontaneously $v = 0$ the vacuum must be invariant $Q_i \mid 0 >= 0$ for $i = 1, \ldots, n^2 - 1$. Otherwise $< 0 \mid [Q_i, \varphi_a] \mid 0 >\neq 0$ for some i. We denote the subset of Q_i's for which $< 0 \mid [Q_i, \varphi_a] \mid 0 >\neq 0$ by $\tilde{Q}_i$. Since $< 0 \mid \tilde{Q}_i\varphi_a \mid 0 > - < 0 \mid \varphi_a\tilde{Q}_i \mid 0 >\neq 0$ we must have $\tilde{Q}_i \mid 0 >=\mid 0' >_i \neq 0$. On the other hand, by the symmetry of the Lagrangian, the generators commute with the Hamiltonian

$$[Q_i\ ,\ H] = 0\ ,\ \ i = 1, \ldots, n^2 - 1.$$

This implies

$$\left[\tilde{Q}_i, H\right] \mid 0 >= \tilde{Q}_i H \mid 0 > -H\tilde{Q}_i \mid 0 >= 0$$

i.e. if $\mid 0 >$ is an eigenstate of H then $\tilde{Q}_i \mid 0 >=\mid 0' >_i$ must be an eigenstate of H with the same eigenvalue if $\tilde{Q}_i \mid 0 >\neq 0$. The vacuum must be degenerate in this case. Another important consequence follows if we consider

$$\begin{aligned} < 0 \mid \left[\tilde{Q}_i, H\right] \mid p > &= < 0 \mid \tilde{Q}_i H \mid p > - < 0 \mid H\tilde{Q}_i \mid p > \\ &= p^0 < 0 \mid \tilde{Q}_i \mid p >= p^0\ {}_i < 0' \mid p >= 0. \end{aligned}$$

for a complete set of eigenstates $\mid p >$ of P^μ.

This implies that the new vacuum $\mid 0' >_i = \tilde{Q}_i \mid 0 >$ is orthogonal to all the states $\mid p >$ belonging to the Hilbert space with vacuum $\mid 0 >$. For each vacuum we get an **inequivalent representation** of the physical states i.e. the states created from different vacua cannot be mapped by unitary transformations. Each choice of a fixed vacuum $\mid 0 >$ yields a physically equivalent description of the system, however.

Next we consider the conditions

$$< 0 \mid \left[\tilde{Q}_i, \varphi_a\right] \mid 0 >= \int d^3x \ < 0 \mid \left[\tilde{j}_i^0(x), \varphi_a\right] \mid 0 >= \tilde{c}_{ia} v \neq 0$$

which imply that

$$< 0 \mid \tilde{j}_i^\mu(x)\varphi_a(y) \mid 0 > \neq 0.$$

$\varphi_a(y)$ contains a creation operator which creates from the vacuum $\mid 0 >$ a state $\mid p >_a$. The above condition thus is equivalent to

$$< 0 \mid \tilde{j}_i^\mu(x) \mid p >_a \neq 0.$$

Using translation invariance and $P^\mu \mid p >_a = p^\mu \mid p >_a$ we obtain

$$\begin{aligned} < 0 \mid \tilde{j}_i^\mu(x) \mid p >_a &= < 0 \mid e^{-iPx}\tilde{j}_i^\mu(0)e^{iPx} \mid p >_a \\ &= < 0 \mid \tilde{j}_i^\mu(0) \mid p >_a \, e^{ipx} \end{aligned}$$

where $< 0 \mid \tilde{j}_i^\mu(0) \mid p >_a = f_{ia}p^\mu$ since it is a function of p only and must be a Lorentz vector. Here it is important that $\mid p >$ is a scalar state. Hence

$$< 0 \mid \tilde{j}_i^\mu(x) \mid p >_a = f_{ia}p^\mu e^{ipx} \quad \text{with} \quad f_{ia} \neq 0.$$

Since $\tilde{j}_i^\mu(x)$ is a conserved current we must have

$$\begin{aligned} \partial_\mu < 0 \mid \tilde{j}_i^\mu(x) \mid p >_a &= < 0 \mid \partial_\mu \tilde{j}^\mu(x) \mid p >_a \\ &= i f_{ia} p^2 e^{ipx} = 0 \end{aligned}$$

and hence $p^2 = 0$. This is a very interesting result saying that the state $\mid p >_a$ must be a mass zero state and $\varphi_a(x)$ must be a massless scalar field. We thus have proven the **Goldstone theorem:** Spontaneous breaking of a continuous symmetry implies the existence of zero mass bosons, so called **Goldstone bosons**. How many Goldstone bosons are there? This question may be answered easily if we inspect the conditions

$$< 0 \mid [Q_i, \varphi_a] \mid 0 >= (T_i)_{an} \, v; i = 1, \ldots, n^2 - 1, a = 1, \ldots, n$$

more closely. Only generators T_i with a non-zero element in the nth column yield a symmetry breaking condition. Using the representation of the T_i's given in Sec. 1 we have **one** of the $n-1$ diagonal T_i' with a non-zero element in the nth column. We denote the corresponding Q_i by $\tilde{Q}_0$ and obtain

$$< 0 \mid \left[\tilde{Q}_0, \varphi_n\right] \mid 0 >= -\frac{n-1}{\sqrt{2n(n-1)}} v \neq 0.$$

In addition, there are $2(n-1)$ off-diagonal matrices which have either a $-i$ or a 1 at one position in the last column of rows $a = 1$ to $n-1$. The corresponding Q_i's we denote by $\tilde{Q}^1_a$ and $\tilde{Q}^2_a$, $a = 1, \ldots, n-1$. Thus, we have

$$< 0 \mid \left[\tilde{Q}^1_a, \varphi_a\right] \mid 0 >= -iv \; ; \;\; < 0 \mid \left[\tilde{Q}^2_a, \varphi_a\right] \mid 0 >= v \; .$$

The remaining $n^2 - 1 - 2(n-1) - 1 = (n-1)^2 - 1$ generators have $< 0 \mid [Q_a, \varphi_a] \mid 0 >= 0$ and hence leave the vacuum invariant. These are precisely the generators of the subgroup $SU(n-1)$ which leaves the vector

$$F_a = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix}$$

invariant:

$$UF = F \;\; \Rightarrow \;\; U = \left(\begin{array}{c|c} \hat{U} & 0 \\ \hline 0 & 1 \end{array}\right) \; , \;\; \hat{U} \in SU(n-1).$$

This group is the stability group (or little group) of the vacuum expectation value $< 0 \mid \varphi_a(x) \mid 0 >= F_a$ of the scalar multiplet $\varphi_a(x)$.

The $2(n-1)+1$ broken generators require $2n-1$ of the $2n$ real fields contained in the n complex fields φ_a to be massless i.e. there must be $2n-1$ Goldstone bosons.

If we introduce real fields by $\varphi_a = \varphi^1_a + i\varphi^2_a \left(\varphi^1_a = Re\varphi_a, \varphi^2_a = Im\varphi_a\right)$ we obtain

$$< 0 \mid [Q_i, \varphi_a] \mid 0 >=< 0 \mid \left[Q_i, \varphi^1_a\right] \mid 0 > + i < 0 \mid \left[Q_i, \varphi^2_a\right] \mid 0 >$$

Now, for a real field (and Hermitian generators $Q^+_i = Q_i$) we have

$$\begin{aligned} < 0 \mid \left[Q_i, \varphi^k_a\right] \mid 0 > \;\; &= \;\; < 0 \mid Q_i\varphi^k_a \mid 0 > - < 0 \mid \varphi^k_a Q_i \mid 0 > \\ &= \;\; < 0 \mid Q_i\varphi^k_a \mid 0 > - < 0 \mid Q_i\varphi^k_a \mid 0 >^* \\ &= \;\; i\, 2\, Im < 0 \mid Q_i\varphi^k_a \mid 0 > \end{aligned}$$

and hence

$$< 0 \mid [Q_i\varphi_a] \mid 0 >= i\, 2\, Im < 0 \mid Q_i\varphi^1_a \mid 0 > -2\, Im < 0 \mid Q_i\varphi^2_a \mid 0 >$$

The non-vanishing expectation values are then

$$\begin{aligned} 2\, Im < 0 \mid \tilde{Q}_0, \varphi^2_n \mid 0 > \;\; &= \;\; \frac{n-1}{\sqrt{2n(n-1)}} v \\ 2\, Im < 0 \mid \tilde{Q}^1_a, \varphi^1_a \mid 0 > \;\; &= \;\; -v \\ 2\, Im < 0 \mid \tilde{Q}^2_a, \varphi^2_a \mid 0 > \;\; &= \;\; -v \end{aligned}$$

and the corresponding fields must be massless by the argument given before.

As a result we have: The complex multiplet $\varphi_a(x)$ in the fundamental representation of $SU(n)$ exhibits $2n$ real fields. In the spontaneously broken phase we can choose the internal space frame such that exactly **one** field, $\varphi_n^1 = Re\varphi_n$ has a non-vanishing vacuum expectation value $< 0 \mid \varphi_n^1 \mid 0 >= v > 0$. The remaining $2n-1$ must be massless Goldstone boson fields. The field φ_n' may have any mass and is not a Goldstone boson even if it would be massless by accident.

A final remark concerning the currents: If a global symmetry is exact the Noether currents

$$j_i^\mu(x) =: j_i^\mu(x) :$$

represented in terms of creation and annihilation operator exhibit terms a^+b only (possibly $b = a$) and no a^+b^+ or $a\,b$ terms. Therefore

$$j_i^\mu(x) \mid 0 >= 0.$$

In the spontaneously broken phase conditions

$$< 0 \mid \tilde{j}_i^\mu(x)\varphi_a(y) \mid 0 >\neq 0$$

must hold, such that $\tilde{j}_i^\mu(x)$ cannot annihilate the vacuum! How can we understand this? In our case $\varphi_n = \varphi_n' + v$ with $< 0 \mid \varphi_n' \mid 0 >= 0$ i.e. φ_n' is the field having "normal" properties if expanded in terms of creation and annihilation operators. The contribution of φ_a to the Noether current is

$$j_i^\mu = i : \varphi_a^* \, (T_i)_{ab} \overset{\leftrightarrow}{\partial}_\mu \varphi_b :$$

If we shift the field $\varphi_n \to \varphi_n + v$ we obviously get terms proportional to $v\partial_\mu\varphi_a$ which are **linear** in the fields. This obviously explains why

$$j_i^\mu(x) \mid 0 >\neq 0$$

now.

There are $2n-1$ currents which get a linear term

$$\begin{aligned}
\tilde{Q}_0 : \quad \tilde{j}_0^\mu \quad &= i : \varphi_a^* \left(\tilde{T}_0\right)_{ab} \overset{\leftrightarrow}{\partial}_\mu \varphi_b : \quad +2\tfrac{n-1}{\sqrt{2n(n-1)}} v\partial_\mu\varphi_n^2 \\
\tilde{Q}_a^1 : \quad \tilde{j}_a^{\mu 1} \quad &= i : \varphi_a^* \left(\tilde{T}_a^1\right)_{ab} \overset{\leftrightarrow}{\partial}_\mu \varphi_b : \quad -2v\partial_\mu\varphi_a^1 \\
\tilde{Q}_a^2 : \quad \tilde{j}_a^{\mu 2} \quad &= i : \varphi_a^* \left(\tilde{T}_a^2\right)_{ab} \overset{\leftrightarrow}{\partial}_\mu \varphi_b : \quad -2v\partial_\mu\varphi_a^2 \, .
\end{aligned}$$

and create $2n-1$ different Goldstone bosons from the vacuum.

2.2 Models of spontaneous symmetry breaking

The first field theory models which exhibited spontaneous symmetry breaking have been invented around the year 1960 as models for describing the pion-nucleon ($\pi - N$) system. Nambu (1960) and Nambu-Jona-Lasinio (1961) proposed the so called **Nambu-Jona-Lasinio model**. At about the same time Gell-Mann-Lévy (1960) proposed the linear **σ-model** as a description of the same system. In both models pions appear as Goldstone bosons. An approximately realized Goldstone

mechanism still is the only way to understand why pions are so much lighter than nucleons [5]. Later Goldstone (1961) noticed that the appearance of massless states is a general phenomenon related to spontaneous symmetry breaking. One distinguishes two different schemes of spontaneous symmetry breaking. In the **linear realization** (prototype linear σ-model) a scalar field, which develops a non-vanishing vacuum expectation value, is introduced as a fundamental field in the Lagrangian. Such models with elementary scalars are often considered to be doubtful as viable physical models because elementary scalars have not yet been found in nature. The other possibility is the **dynamical symmetry breaking** scheme (prototype Nambu-Jona-Lasinio model) which has no elementary scalars. The order parameter is a composite field like $\bar{\psi}\psi$ and the Goldstone bosons are composite fermion – antifermion states similar to the real pions (if they would be massless). The problem with such models is that our unfortunate inability to treat the relativistic bound state problem makes it hard to make unambiguous predictions.

In the following we consider models with elementary scalars for illustration of spontaneous symmetry breaking and the Goldstone mechanism.

2.2.1 A model with spontaneous breaking of a discrete symmetry

Consider a real scalar field with self-interaction described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 - V(\varphi) = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \frac{\mu^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 \,.$$

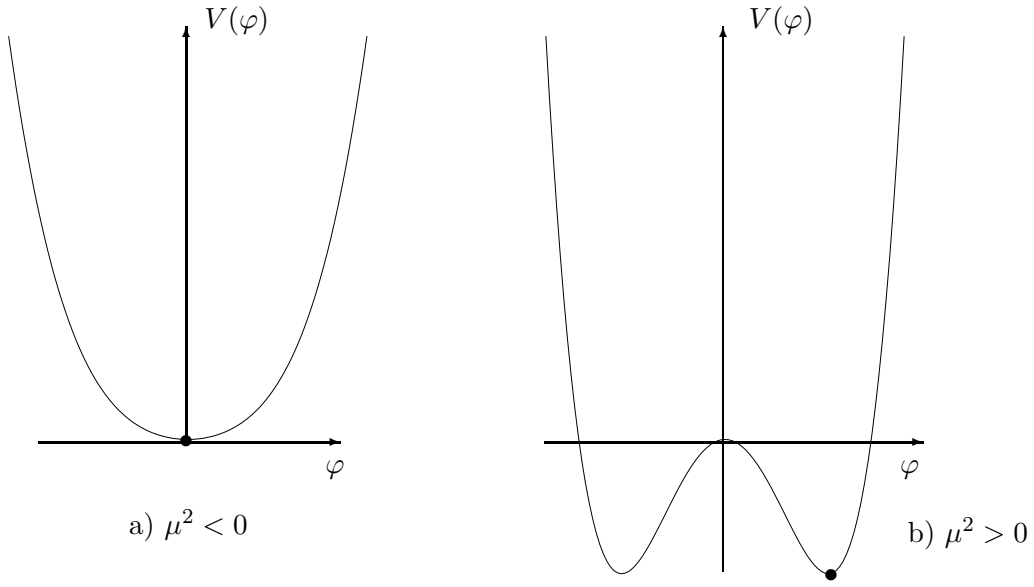


Figure 2.1: Scalar potential a) in the symmetric and b) in the spontaneously broken phase.

[5] In modern QCD language the Goldstone boson picture of the pions is realized in the **chiral limit** of vanishing light quark masses. In this limit the pions are bona fide massless Goldstone particles

Stability of the system requires $\lambda > 0$. The equation of motion for the field φ is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} = \frac{\partial \mathcal{L}}{\partial \varphi} \quad \text{or} \quad \Box\varphi = -\frac{\partial V}{\partial \varphi} = \mu^2 \varphi - \frac{\lambda}{3!}\varphi^3 \; .$$

The Lagrangian ha a discrete symmetry $Z_2 = \{\pm 1\} : \varphi \to -\varphi$.

Let us treat φ as a classical field for the moment. The equilibrium solution of the system is the one for which $\varphi(x)$ is a constant determined by $\frac{\partial V}{\partial \varphi} = 0$. If $\mu^2 < 0$, we have a unique ground state solution $\varphi(x) = \varphi_0 = 0$. In the quantized version we then have a unique vacuum $\mid 0 >$ and $< 0 \mid \varphi(x) \mid 0 >= 0$. $m^2 = -\mu^2 > 0$ is the physical mass of the field φ. Now choose $\mu^2 > 0$. Then

$$\frac{\partial V}{\partial \varphi} = \frac{\lambda}{3!}\varphi^3 - \mu^2 \varphi = 0$$

has besides the trivial solution $\varphi_0 = 0$ two non-trivial solutions (see Fig. 2.1)

$$\varphi_\pm = v_\pm = \pm\sqrt{\frac{6\mu^2}{\lambda}} \; .$$

Only these two solutions correspond to a minimum of the potential and hence to a ground state solutions. Obviously now we have two degenerate ground states.

If $\varphi(x)$ is treated as a quantum field we have to choose **one** of the ground states as the vacuum $\mid 0 >$. This choice is fixed if we require, in first approximation the vacuum expectation value to be equal to one of the classical results

$$< 0 \mid \varphi(x) \mid 0 >= v_+ = v > 0$$

for example.

Notice that a perturbation expansion based on the splitting $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ with $\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \frac{\mu^2}{2}\varphi^2$ and $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\varphi^4$ does not make sense because this would correspond to an expansion in terms of negative $m^2 = -\mu^2 < 0$ solutions. Such Tachyons, neither satisfy the spectrum condition nor local causality. As we shall see there is a simple way to circumvent this problem. If we would perform a perturbation expansion about the fake free field solution, in every finite order of perturbation expansion, we would have a tachyon as an artifact of a nonsensical expansion. Only be infinite resummation techniques we would be able to recover the right physical answer. A field which allows for a normal particle interpretation must satisfy

$$< 0 \mid \varphi'(x) \mid 0 >= 0.$$

Such a field we simply obtain by a shift

$$\varphi = \varphi' + v$$

from the original field φ (B. W. Lee 1969). When we rewrite the Lagrangian in terms of φ' we obtain

$$\mathcal{L}(\varphi) \quad = \quad \mathcal{L}(\varphi' + v) = \mathcal{L}'(\varphi')$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\partial_\mu\varphi'\right)^2 + \frac{\mu^2}{2}\left(\varphi'+v\right)^2 - \frac{\lambda}{4!}\left(\varphi'+v\right)^4 \\
&= \frac{1}{2}\left(\partial_\mu\varphi'\right)^2 + \frac{1}{2}\left(\mu^2 - \frac{\lambda v^2}{2}\right)\varphi'^2 \\
&\quad -\frac{\lambda}{4!}\varphi'^4 - \frac{\lambda v}{3!}\varphi'^3 + \left(\mu^2 v - \frac{\lambda v^3}{6}\right)\varphi' + \frac{\mu^2 v^2}{2} - \frac{\lambda v^4}{4!} \quad .
\end{aligned}
$$

The mass term of φ' now is given by

$$
m_{\text{eff}}^2 = \frac{\lambda v^2}{2} - \mu^2 = \frac{\lambda v^2}{2} - \frac{\lambda v^2}{6} = \frac{\lambda v^2}{3} > 0 \ !
$$

where we have used $v = \sqrt{\frac{6\mu^2}{\lambda}}$ such that $\lambda v^2 = 6\mu^2$.
Now we can set up a perturbation expansion with

$$
\begin{aligned}
\mathcal{L}_0(\varphi') &= \frac{1}{2}\left(\partial_\mu\varphi'\right)^2 - \frac{m_{\text{eff}}^2}{2}\varphi'^2 \\
\mathcal{L}_{\text{int}}\left(\varphi'\right) &= -\frac{\lambda}{4!}\varphi'^4 - \frac{\lambda v}{3!}\varphi'^3 - c_1\varphi' + c_0
\end{aligned}
$$

and obtain the Feynman rules depicted in Fig. 2.2.

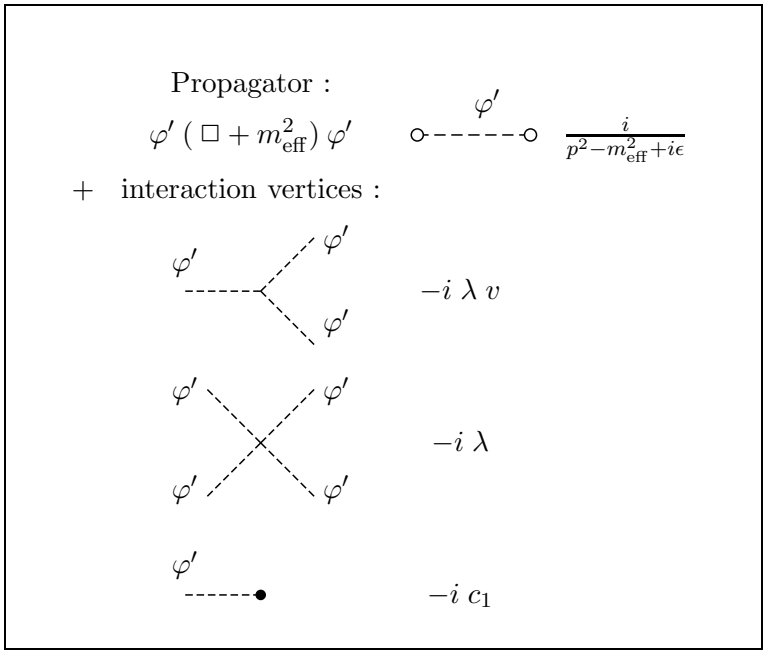


Figure 2.2: Feynman rules for $\mathcal{L}\left(\varphi'\right)$

The vacuum energy must be adjusted to zero: $c_0 = 0$. The free parameters are λ and m_{eff}^2 as in the symmetric case. The vacuum expectation value is determined by

$$
v = +\sqrt{\frac{3m_{\text{eff}}^2}{\lambda}}
$$

and $c_1 = v\left(\frac{\lambda v^2}{6} - \mu^2\right) = 0$! by the condition that $< 0 \mid \varphi'(x) \mid 0 >= 0$.

The symmetry $\varphi' \to -\varphi'$ is broken now and we have learned that spontaneous symmetry breaking can **reveal a physical mass** to a particle.

2.2.2 The Goldstone model

The Goldstone model illustrates spontaneous breakdown of a continuous symmetry. We may obtain it from the previous model by replacing the real field by a complex field $\varphi \neq \varphi^*$. The Lagrangian has the form

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(\varphi\varphi^*) = \partial_\mu \varphi^* \partial^\mu \varphi + \mu^2 \varphi^* \varphi - \frac{\lambda}{3!}(\varphi^* \varphi)^2$$

and is symmetric under **global** $U(1)$ transformations. We use the equivalent representation in terms of a doublet of two real fields

$$\hat{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \; ; \;\; \varphi_i = \varphi_i^* \;\; (i = 1, 2)$$

which transforms under $O(2) \simeq U(1)$ rotations. The complex (charged) field is then given by

$$\varphi = \frac{1}{\sqrt{2}} (\varphi_1 + \, i \, \varphi_2) \; .$$

With $\hat{\varphi}^2 = \varphi_1^2 + \varphi_2^2 = 2\varphi^* \varphi$ the Lagrangian takes the form

$$\mathcal{L}(\hat{\varphi}) = \frac{1}{2} (\partial_\mu \hat{\varphi})^2 + \frac{\mu^2}{2} \hat{\varphi}^2 - \frac{\lambda}{4!} \hat{\varphi}^4 \; ,$$

which obviously is symmetric under 2-dimensional rotations

$$\hat{\varphi} \to \hat{\varphi}' = \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} .$$

The equation of motion for $\hat{\varphi}$ is

$$\Box \hat{\varphi} = \mu^2 \hat{\varphi} - \frac{\lambda}{3!} \hat{\varphi}^2 \hat{\varphi} \; .$$

Again for $\mu^2 < 0$ we find a unique ground state solution $\varphi_1 = \varphi_2 = 0$ and $m^2 = -\mu^2$ is the common mass of φ_1 and φ_2 . If $\mu^2 > 0$, the solution $\hat{\varphi}_0 = \binom{0}{0}$ is unstable. The potential has a minimum at (see Fig. 2.3)

$$\hat{\varphi}_0^2 = \frac{6\mu^2}{\lambda}$$

and we have a continuous orbit of ground state solutions related by rotations. We now choose, ad hoc, **one** particular ground state solution as the vacuum $\mid 0 >$. $O(2)$-invariance is spontaneously

broken now. The particular choice of $\mid 0 >$ is fixed by specifying the vacuum expectation value of $\hat{\varphi}$:

$$< 0 \mid \hat{\varphi}(x) \mid 0 >= \begin{pmatrix} 0 \\ v \end{pmatrix} \; ; \;\; v > 0 \; ; \;\; v = \sqrt{\frac{6\mu^2}{\lambda}}$$

The physical effect of this breaking again becomes transparent if we write the Lagrangian in terms of the shifted field

$$\hat{\varphi}'(x) = \hat{\varphi}(x) - \begin{pmatrix} 0 \\ v \end{pmatrix} \text{ i.e. } \begin{array}{l} \varphi_1' = \varphi_1 \\ \varphi_2' = \varphi_2 - v \;\; . \end{array}$$

We obtain

$$\begin{aligned} \mathcal{L}(\hat{\varphi}) &= \mathcal{L}(\hat{\varphi}' + \begin{pmatrix} 0 \\ v \end{pmatrix}) = \mathcal{L}'(\hat{\varphi}') \\ &= \frac{1}{2}\left(\partial_\mu \hat{\varphi}'\right)^2 - \frac{1}{2}\left(\frac{\lambda v^2}{6} - \mu^2\right)\varphi_1'^2 - \frac{1}{2}\left(\frac{\lambda v^2}{2} - \mu^2\right)\varphi_2'^2 \\ &\quad -\frac{\lambda}{4!}\left(\hat{\varphi}'\right)^4 - \frac{\lambda v}{2 \cdot 3!}\hat{\varphi}'^2\varphi_2' - c_1\varphi_2' + c_0 \end{aligned}$$

with the mass terms

$$\begin{aligned} m_{1\text{eff}}^2 &= \frac{\lambda v^2}{6} - \mu^2 = \frac{\lambda v^2}{6} - \frac{\lambda v^2}{6} = 0 \\ m_{2\text{eff}}^2 &= \frac{\lambda v^2}{2} - \mu^2 = \frac{\lambda v^2}{2} - \frac{\lambda v^2}{6} = \frac{\lambda v^2}{3} > 0 \;\; ! \end{aligned}$$

This is a remarkable result: By spontaneous symmetry breaking the particles have acquired different masses and one of the particles is massless. As we know, the appearance of a Goldstone boson is a necessary consequence of spontaneous breaking of a continuous symmetry and is not peculiar to the particular model.

The original Lagrangian which seems to describe the two fields φ_1 and φ_2 in a completely symmetric way, has been shown to describe two neutral scalar particles of unequal mass. This is a clear manifestation of symmetry breaking.

This symmetry breaking is very different, however, from an explicit breaking of the symmetry which would result if we would add ad hoc two independent mass terms for the fields φ_1 and φ_2. This would yield a model with three independent parameters m_1, m_2 and λ. In the spontaneously broken case one of the fields **must** be massless and the model has the original number of parameter λ and m_2 in spite of the fact that new interaction vertices have been generated by the shift of the field. v is determined by $v = \sqrt{\frac{3m_2^2}{\lambda}}$ and $c_1 = v\left(\frac{\lambda v^2}{6} - \mu^2\right) = 0$.

There is an intuitive way of understanding the existence of a Goldstone boson from the fact that the vacuum is not unique. Since the vacuum is a state of zero energy and momentum different vacuum states can differ only by the presence of a Bose condensate i.e. the presence of an unspecified number of quanta of zero energy and momentum. Such quanta are possible only if there exist massless particles with the quantum numbers of the vacuum. In our example we can understand intuitively that there will be **one** such type of Goldstone bosons. The different

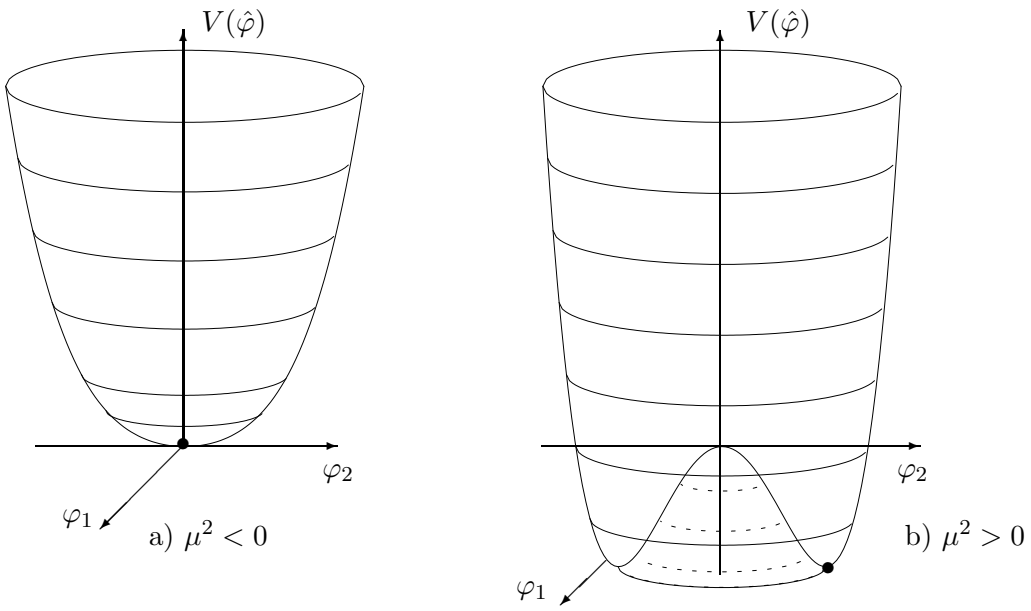


Figure 2.3: Potential of the Goldstone model: a) in the symmetric and b) in the spontaneously broken phase.

possible values of $< 0 \mid \hat{\varphi}(x) \mid 0 >$, lie on a circle of radius v. Each point on the circle correspond to a particular choice of the vacuum state. Thus there is one degree of freedom (motion along the circle) by which different vacuum states may be connected, and hence **one** type of massless bosons.

The curvature of the potential

$$\left(\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_k}\right)\Big|_{\hat{\varphi}=\hat{\varphi}_0}$$

at the classical minimum actually represents the mass matrix of the system. If we diagonalize the matrix we find an eigenvalue zero corresponding to zero curvature of the potential in direction tangential to the ground state orbit.

A final remark may be in order here: we have shown in Subsec. 2.1 that the different degenerate vacua carry physical Hilbert spaces which are totally orthogonal (inequivalent representations). Thus the existence of the valley in the potential which seems to explain the come about of a Goldstone boson as the mode which moves along the bottom of the valley is misleading. There are no Goldstone bosons moving from one vacuum to another one. Note that all Goldstone bosons move at the speed of light and carry some momentum. The fact that this momentum may be arbitrarily small does not change the fact that there cannot be physical transitions between different degenerate vacua. Similarly, if we consider a Heisenberg ferromagnet which has a net spontaneous magnetization in a given direction (always in the thermodynamic limit, i.e., at infinite volume) the spin waves which correspond to the Goldstone excitations of course do not affect (i.e., rotate) the given ground state of the system.

Exercises: Section 2

① Discuss the symmetry breaking of $SU(2)_{\text{flavor}}$ (Isospin) and $SU(3)_{\text{flavor}}$ (Isospin and

Strangeness) in the spin 1/2 baryon octet. Comment the decays $\Sigma^0 \to \Lambda\gamma$ and $\Sigma^+ \to p\gamma$ and compare them to the strong decays $\Sigma^- \to \Lambda\pi^-$(?) and $\Sigma^- \to n\pi^-$. Use the quark model schema for the discussion.

② In the Goldstone model of spontaneous symmetry breaking two real fields $\hat{\varphi} = \binom{\varphi_1}{\varphi_2}$ interact by the Lagrangian $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \hat{\varphi}\right)^2 + \frac{\mu^2}{2}\hat{\varphi}^2 - \frac{\lambda}{4!}\hat{\varphi}^4$. For $\mu^2 > 0$ let the ground state be given by $\hat{\varphi}_0 = \binom{0}{v}, v > 0$. Show that the curvature of the potential $\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_k}$ at the ground state $\hat{\varphi}_0$ represents the mass matrix of the system.

③ Write down the Feynman rules for the Goldstone model (see Fig. 2.2).

③ Calculate the Noether currents for the Goldstone model, with respect to the unbroken global symmetry. Discuss how it is possible that a current couples to the vacuum, i.e. why it can bee that $\langle 0|j_i^\mu(x)|0\rangle \neq 0$.

3 Chiral symmetry and quark flavor mixing

3.1 Prolog: QCD as a part of the Standard Model of fundamental interactions

According to present day knowledge, the known "fundamental" interactions of "elementary" particles follow from a local gauge principle (*Weyl, Yang-Mills*) with the gauge group

$$G^{\rm SM}_{\rm local} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \tag{3.1}$$

which is broken through a Higgs mechanism to $SU(3)_c \otimes U(1)_{\rm em}$. $SU(3)_c$ is the color gauge group which determines the interaction of the color triplets of quarks via an octet of colored gluons. This unbroken color gauge theory is called quantum chromodynamics (QCD) (Fritzsch, Gell-Mann, Leutwyler 1973) and determines strong interaction physics, and in particular, the hadron spectrum and the residual strong interaction between hadrons. Quarks originally emerged as the building blocks of hadrons in the attempt to classify the hadronic states according to their flavor (Gell-Mann, Ne'eman, Zweig 1964). The quark model hypothesis required the hadrons to be composite states of quarks with the quarks being permanently confined in the hadrons. The confinement hypothesis declares **hadrons** to be **composite elementary particles**. More specifically, baryons are three quark states, mesons are quark-antiquark states

Baryons:, $p, n, \Lambda, \Sigma, \Delta, \ldots$

$$\begin{array}{lllllll} B = (q_1 q_2 q_3): & \underline{s=1/2}: & p: & uud & \underline{s=3/2}: & \Delta^{++}: & uuu \\ & & n: & udd & & \vdots & \vdots \\ & & & & & \Delta^{-} : & ddd \end{array}$$

Mesons: $\pi, K, \eta, \rho, \omega, \ldots$

$$\begin{array}{lllllll} M = (q_1 \bar{q}_2): & \underline{s=0}: & \pi^+: & u\bar{d} & \underline{s=1}: & \rho^+: & u\bar{d} \\ & & \pi^0 : & \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) & & \rho^0 : & \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \\ & & \pi^-: & d\bar{u} & & \rho^-: & d\bar{u} \\ & & \eta : & \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) & & \omega : & \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \end{array}$$

Quarks must then be fractionally charged: $Q_u = 2/3, Q_d = -1/3$. A crucial problem showed up for the states which are totally symmetric under permutations of spin and flavor (e.g. Δ^{++} : $u(\uparrow)u(\uparrow)u(\uparrow)$). The spin-statistic theorem requires both the quarks ($s = 1/2$) and the baryons ($s = 1/2, 3/2, \ldots$) to satisfy the Pauli principle. This requires a totally antisymmetric quark wave function for the quarks in the baryon. This **spin-statistics crisis** could be solved only by assigning a new quantum number, called **color**, to quarks. Each quark must exist in three copies, the red(r), green(g), and blue(b) quarks. Since color never has been observed it was natural to require $SU(3)_c$ color symmetry to be a local symmetry, which implies that colors are indistinguishable. The confinement hypothesis now requires that **physical states** must be **color singlets**. The singlet condition leads in a very natural way to the baryons

$$(q_1 q_2 q_3)_{\text{color singlet}} = \frac{1}{\sqrt{3!}} \varepsilon_{c_1 c_2 c_3} q_{1c_1} q_{2c_2} q_{3c_3}$$

and mesons

$$(q_1 \bar{q}_2)_{\text{color singlet}} = \frac{1}{\sqrt{3}} \delta_{c_1 c_2} q_{1c_1} \bar{q}_{2c_2}$$

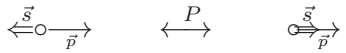
The wave functions of hadronic states are given now by a product of spatial, color, flavor and spin wave functions. The ground state hadrons have no orbital angular momentum, such that the spatial wave function is symmetric under permutations of the constituents. Simultaneous eigenstates of color (singlet), flavor and spin are then easily constructed by considering factors. The calculation of the hadrons as bound states of quarks (i.e. the actual calculation of the spatial wave function) is an unsolved problem. After this digression to hadronic physics we come back to the other part of of $G^{\rm SM}_{\rm local}$.

The other unbroken subgroup $U(1)_{\rm em}$ of $SU(2)_L \otimes U(1)_Y$ defines quantum electrodynamics (QED), describing the interaction of all charged particles with the photon. $SU(2)_L$ is the **weak isospin** gauge group which determines the interaction of the left–handed $(V-A)$ fermion currents, known from weak interaction processes, with a triplet of weak gauge bosons also called intermediate vector bosons. Finally, the $U(1)_Y$ subgroup is needed in order to recover $U(1)_{\rm em}$ with $Q = T_3 + \frac{Y}{2}$ in the broken phase as we shall see below.

The standard model (SM) is determined essentially by specifying the matter fields and their transformation properties under local gauge transformations. A way to "understand" the emergence of the SM is described in the following.

3.1.1 The matter fields

The "real world" is build from **massless** spin 1/2 particles the quarks and leptons. Spin 1/2 particles in a sense are more fundamental than other particles because they allow to compose particles of any spin (e.g. $1/2 \otimes 1/2 = 0 \oplus 1$ etc.). Massless Fermi fields have fixed handedness called chirality or helicity. If ψ is a massless Dirac field the left–handed field $\psi_L = \frac{1-\gamma_5}{2}\psi$ and the right–handed field $\psi_R = \frac{1+\gamma_5}{2}\psi$ do not mix under Lorentz-transformations, rotations and space-time translation.



ψ_L ψ_R

The fields ψ_L and ψ_R are interchanged under parity transformations. Since ψ is a local field satisfying Einstein causality the left–handed Dirac field of a massless electron denoted by e^-_L describes a left–handed electron and, simultaneously, a right–handed positron. Similarly, if e^-_R is the local field describing a right–handed electron, this field also describes a left–handed positron. Thus $e^-_R \equiv e^+_L$. Therefore, we may consider all massless fields to be left–handed.

According to todays knowledge, matter is made out of colored quarks and leptons which are grouped into three families[6].

As we have argued earlier quarks of different flavors must show up in three replica of red(r), green(g), and blue(b) color. The first family of fermions are

$$\nu_{eL}, \bar{\nu}_{eL}, e^-_L, e^+_L, u_{Lr}, u_{Lg}, u_{Lb}, u^c_{Lr}, u^c_{Lg}, u^c_{Lb}, d_{Lr}, d_{Lg}, d_{Lb}, d^c_{Lr}, d^c_{Lg}, d^c_{Lb}$$

where u^c denotes the antiparticle of the u-quarks and so on. All **stable matter** is built from these first family quarks and leptons. The two additional families (who has ordered them?) we obtain by replacing (ν_e, e) with (ν_μ, μ) and (u, d) with (c, s) and (ν_e, e) with (ν_τ, τ) and (u, d)

[6]The not too long ago observed neutrino oscillations require the neutrinos to have a tiny mass which must be different for the different flavors. This requires the existence of right–handed neutrinos $\nu_{\ell R} \equiv \bar{\nu}_{\ell L}$ in spite of the fact that they do not couple directly to gauge fields (i.e., they are singlets with respect to the SM gauge group).

with (t, b) . They supplement Nature by forms of unstable matter and allow for the phenomena of flavor mixing and *CP*-violation. Altogether we know 3×16 degrees of freedom. If there would be no interactions the (free) Lagrangian of the world would be

$$\mathcal{L}_{\text{matter}} = \sum_a \bar{\psi}_{La} i\gamma^\mu \partial_\mu \psi_{La}$$

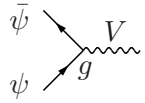
which exhibits a huge symmetry of global $U(48)$. In nature a subgroup of this large global symmetry group turns out to be a local symmetry

$$\psi_{La} \to U(x)_{ab} \psi_{Lb} \; ; \;\; U(x) \in G_{\text{local}}$$

where $G_{\text{local}} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. This requires the fields to couple to massless spin 1 gauge fields via minimal coupling

$$\partial_\mu \psi_L \to D_\mu \psi_L = \left(\partial_\mu - i \sum_\alpha g_\alpha T_{\alpha i} V_{\mu \alpha i} \right) \psi_L$$

with interaction vertices:



Here, α labels the factors $SU(3)_c, SU(2)_L$ and $U(1)_Y$ to which the Yang-Mills construction applies individually. The emergence of G_{local} as a direct product of two simple Lie groups and the Abelian group $U(1)_Y$ is closely related to the phenomenological appearance of strong, weak and electromagnetic interactions as separate phenomena differing in strength and symmetry. The $SU(3)_c$ which determines strong interactions distinguishes between triplets of quarks, "antitriplets" of antiquarks and singlets of leptons and neutrinos. Since the latter do not carry color they do not participate in strong interactions i.e. they do not talk to quarks and gluons, the gauge quanta of $SU(3)_c$. The $SU(2)_L$ distinguishes between doublets of left–handed particles $\binom{\nu_e}{e^-}_L, \ldots$ and singlets of left–handed antiparticles which are usually identified by the right–handed particles $e_L^+ \equiv e_R^-, \ldots$. This undemocratic treatment of particles and antiparticles is what we know as maximal parity violation of weak interactions. The Abelian $U(1)_Y$ only affects the phases of the fields according to the **weak hypercharge** assignment $Y = 2(Q - T_3)$.

Notice that a fermion mass term

$$\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$$

cannot be $SU(2)_L \otimes U(1)_Y$ invariant. It is the parity violating nature of weak interactions which forbid fermion masses. We summarize the local multiplet structure in the following tables:

"Weak quantum numbers": $Q = T_3 + \frac{1}{2}Y$

	Doublets				Singlets			
	$(\nu_\ell)_L$	$(\ell^-)_L$	$(u,c,t)_L$	$(\tilde{d},\tilde{s},\tilde{b})_L$	$(\nu_\ell)_R$	$(\ell^-)_R$	$(u,c,t)_R$	$(d,s,b)_R$
Q	0	-1	$2/3$	$-1/3$	0	-1	$2/3$	$-1/3$
T_3	$1/2$	$-1/2$	$1/2$	$-1/2$	0	0	0	0
Y	-1	-1	$1/3$	$1/3$	0	-2	$4/3$	$-2/3$

Table 1: Matter fields and their $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ quantum numbers.

group	representation		
$SU(3)_c$	3	$\begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix}$	quark color triplets
	3^*	$\begin{pmatrix} q^c_{\bar{r}} \\ q^c_{\bar{g}} \\ q^c_{\bar{b}} \end{pmatrix}$	antiquark color triplets
	1		leptons, neutrinos
$SU(2)_L$	$2 = 2^*$	$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} u \\ \tilde{d} \end{pmatrix}_L$ $\begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} c \\ \tilde{s} \end{pmatrix}_L$ $\begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \begin{pmatrix} c \\ \tilde{b} \end{pmatrix}_L$	left–handed weak isospin doublets of leptons and quarks (flavor doublets)
	1	$\nu_{eR}, e^-_R, u_R, d_R$ $\nu_{\mu R}, \mu^-_R, c_R, s_R$ $\nu_{\tau R}, \tau^-_R, t_R, b_R$	right–handed weak isospin singlets of leptons and quarks
$U(1)_Y$	phase transformations	weak hypercharge	$Y = 2(Q - T_3)$

Notice:
- $\nu_{eR}, \nu_{\mu R}$ and $\nu_{\tau R}$ all have zero quantum numbers with respect to G_{local} (i.e. no couplings to gauge fields).
- the bottom components of the quark doublets are Cabibbo-Kobayashi-Maskawa rotated fields $\tilde{d}, \tilde{s}, \tilde{b}$ (see below).

In the following we denote the weak doublets by L_ℓ ($\ell = e,\ \mu,\ \tau$) for the leptons and by L_q for the quarks.

3.1.2 The gauge fields

For each factor group of G_{local}, local gauge invariance requires the existence of a set of massless gauge fields in the adjoint representation, which couple minimally to the matter fields. We denote the gauge fields $V_{\mu\alpha i}(x)$ and the gauge couplings as follows:

group		**gauge fields**	**name**	**coupling**
$SU(3)_c$	:	$G_{\mu i}$; $i = 1, \ldots, 8$	gluons	g_s
$SU(2)_L$	:	$W_{\mu a}$; $a = 1, 2, 3$	weak gauge bosons	g
$U(1)_Y$	:	B_μ		g'

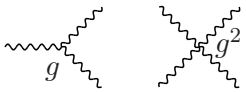
The Yang-Mills Lagrangian is given by

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} G_{\mu\nu i} G_i^{\mu\nu} - \frac{1}{4} W_{\mu\nu a} W_a^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

with field strength tensors

$$\begin{aligned} SU(3)_c &: \; G_{\mu\nu i} = \partial_\mu G_{\nu i} - \partial_\nu G_{\mu i} + g_s f_{ikl} G_{\mu k} G_{\nu l} \\ SU(2)_L &: \; W_{\mu\nu a} = \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g \varepsilon_{abc} W_{\mu b} W_{\nu c} \\ U(1)_Y &: \; B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \; . \end{aligned}$$

The non-abelian fields must exhibit self–interactions described by the vertices



and with coupling strength equal the matter field coupling

$$\mathcal{L}_{\text{matter, int}} = g \bar{\psi}_L \gamma^\mu T_a \psi_L W_{\mu a}$$

where $J_a^\mu = \bar{\psi}_L \gamma^\mu T_a \psi_L$ are the fermion currents associated with the gauge group.

The non–Abelian gauge field interaction terms for the electroweak gauge group $SU(2)_L \otimes U(1)_Y$ have been experimentally confirmed by LEP experiments [16]. The corresponding test for the QCD part will be considered later.

3.2 Chiral transformations, chiral symmetry and the axial-vector anomaly

3.2.1 Chiral fields and the $U(1)$-axial current

In the zero mass limit a free Dirac particle decouples into two chiral states described by the Weyl fields ψ_L and ψ_R. This becomes evident if we write the free Dirac Lagrangian in terms of ψ_L and

$\psi_R : \psi = \psi_L + \psi_R$[7]

$$\begin{aligned}\mathcal{L} &= \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \\ &= \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R - m\left(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L\right)\end{aligned}$$

If $m = 0$ we observe that $\mathcal{L}$ decomposes into two independent Lagrangians for the fields ψ_L and ψ_R. In this case $\mathcal{L}$ is not only invariant under global phase transformations

$$\psi \to e^{i\alpha}\psi \; ; \;\; e^{i\alpha} \in U(1)_V \quad \text{vector group}$$

(infinitesimal: $\delta\psi = i\alpha\psi, \;\; \delta\bar{\psi} = -i\alpha\bar{\psi}$) but also under global **chiral transformations**

$$\psi \to e^{i\beta\gamma_5}\psi \; ; \;\; e^{i\beta\gamma_5} \in U(1)_A \quad \text{axial group}$$

(infinitesimal: $\delta\psi = i\beta\gamma_5\psi, \delta\bar{\psi} = i\beta\bar{\psi}\gamma_5$; note the change of sign for $\delta\bar{\psi}$!). Since

$$\gamma_5 \; \psi_L = -\psi_L \; , \;\; \gamma_5 \; \psi_R = \psi_R$$

the chiral transformation for the fields ψ_L and ψ_R reads:

$$\begin{aligned}\psi_L \to e^{i\beta\gamma_5}\psi_L &= e^{-i\beta}\psi_L \\ \psi_R \to e^{i\beta\gamma_5}\psi_R &= e^{\;i\beta}\psi_R\end{aligned}$$

and hence the "chiral" fields transform with opposite chirality (handedness). The conserved Noether currents

$$j^\mu(x) = -\left\{\frac{\delta\bar{\psi}}{\delta\omega}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\frac{\delta\psi}{\delta\omega}\right\}$$

for the $U(1)_V \otimes U(1)_A$ symmetry group are the vector current $\left(\frac{\delta\psi}{\delta\omega} = i\psi\right)$[8]

$$j^\mu = \bar{\psi}\gamma^\mu\psi \; ; \;\; \partial_\mu j^\mu = 0$$

and the axial-vector current $\left(\frac{\delta\psi}{\delta\omega} = i\gamma_5\psi\right)$

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \; ; \;\; \partial_\mu j_5^\mu = 0 \; .$$

Both vectors $V^\mu = \bar{\psi}\gamma^\mu\psi$ and axial vectors $A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ are invariant under chiral transformations whereas scalars $S = \bar{\psi}\psi$, pseudoscalars $P = i\bar{\psi}\gamma_5\psi$ and tensors $T^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$ are not.

[7]Remember: γ_5 is Hermitian and anticommutes with all γ-matrices. Therefore $\Pi_\pm = (1 \pm \gamma_5)/2$ are Hermitian projection operators: $\Pi_+ + \Pi_- = 1$, $\Pi_\pm^2 = \Pi_\pm$, $\Pi_+\Pi_- = \Pi_-\Pi_+ = 0$. Since by definition $\psi_L = \Pi_-\psi$ and $\psi_R = \Pi_+\psi$ we have $\bar{\psi}_L = \psi_L^+\gamma^0 = \psi^+\Pi_-\gamma^0 = \psi^+\gamma^0\Pi_+ = \bar{\psi}\Pi_+$ and similarly $\bar{\psi}_R = \bar{\psi}\Pi_-$. Therefore $\bar{\psi}_L\gamma^\mu\psi_R = \bar{\psi}_R\gamma^\mu\psi_L = 0$ and $\bar{\psi}_L\psi_L = \bar{\psi}_R\psi_R = 0$.

[8]For massive fields and charged (non-diagonal) currents $j^\mu = \bar{\psi}_1\gamma^\mu\psi_2$ and $j_5^\mu = \bar{\psi}_1\gamma^\mu\gamma_5\psi_2$ the divergences of the currents can be easily calculated for free fields: By the Dirac equation $\gamma^\mu\partial_\mu\psi_i = -im_i\psi_i$, $\partial_\mu\bar{\psi}_i\gamma^\mu = im_i\bar{\psi}_i$ and hence

$$\partial_\mu j^\mu = \left(\partial_\mu\bar{\psi}_1\right)\gamma^\mu\psi_2 + \bar{\psi}_1\gamma^\mu\left(\partial_\mu\psi_2\right) = i(m_1 - m_2)\bar{\psi}_1\psi_2$$

and

$$\partial_\mu j_5^\mu = \left(\partial_\mu\bar{\psi}_1\right)\gamma^\mu\gamma_5\psi_2 + \bar{\psi}_1\gamma^\mu\gamma_5\left(\partial_\mu\psi_2\right) = i(m_1 + m_2)\bar{\psi}_1\gamma_5\psi_2 \; .$$

Thus j^μ is conserved only if $m_1 = m_2$, for j_5^μ to be conserved we must require $m_1 = m_2 = 0$.

The conserved "charges" which correspond to these conserved currents for *free fermions* are

$$\text{(a)} \quad Q = \int d^3x\, j^0(x^0, \vec{x}\,) = N_R + N_L \ , \quad \text{(b)} \quad Q_5 = \int d^3x\, j^0(x^0, \vec{x}\,) = N_R - N_L \ ,$$

where N_R is the number of right handed fields and N_L the number of left handed fields. Conservation of Q corresponds to fermion number conservation, conservation of Q_5 means that $N_R - N_L$ is preserved.

In reality fermions are interacting, in particular the charged one's interact electromagnetically. Suppose that ψ describes a massless charged Dirac particle which couples to photons. The massless QED Lagrangian

$$\mathcal{L} = \bar{\psi}\gamma^\mu \left(\partial_\mu - ieA_\mu\right)\psi$$

formally still looks chirally invariant. In fact, however, global chiral symmetry is broken now by the **axial-vector anomaly** *(Adler, Bell and Jackiw, 1969)* [1, 2]

$$\partial_\mu j_5^\mu = \frac{e^2}{8\pi^2}\tilde{F}_{\mu\nu}F^{\mu\nu} \neq 0 \tag{3.2}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field strength tensor and $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ its dual pseudotensor (parity odd). The pseudoscalar density is a divergence of a gauge dependent pseudovector [9]

$$\tilde{F}_{\mu\nu}F^{\mu\nu} = \partial^\mu K_\mu \ ; \ \ K_\mu = 2\epsilon_{\mu\rho\nu\sigma}A^\rho\partial^\nu A^\sigma. \tag{3.3}$$

This anomaly is a quantum effect which cannot be removed. In particular it cannot be compensated by adding a counterterm to the Lagrangian which would restore the symmetry. For QED the non-conservation of the j_5^μ current poses no problems because photons do not couple to an axial-vector current. For gauge theories for which gauge fields couple to axial-vector currents γ_5-anomalies are disastrous. If they are present they destroy renormalizability and unitarity. The problems are evident if we consider, for example, the Abelian subgroup $U(1)_Y$ of hypercharge Y of the electroweak standard model. The latter is not parity conserving and thus does not treat left-handed and right-handed fields in a democratic way. As we shall see in Sec. 3.1, the leptonic hypercharge current has the form

$$j_Y^\mu = \frac{1}{2}\left(\bar{\ell}_L\gamma^\mu\ell_L + \bar{\nu}_{\ell L}\gamma^\mu\nu_{\ell L}\right) + \bar{\ell}_R\gamma^\mu\ell_R$$

where ℓ denotes a lepton field and ν_ℓ its associated neutrino field. This current contributes to the action a term

$$\mathcal{A}_{int}^{(Y)} = g' \int d^4x\, j_Y^\mu(x)\, B_\mu(x) \ .$$

B_μ is the $U(1)_Y$ gauge field and g' the corresponding gauge coupling. Under a local gauge transformation

$$B_\mu(x) \to B'_\mu(x) = B_\mu(x) - \partial_\mu\omega(x)$$

the action changes by

$$\delta\mathcal{A}_{int}^{(Y)} = -g' \int d^4x\, j_Y^\mu(x)\, \partial_\mu\omega(x)$$

and after a partial integration

$$\delta\mathcal{A}_{int}^{(Y)} = -g' \int d^4x\, \left(\partial_\mu j_Y^\mu(x)\right)\, \omega(x)$$

we see that the action cannot be gauge invariant unless the current is conserved. In fact

$$\left(\partial_\mu j^\mu_Y(x)\right) \neq 0$$

has an anomaly as we shall see below. In any case, for a gauge theory in order to be renormalizable we must make sure that axial anomalies are absent. For QED and QCD, which are parity conserving, anomalies are absent because the gauge fields couple to vector currents only. The $SU(2)_L \otimes U(1)_Y$ electroweak theory is **not** anomaly save. It can be rendered renormalizable only by **anomaly cancellation** between leptons and quarks [3]. This leads to **lepton-quark duality**: the electroweak standard model is renormalizable only if fermions appear in lepton-quark families.

3.2.2 The chiral group $U(n)_V \otimes U(n)_A$

In general we have to consider currents associated with non-Abelian symmetries. What we said about Abelian models carries over to the non-Abelian ones. Of particular interest is the so called **chiral group**

$$\mathcal{G}_F = U(N_F)_V \otimes U(N_F)_A \simeq SU(N_F)_V \otimes SU(N_F)_A \otimes U(1)_V \otimes U(1)_A$$

with N_F the number of quark flavors. As we know the pions have odd inner parity which means that an effective pion field has the transformation properties of a $SU(2)_I$-triplet

$$\vec{\pi}(x) = \bar{\Psi} i\gamma^5 \frac{\vec{\tau}}{2} \Psi(x) \; ; \;\; \Psi \simeq \begin{pmatrix} p \\ n \end{pmatrix}$$

where Ψ is a $SU(2)_I$-doublet of fermions with the quantum numbers of the nucleons, as indicated. This shows that although QCD is parity conserving non-trivial chiral properties are crucial in the theory of strong interaction. QCD distinguishes between leptons (which do not participate in strong interactions) and colored triplets of quarks and antiquarks it does not distinguish between different flavors however. Strong interactions therefore exhibit a flavor symmetry. For free quarks known are the f=u, d, c, s, t and b (6 flavors) quarks which show up in three colors each c=red (r), green (g) and blue(b), we have a Lagrangian

$$\mathcal{L}^q = \sum_{f=1}^{6} \sum_{c=1}^{3} \left(\bar{\psi}_{cf} i \, \gamma^\mu \partial_\mu \psi_{cf} - m_f \bar{\psi}_{cf} \psi_{cf} \right)$$

To the extend that we can neglect the quark masses this Lagrangian has a global $U(18)_V \otimes U(18)_A$ symmetry. If strong interaction is switched on by the minimal substitution

$$\partial_\mu \to D_\mu = \partial_\mu \delta_{cc'} - i g_s \left(\frac{\lambda_i}{2} \right)_{cc'} G_{i\mu} \; ,$$

where the gauge fields $G_{i\mu}$ $(i = 1, \ldots, 8)$ are called gluons, g_s is the QCD coupling constant and the λ_i are the Gell-Mann matrices (see Sec. 1), the subgroup $SU(3)_c$ of color is promoted to a local symmetry. QCD requires the quark masses to be degenerate in color space. Again, provided we discard quark masses, there is a symmetry flavor symmetry

$$[\mathcal{G}_F, SU(3)_c] = 0$$

where $N_F = 6$. As we increase the number of flavors from $N_F = 2$ to 6, the above symmetry is broken more and more by increasingly heavy quark masses

quark flavor	u	d	s	c	b	t
mass (MeV)	$\sim$5	$\sim$9	190	1650	4750	172600

In practice at a given energy scale only those flavors are effective in a physical process which have masses lower than the given energy. Degrees of freedom with masses heavier than the given energy scale do not influence the which means that the flavor symmetry is preserved the better the lower the number of flavors is which participate in a given physical process.

Chiral symmetry plays a key role in the physics of the low lying hadrons composed of the light quarks u, d and s. In the isospin sector, the relevant Noether currents, conserved in the chiral limit, are given by

$$V_\mu^a = \sum_c \bar{\Psi}_c \gamma_\mu \frac{\tau^a}{2} \Psi_c \; ; \quad A_\mu^a = \sum_c \bar{\Psi}_c \gamma_\mu \gamma_5 \frac{\tau^a}{2} \Psi_c \tag{3.4}$$

where

$$\Psi_c = \begin{pmatrix} u_c \\ d_c \end{pmatrix}$$

is the light quark isospin doublet, c denotes color in the current has to be which is summed over [color singlet currents]. In the following color summation is always understood implicitly. The weak hadronic currents are not diagonal in flavor. Prototype is the neutron β–decay $n \to p + e^- + \bar{\nu}_e$ which in the hadronic part is mediated by a $d \to u$ quark transition. The currents thus are of the form

$$V_{12}^\mu = \bar{\psi}_1(x)\, \gamma^\mu \psi_2(x) \; ; \quad A_{12}^\mu = \bar{\psi}_1(x)\, \gamma^\mu \gamma_5 \psi_2(x) \, ,$$

where ψ_1 and ψ_2 are field of different masses m_1 and m_2, respectively. In this cases the Dirac equation yields

$$\partial_\mu V_{12}^\mu(x) = \mathrm{i}\,(m_1 - m_2)\, \bar{\psi}\psi(x) \; ; \quad \partial_\mu A_{12}^\mu(x) = \mathrm{i}\,(m_1 + m_2)\, \bar{\psi}\gamma_5\psi(x) \, ,$$

i.e. the vector currents are conserved only if masses are equal, the axial currents only if masses are zero, i.e. in the chiral limit. In particular in isospin $SU(2)$ where $m_u \sim 5$ MeV and $m_d \sim 7$ MeV the vector current is practically conserved [exact in the isospin limit $m_u = m_d$] which goes under the label "conserved vector current" (CVC). But due to the smallness of the light quark masses also the axial current is approximately conserved, which goes under the label "partially conserved axial vector current" (PCAC). If the axial isospin current would be conserved the charged pions could not decay. In fact the PCAC relation may be formulated in terms of the pion decay constant as follows.

For the chiral $SU(2)$ currents (3.4), in the In Cartesian notation (real pseudoscalar fields) we have the one pion matrix element of the divergence of the axial current

$$\langle 0|\partial^\mu A_\mu^a(0)|\pi^b(p)\rangle = F_\pi m_\pi^2 \delta^{ab}$$

and correspondingly for the pion field

$$\langle 0|\varphi^a(0)|\pi^b(p)\rangle = \delta^{ab} \; ,$$

such that

$$\langle 0|\partial^\mu A_\mu^a(0)|\pi^b(p)\rangle = F_\pi m_\pi^2 \, \langle 0|\varphi^a(0)|\pi^b(p)\rangle \; .$$

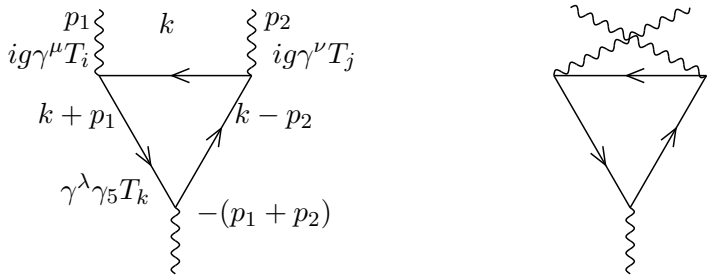


Figure 3.1: Triangle diagram exhibiting the axial anomaly

The PCAC relation requires this relation between matrixelements (which is nothing but the definition of F_π and that $\varphi^a(x)$ is an interpolating field for the corresponding pion state $|\pi^a(p)\rangle$) to hold as an operator relation:

$$\partial^\mu A^a_\mu(x) = F_\pi m_\pi^2 \, \varphi^a(x) \ .$$

The charged fields we obtain as

$$A^+_\mu = \left(A^1_\mu + \mathrm{i}\, A^2_\mu\right)/\sqrt{2}\,; \ \ \varphi^+ = \left(\varphi^1 + \mathrm{i}\,\varphi^2\right)/\sqrt{2} \ .$$

The hadronic weak charged axial currents thus reads

$$A^+_\mu = \sum_c (\bar{\psi}_{u_c}\gamma_\mu\gamma_5\psi_{d_c})/\sqrt{2} \ .$$

The corresponding axial nucleon current we denote $J^a_{5\mu} = \bar{\Psi}_N\gamma_\mu\gamma_5\frac{\tau^a}{2}\Psi_N$ and the charged axial nucleon current is given by

$$J^+_{5\mu} = (\bar{\psi}_p\gamma_\mu\gamma_5\psi_n)/\sqrt{2} \ .$$

3.2.3 The Adler Bell, Jackiw triangle anomaly

In the following we will discuss the general condition for anomaly freedom of a theory.

In perturbation theory the axial anomaly shows up in closed fermion loops with an odd number of axial-vector couplings if a non-vanishing γ_5-odd trace of γ-matrices like [9]

$$Tr\ (\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = 4i\epsilon^{\mu\nu\rho\sigma}$$

is involved and if the corresponding Feynman integral is not ultraviolet convergent such that it requires regularization. The simplest diagram exhibiting the axial anomaly is the triangle diagram Fig. 3.1 which leads to the amplitude (1st diagram)

$$\tilde{T}^{\mu\nu\lambda}_{ijk}(p_1,p_2) = (-1)\, i^5\ Tr\,(T_jT_iT_k)\frac{g^2}{(2\pi)^4}\int d^4k\ Tr\ \left(\frac{1}{\not{k}-\not{p}_2+i\epsilon}\gamma^\nu\frac{1}{\not{k}+i\epsilon}\gamma^\mu\frac{1}{\not{k}+\not{p}_1+i\epsilon}\gamma^\lambda\gamma_5\right) \quad (3.5)$$

If we include the Bose symmetric contribution (2^{nd} diagram)

$$T^{\mu\nu\lambda}_{ijk}(p_1,p_2) = \tilde{T}^{\mu\nu\lambda}_{ijk}(p_1,p_2) + \tilde{T}^{\nu\mu\lambda}_{jik}(p_2,p_1) \quad (3.6)$$

[9] Notice that $Tr\ \left(\prod_{i=1}^n \gamma^{\mu_i}\gamma_5\right) = 0$ for $n < 4$ and for all $n =$ odd.

and **impose** vector current conservation

$$p_{1\mu}T^{\mu\nu\lambda}_{ijk}(p_1,p_2) = p_{2\nu}T^{\mu\nu\lambda}_{ijk}(p_1,p_2) = 0 \tag{3.7}$$

we obtain the unambiguous regularization independent result

$$-(p_1+p_2)_\lambda\, T^{\mu\nu\lambda}_{ijk}(p_1,p_2) = i\,\frac{g^2}{16\pi^2} D_{ijk}\; 4\,\epsilon^{\mu\nu\rho\sigma} p_{1\rho} p_{2\sigma} \neq 0 \tag{3.8}$$

with $D_{ijk} = Tr\ (\{T_i,T_j\}\,T_k)$.

This result is independent on the masses of the fermion lines and is not changed by higher order corrections. Therefore the result is **exact** beyond perturbation theory! *(Adler and Bardeen, 1969)* [4].

We may represent the result as an operator identity in configuration space: $p_{1\rho}(p_{2\sigma})$ corresponds to a derivative $i\partial_\rho(i\partial_\sigma)$ acting on a gauge field (external leg) $V_{i\mu}(x)\,(V_{j\nu}(x))$ while $-(p_1+p_2)_\lambda$ corresponds to the vertex $i\partial_\lambda j^\lambda_5(x)$. Because of the permutation symmetry in the two gauge fields we have to divide by a factor 2. We then obtain:

$$\partial_\lambda j^\lambda_{5k}(x) = -\frac{g^2}{32\pi^2} D_{ijk}\; 4\,\epsilon^{\mu\nu\rho\sigma}\,(\partial_\rho V_{i\mu}(x))\,(\partial_\sigma V_{j\nu}(x))\,.$$

Using the antisymmetry of the ϵ-tensor and renaming summation indices we may rewrite this result

$$\begin{aligned}\partial_\lambda j^\lambda_{5k}(x) &= -\frac{g^2}{32\pi^2} D_{ijk}\;\epsilon^{\mu\nu\rho\sigma}\,(\partial_\rho V_{i\mu} - \partial_\mu V_{i\rho})\,(\partial_\sigma V_{j\nu} - \partial_\nu V_{j\sigma}) \\ &= \frac{g^2}{16\pi^2} D_{ijk}\;\overset{\circ}{\tilde{G}}{}^{\mu\nu}_i \overset{\circ}{G}_{j\mu\nu}\end{aligned}$$

with $\overset{\circ}{G}_{j\mu\nu} = \partial_\mu V_{j\nu} - \partial_\nu V_{j\mu}$ and $\overset{\circ}{\tilde{G}}{}^{\mu\nu}_i = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\ \overset{\circ}{G}_{i\rho\sigma}$. The expression for $-(p_1+p_2)_\lambda\, T^{\mu\nu\lambda}_{ijk}(p_1,p_2)$ is a matrix element of $i\partial_\lambda j^\lambda_5(0)$. If terms from other diagrams contributing to other possible matrix-elements of the axial current are included one finds for the final form of the anomaly

$$\partial_\lambda j^\lambda_{5k}(x) = \frac{g^2}{16\pi^2} D_{ijk}\;\tilde{G}^{\mu\nu}_i(x) G_{j\mu\nu}(x) \tag{3.9}$$

where $G_{i\mu\nu}(x)$ is the non-Abelian field strength tensor and $\tilde{G}^{\mu\nu}_i$ its dual pseudotensor. As a result the condition for the absence of an anomaly reads

$$D_{ijk} = Tr\ (\{T_i,T_j\}\,T_k) = 0\ \ \forall\ \ (ijk)\ . \tag{3.10}$$

The matrices T_i are the generators of a gauge group in a representation R under which the fermions transform. What are the general conditions for the absence of anomalies? To answer this question we need the following basic properties of traces:

i) trace of the transpose A^T of a matrix A:

$$Tr\,(A) = \sum_i A_{ii} = Tr\,(A^T)$$

ii) trace of the equivalent $A' = SAS^{-1}$ of a matrix A

$$Tr\,(SAS^{-1}) = Tr\,(S^{-1}SA) = Tr\,(A)$$

since $Tr\,(AB) = Tr\,(BA)$. S must be nonsingular.

We have to distinguish two different types of representations, real and complex ones. A representation R is called **real** if its **complex conjugate** representation R^* is equivalent to $R : R^* \simeq R$ i.e. $R^* = SRS^{-1}$ for some fixed non-singular matrix S. Non-real representation $R \not\simeq R^*$ are called **complex**. The following general statements hold:

A) Real representations are anomaly free: $D_{ijk} \equiv 0$. Since $R^* \simeq R$ we have $D_{ijk}(R^*) = D_{ijk}(R)$. On the other hand in the canonical basis $T_i^+ = T_i$, if T_i are the generator of R then $-T_i^* = -T_i^T$ are the generators of R^*. Therefore

$$D_{ijk}(R^*) = -D_{ijk}(R).$$

Consequently $D_{ijk}(R) \equiv 0$ for real representation. The groups $SO(2\ell{+}1), \ell > 1, Sp(2\ell), G_2, F_4, E_7$ and E_8 have only real representations and hence are anomaly free. The groups $SO(2\ell), \ell > 1$, except $SO(6) \simeq SU(4)$ are also anomaly free.

B) Groups for which the fundamental representation is real are anomaly free.

One can show that for any representation R

$$D_{ijk}(R) = D_{ijk}(R_0) \cdot K(R)$$

where R_0 is the fundamental representation. Only the invariant quantity $K(R)$ depends on the representation, $K(R_0) = 1$.

For the groups $SU(2) \simeq SO(3)$ and E_6 one has $D_{ijk}(R_0) = 0$; and thus all representations of these groups are anomaly free.

C) The groups $SU(n), n \geq 3$ have complex representations and $D_{ijk}(R_0) \neq 0$. Fermions transforming under representations of these groups in general lead to anomalies. In order to avoid anomalies in this case one has to find those representations R for which $K(R) = 0$ i.e. for the $SU(n)$, $n \geq 3$, groups one can avoid anomalies only by putting the fermions into particular representations. One can easily find these representations. We always can write the fermion currents j_i^μ which couple to the gauge fields in terms of left-handed and right-handed fields

$$\begin{aligned} j_i^\mu &= \bar{\psi}_L \gamma^\mu T_{Li} \psi_L + \bar{\psi}_R \gamma^\mu T_{Ri} \psi_R \\ &= \bar{\psi} \gamma^\mu \frac{1-\gamma_5}{2} T_{Li} \psi + \bar{\psi} \gamma^\mu \frac{1+\gamma_5}{2} T_{Ri} \psi. \end{aligned}$$

Since $\bar{\psi}_L \gamma_\mu \psi_R = \bar{\psi}_R \gamma_\mu \psi_L = \bar{\psi}_L \gamma_\mu \gamma_5 \psi_R = \bar{\psi}_R \gamma_\mu \gamma_5 \psi_L = 0$ the contributions from the left-handed and right-handed fields in closed fermion loops decouple. The contribution to the anomaly thus is given by the sum of the contributions from left-handed fields and from right-handed fields. If a particular left-handed loop gives an anomaly proportional to $D_{ijk}(R_L)$ then the right-handed loop gives an anomaly contribution proportional to $-D_{ijk}(R_R)$ because the two contributions differ by a sign at the $\gamma^\mu\gamma_5$ vertex as $\gamma_5 \frac{1\pm\gamma_5}{2} = \pm\frac{1\pm\gamma_5}{2}$. Thus the condition that no anomalies arise from gauge interactions is that for all i, j, k :

$$Tr\,(\{T_{Li}, T_{Lj}\}\, T_{Lk}) - Tr\,(\{T_{Ri}, T_{Rj}\}\, T_{Rk}) = 0\,. \tag{3.11}$$

For $SU(2)$ the terms are individually zero for any representation. For $SU(3)$ there is no anomaly if the ψ_L and ψ_R transform under the same representation. This is the case for QCD where due to $T_{Ri} = T_{Li}$ no axial currents couple to the gluons. Since the left-handed antiquarks are in the complex conjugate representation 3^* of the fundamental representation 3 the corresponding right-handed particle fields also transform under 3: This follows, because under antiparticle conjugation

$$\psi_L \stackrel{C}{\to} \psi_L^c = i\gamma^2 \psi_R^* \; .$$

D) The Abelian group $U(1) \simeq SO(2)$ are not anomaly save. The previous argument for $SU(n)$ groups for the $U(1)$ group leads to

$$D = Tr\, T_{L0}^3 - Tr\, T_{R0}^3 = 0$$

as a condition for anomaly cancellation. Here T_0 denotes the Abelian generator of hyper charge.

For non-simple groups the generators of the factor groups commute with each other. Therefore, the anomaly is given by the sum of the anomalies of the individual subgroups. This means that anomalies must be absent for each factor group.

For the standard model only the $U(1)_Y$ yields a non trivial condition which must be satisfied in order to have an anomaly free theory! Let us check this condition now.

The matter field are in

$$\text{doublets} \quad \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)_L \quad \text{and singlets} \quad \psi_{R1}, \psi_{R2} \; .$$

The hypercharge and charge assignments satisfy:

$$Y_i = 2(Q_i - T_{3i}) \; ; \;\; Q_{Ri} = Q_{Li} = Q_i \; ; \;\; Q_1 - Q_2 = 1$$

such that we have

$$\begin{aligned} Y_{L1} &= 2(Q_1 - \frac{1}{2}) = 2Q_1 - 1 \; , \;\; Y_{R1} = 2Q_1 \\ Y_{L2} &= 2(Q_2 + \frac{1}{2}) = 2Q_1 - 1 \; , \;\; Y_{R2} = 2Q_2 = 2Q_1 - 2 \; . \end{aligned}$$

Thus the anomaly contribution per doublet is given by

$$D = Y_{L1}^3 + Y_{L2}^3 - Y_{R1}^3 - Y_{R2}^3 = 2\,(2Q_1 - 1)^3 - (2Q_1)^3 - (2Q_1 - 2)^3 = -12\,Q_1 + 6 \; .$$

Consequently, the $SU(2)_L \otimes U(1)_Y$ electroweak model with leptons only is **not** renormalizable (not anomaly free). The anomaly must be canceled by a contribution of opposite size coming from the quarks!

For the anomaly factor D we get for a lepton doublet and $N_c = 3$ colored quark doublets and the associated singlets

$$D_{\text{leptons}} + D_{\text{quarks}} = 6 - 6N_c\,(2Q_1^q - 1) = 0$$

and since

$$Q_1^q = \frac{1}{2}\left(1 + \frac{1}{N_c}\right) = 2/3 \quad \text{for} \quad N_c = 3 \ .$$

Notice that one could satisfy the anomaly condition also with the nucleon doublet: $N_c = 1$, $Q_1 = 1$ i.e.

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} p \\ n \end{pmatrix} .$$

This is an example of 't Hooft's anomaly condition: A composite particle must reproduce the axial anomaly of its fermionic constituents [11]. This may be understood as a consequence of the Adler-Bardeen theorem which states that the anomaly is not renormalized by higher order effects and hence that the axial anomaly is a known non-perturbation effect.

As a result we find: The electroweak standard model is renormalizable only if fermions are grouped into lepton-quark families.

Exercises: Section 3.2

① In the SM the leptonic hypercharge current has the form

$$j_Y^\mu = \frac{1}{2}\left(\bar{\ell}_L \gamma^\mu \ell_L + \bar{\nu}_{\ell L}\gamma^\mu \nu_{\ell L}\right) + \bar{\ell}_R \gamma^\mu \ell_R \ .$$

Write it in terms of vector plus axialvector contributions.

② Calculate the trace and the Feynman integral (3.5) and verify (3.8).

3.3 Yukawa interaction of quarks and quark flavor mixing

So far we have considered the Higgs–fermion couplings for each quark–lepton family separately. The fact that there exist three families of fermions which are made up of fields with identical $SU(2)_L \otimes U(1)_Y$ transformation properties allows us to form invariant Yukawa couplings for arbitrary combinations of fields from the different families. Such **flavor mixing** is known to occur in the quark sector while for leptons all searches for family mixing have been negative so far.

For the quarks we have four **horizontal vectors** in "family space" with identical quantum numbers with respect to the local gauge group. These are the left–handed and right–handed versions of the up and down family vectors $u_i = (u, c, t)$ and $d_i = (d, s, b)$ and we denote them by u_{iL}, d_{iL}, u_{iR}, d_{iR} . The general form of the Yukawa term reads

$$\mathcal{L}^q_{\text{Yukawa}} \ = \ -\sum_{i,j=1}^{3} \left[G^{ij}_{tq} \bar{L}_{qi} \Phi_t u_{jR} + G^{ij}_{bq} \bar{L}_{qi} \Phi_b d_{jR} + \text{h.c.} \right]$$

with G^{ij}_{tq} and G^{ij}_{bq} arbitrary complex 3 x 3 matrices. The mass–matrix we obtain by inserting $\Phi_b = \frac{v}{\sqrt{2}}\binom{0}{1}$ and $\Phi_t = \frac{v}{\sqrt{2}}\binom{1}{0}$. Thus

$$\mathcal{L}^q_{\text{mass}} \ = \ -\frac{v}{\sqrt{2}} \sum_{i,j=1}^{3} \left[G^{ij}_{tq} \bar{u}_{iL} u_{jR} + G^{ij}_{bq} \bar{d}_{iL} d_{jR} + \text{h.c.} \right]$$

$$= \; -\sum_{i,j=1}^{3} \left[m_{tq}^{ij} \bar{u}_{iL} u_{jR} + m_{bq}^{ij} \bar{d}_{iL} d_{jR} + \text{h.c.} \right]$$

with

$$m_{\cdot}^{ij} = \frac{v}{\sqrt{2}} G_{\cdot}^{ij} \quad .$$

The quark fields considered up to now are in the **weak interaction basis** in which the matter field Lagrangian is diagonal in the families. We now have to find the **physical fields** in the **mass eigenstate basis** in which the mass–matrix is diagonal such that each field has a fixed mass[10]. To this end we have to perform global unitary transformation on the horizontal vectors. Unitary because they have to leave the kinetic terms of the quarks invariant and global because we have to diagonalize the constant coupling matrices $G_{\cdot\, q}$.

We first perform the transformations

$$\begin{aligned} u_{jR} &\rightarrow (V_{uR})_{jk}\, u_{kR} \\ d_{jR} &\rightarrow (V_{dR})_{jk}\, d_{kR} \\ \begin{pmatrix} u \\ d \end{pmatrix}_{jL} &\rightarrow (V_L)_{jk} \begin{pmatrix} u \\ d \end{pmatrix}_{kL} \end{aligned}$$

of the singlets and the doublet. Since the doublet fields are transformed as a doublet these unitary transformations do not change the matter field Lagrangian which exhibits terms of the form $\bar{L}_{qi} \cdots L_{qi}$, $\bar{u}_{iR} \cdots u_{iR}$ and $\bar{d}_{iR} \cdots d_{iR}$! For the mass matrices we obtain

$$\begin{aligned} m_{tq} &\rightarrow V_L^{+}\, m_{tq}\, V_{uR} \quad \text{real diagonal} \\ m_{bq} &\rightarrow V_L^{+}\, m_{bq}\, V_{dR} \end{aligned}$$

and the transformed matrices are required to be **real diagonal** each with 3 free parameters (the 3 independent masses) left out of the 18 we started with. A unitary 3 x 3 matrix has 9 parameters and with the two matrices V_{uR} and V_L we have enough parameters to satisfy the 15 conditions to make G_{tq} real diagonal. However, the remaining 3 parameters not used up from V_L plus the 9 of V_{dR} are not sufficient to diagonalize G_{bq}. We thus have to transform the (by convention) lower components of the doublets by an independent unitary transformation:

$$d_{jL} \;\rightarrow\; \tilde{d}_{jL} = (U_{\text{CKM}})_{jk}\, d_{kL}$$

[10]Notice that the mass-matrices m_{bq} and m_{tq} are not Hermitian in general. We make use of the fact that any square matrix can be diagonalized with the help of two unitary transformations. In our case we thus need four matrices such that

$$\begin{aligned} V_{uL}^{+} m_{tq} V_{uR} &= D_{tq} = diag\,(m_u, m_c, m_t) \\ V_{dL}^{+} m_{bq} V_{dR} &= D_{bq} = diag\,(m_d, m_s, m_b) \end{aligned}$$

The matrices may be determined if we multiply the equations with its Hermitian conjugate, for example,

$$V_{uL}^{+} m_{tq} V_{uR} V_{uR}^{+} m_{tq}^{+} V_{uL} = V_{uL}^{+} m_{tq} m_{tq}^{+} V_{uL} = D_{tq}^2 = diag\,(m_u^2, m_c^2, m_t^2)$$

and we see that V_{uL} is the matrix which diagonalizes the Hermitian matrix $m_{tq} m_{tq}^{+}$. This reduces our problem to a standard eigenvalue problem for Hermitian operators.

This however changes the form of $\mathcal{L}_{\rm matter}$ and generates a family–mixing in the charged current. This leads us to the form of the charged current

$$J_\mu^{CC} = (\bar{u}, \bar{c}, \bar{t})\, \gamma_\mu\, (1-\gamma_5)\, U_{\rm CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

given earlier with the unitary 3×3 matrix

$$U_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} .$$

Part of the parameters are used up to diagonalize

$$m_{bq} \;\; \rightarrow \;\; U_{\rm CKM}^+\, V_L^+\, m_{bq}\, V_{dR} \quad \text{real diagonal} \;\; .$$

After diagonalization we always assume the fields to be relabeled such that the fields with identical quantum numbers are ordered with increasing mass:

$$m_u \leq m_c \leq m_t \;\; \text{and} \;\; m_d \leq m_s \leq m_b \;\; .$$

How many observable parameters are there left ? The phase of a charged field is not observable. In the charged current only left–handed fields are present. If we change the phases of the left–handed fields

$$q_L \rightarrow e^{i\phi_q} q_L \;\; , \;\; \phi_q \quad \text{an arbitrary real number}$$

the CKM-matrix changes according to

$$V \rightarrow \begin{pmatrix} e^{-i\phi_u} & 0 & 0 \\ 0 & e^{-i\phi_c} & 0 \\ 0 & 0 & e^{-i\phi_t} \end{pmatrix} \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} e^{i\phi_d} & 0 & 0 \\ 0 & e^{i\phi_s} & 0 \\ 0 & 0 & e^{i\phi_b} \end{pmatrix}$$

and thus

$$V_{u_i d_j} \rightarrow \exp\, -i\, (\phi_{u_i} - \phi_{d_j})\, V_{u_i d_j} \;\; .$$

This generalizes to any number N of families. If all of the 2N fields are transformed with the same phase $V_{u_i d_j}$ is not affected and hence there are 2N-1 phases which may be transformed away by rephasing the left–handed fields. This means that 2N-1 phases of $U_{\rm CKM}$ are not measurable. This is only true if the remaining parts of the Lagrangian are not affected. The neutral current is not affected by this rephasing because it is diagonal in flavor and handedness. The Yukawa Lagrangian obviously is not invariant under left–handed rephasing because it necessarily connects left–handed and right–handed fields. However, we have the phases of the right–handed fields at our disposal. We may choose these phases to be the same as the one's of the left–handed fields (for each individual flavor) such that $\mathcal{L}_{\rm Yukawa}$ remains invariant.

We may count now the number of free parameters which affect the physics. Let us consider N families. A unitary N × N matrix has N^2 parameters. We may compare it with an orthogonal N × N matrix, describing a rotation in N dimensional Euclidean space and exhibiting N(N-1)/2

parameters, which may be taken to be the Euler angles. We may parametrize the unitary matrix by the N(N-1)/2 Euler angles, describing the rotations, plus N^2-N(N-1)/2 phases. As we have argued before 2N-1 of these phases are unobservable. Thus we end up with (N-1)(N-2)/2 physical phases:

number of families N	number of angles N(N-1)/2	number of phases (N-1)(N-2)/2
2	1	0
3	3	1
4	6	3

For N=3 we have 4 relevant parameters. Besides 3 (real) 3-dimensional rotations there remains one phase undetermined. Thus $U_{\rm CKM}$ may be parametrized in terms of 3 rotation angles and one phase[11]:

$$U_{\rm CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} .$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ with i and j being the family labels. Without loss of generality one may assume all c_{ij} and s_{ij} to be positive and the phase δ_{13} to lie in the range $0 \leq \delta_{13} < 2\pi$.

A non-degenerate phase

$$\delta_{13} \neq 0 \ , \ \ \pi$$

leads to complex effective couplings in the charge changing current which violate *CP*-invariance. It is important to notice that this specific kind of "standard model *CP*-violation" is only possible for more than two families. In fact, the mixing matrix has to satisfy a number of conditions in order that CP-violation occurs. The basic observation is that a unitary 3 by 3 matrix which has a zero matrix element somewhere is necessarily real. Therefore mixing must be non-degenerate:

$$\theta_{ij} \neq 0 \ , \ \ \frac{\pi}{2} \quad ij = 12, 23, 13 \ \ .$$

For the same reason, *CP*-violation can only occur this way if all the states are distinguishable. This means that **all states with the same charge must have different masses**:

$$m_u < m_c < m_t \quad \text{and} \quad m_d < m_s < m_b \ \ .$$

which happens to be so in Nature. Otherwise, suppose the s and the b quarks would have the same mass, for example, then the Lagrangian would be invariant under $U(2)$ in (s,b)-flavor space. If we perform the $U(2)$ rotation

$$\begin{pmatrix} s' \\ b' \end{pmatrix} = \frac{1}{X} \begin{pmatrix} V_{us} & V_{ub} \\ |V_{ub}|e^{i\delta'} & |V_{us}|e^{i\delta} \end{pmatrix} \begin{pmatrix} s \\ b \end{pmatrix}$$

[11] There are many equivalent parametrizations, here we present the one advocated by the Particle Data Group.

with $X = \sqrt{|V_{us}|^2 + |V_{ub}|^2}$ and the phases constraint by unitarity, $\delta' = \delta + \delta_{us} - \delta_{ub} + \pi$ we obtain

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \begin{pmatrix} V_{ud} & X & 0 \\ V_{cd} & V'_{cs} & V'_{cb} \\ V_{td} & V'_{ts} & V'_{tb} \end{pmatrix} \begin{pmatrix} d \\ s' \\ b' \end{pmatrix}$$

the u-quark only couples to d and s' but no longer to the b' such that the CKM-matrix has a zero element and hence would be real.

Due to unitarity, there is no mixing effect in the neutral current. In the weak interaction basis the neutral current is diagonal in helicities, flavors and families and the unitary CKM-transformation

$$\sum_{i=1}^{3} \bar{d}_{iL} \cdots d_{iL} \;\rightarrow\; \sum_{i=1}^{3} \bar{\tilde{d}}_{iL} \cdots \tilde{d}_{iL} \equiv \sum_{i=1}^{3} \bar{d}_{iL} \cdots d_{iL}$$

leaves its form invariant. This is called the GIM-mechanism explaining the absence of flavor-changing neutral currents (FCNC). In fact, in order to explain the absence of FCNC's, Glashow, Iliopoulos and Maiani had to propose, in 1970, the existence of a fourth quark, the charm quark c as a doublet partner of the s quark [25]. At that time only three quarks where known.

The discovery of the charmonium ($c\bar{c}$) state J/ψ in 1974 revealed the completeness of the 2nd family with the charm quark c. The first 3rd family member showed up in 1975 with the discovery of the τ . With the observation of the bottomonium ($b\bar{b}$) state Υ the existence of the b quark could be established. First indications of an unexpectedly heavy top quark came from the first observation of $B^0 \leftrightarrow \overline{B^0}$ oscillations by the ARGUS Collaboration in 1987 [12]. LEP precision measurements together with SM fits in 1995 had constrained the top mass to

$$m_t = 170 \pm 10^{+17}_{-19} \text{ GeV}$$

assuming a Higgs mass in the range 60 GeV $< m_H <$ 1000 GeV. Shortly after in 1995 the top was discovered at the Tevatron (CDF and D0 at Fermilab) by $t\bar{t}$–production [13]. More recent top mass measurement at the Tevatron [14] determined the rather precise result

$$m_t = 172.6 \pm 1.4 \text{ GeV}\,, \tag{3.12}$$

in excellent agreement with the final indirect determination $m_t = 172.3^{+10.2}_{-7.6}$ GeV from the LEP precision measurements of Z resonance parameters [15].

We summarize the following **important consequences:**

- i) all masses of quarks, leptons and neutrinos are independent
- ii) the coupling of the Higgs boson to the fermions is universally proportional to each fermion mass, for bosons proportional to the square of each boson mass
- iii) there is quark flavor violation in charge exchange weak interactions, similarly, neutrinos exhibit flavor mixing in the charged currents
- iv) the phase in U_{CKM} is CP-violating and in fact capable of explaining the observed CP-violation in K-decays[12] (Cronin and Fitch 1964). At least 3 families are needed to "explain" CP-violation in this way.

[12]The particle-antiparticle mixing of the neutral kaons $K^0 \leftrightarrow \bar{K}^0$ (Gell-Mann und Pais 1955) played a key role in revealing CP violation as an observable effect. In the B-meson system $B^0 \leftrightarrow \bar{B}^0$ mixing plays an analogous role.

- v) flavor is conserved in neutral currents (GIM mechanism). This is strikingly supported by experiment, at least for the light quark flavors.

Note on the actual values of fermion masses:
Leptons masses are

$$m_e = 0.510998902 \pm 0.000000021\text{MeV} \; ; \quad m_\mu = 105.6583568 \pm 0.0000052\text{MeV} \; ; \quad m_\tau = 1776.99^{+0.29}_{-0.26}\text{MeV}(3.13$$

Current quark masses for the light quarks evaluated in the $\overline{\text{MS}}$ scheme at the scale $\mu = 2$ GeV are:

$$m_u \approx 1.5 - 4.5\text{MeV} \; ; \quad m_d \approx 5.0 - 8.5\text{MeV} \; ; \quad m_s \approx 80 - 155\text{MeV} \; . \tag{3.14}$$

The c-quark mass is estimated from charmonium and D masses. The "running" mass in the $\overline{\text{MS}}$ scheme is $m_c(\mu = m_c) = 1.290^{+0.040}_{-0.045}$. The range $1.0 - 1.4$ GeV for the $\overline{\text{MS}}$ mass corresponds to $1.47 - 1.83$ GeV for the pole mass [converted by two-loop perturbative QCD with $\alpha_s(\mu = m_c) = 0.39$.]. The b-quark mass is estimated from bottomonium and B masses. The "running" mass in the $\overline{\text{MS}}$ scheme is $m_b(\mu = m_b) = 4.206 \pm 0.031$. The range $4.0 - 4.5$ GeV for the $\overline{\text{MS}}$ mass corresponds to $4.6 - 5.1$ GeV for the pole mass [converted by two-loop perturbative QCD with $\alpha_s(\mu = m_b) = 0.22$.]. The top mass given above is the pole mass.

Empirically the CKM matrix elements may be expanded in $\lambda = \sin\theta_{\text{Cabibbo}} \simeq 0.22$ with the following approximate sizes of the elements

$$|V| \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \; .$$

The leptonic CC has some very special properties, which derive from the absence of right-handed neutrinos in them. Among the unsolved neutrino-puzzles we mention: Why are neutrino masses so small (see saw mechanism?)? Do neutrinos have unusual magnetic moments? Are there neutrinos which are their own antiparticles (Majorana neutrinos)?

The properties of the weak currents have been established in a long history which started with Fermi in 1934. Here, we only mention some more recent of the fundamental experimental tests:

- V-A structure of the CC:
  μ-decay provides the most sensitive clean direct tests for right-handed currents (e.g. $SU(2)_R \otimes SU(2)_L \otimes U(1)_{B-L}$ extension of the SM). The best limit for the transition amplitude is

$$\frac{A_{V+A}}{A_{V-A}} < 0.029 \quad (90\%CL)$$

- absence of flavor-changing NC at tree level:

$$\begin{aligned} \Gamma(K_L \to \mu^+\mu^-)/\Gamma(K_L \to all) &= (9.5^{+2.4}_{-1.5}) \times 10^{-9} \\ \Gamma(D^0 \to \mu^+\mu^-)/\Gamma(D^0 \to all) &< 1.1 \times 10^{-5} \\ \Gamma(B^0 \to e^+e^-)/\Gamma(B^0 \to all) &< 3 \times 10^{-5} \end{aligned}$$

  Flavor-changing NC processes are allowed in higher orders (rare processes).

For comment about the status of lepton number conservation and direct bounds on neutrino masses we refer to Sect. 1.6.

Neutrino mixing (ν-oscillations $\nu_\ell \leftrightarrow \nu_{\ell'}$) turned out to be very hard to establish. Ater decades of intense research finally neutrino oscillations were confirmed [18–22], which means that neutrinos have tiny non–degenerate masses and mix [24] similar to the quarks, although the mixing pattern looks very different from quark-mixing[13]. This also requires the right–handed singlet neutrinos to exist, which, however, are sterile with respect to any gauge interaction. In 2002 R. Davis Jr. and M. Koshiba awarded the Nobel Prize for pioneering contributions to astrophysics, in particular for the detection of cosmic neutrinos.

At about the same time, when the neutrino puzzle was resolved, the measurements of direct CP-violation (ε') in the K-meson system [CP-LEAR[NA48]/CERN (2002), kTEV/FNL (2008)] and CP-violation in the B-meson system [B factories BaBar/SLAC and Belle/KEK (2001)] was experimentally proven to follow excatly the pattern of the Kobayashi-Maskawa 3–family quark mixing and the resulting CP violation mechanism [23]. The experimental result is $\sin(2\phi_1) = 0.78 \pm 0.08$ where $\phi_1 = \arg(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*})$ [17]. M. Kobayashi and T. Maskawa were awarded the Nobel Prize in 2008, for their prediction (made in 1973) of $O(1)$ (in units of Cabibbo's $\lambda = \sin\theta_c$) CP-violaton in the B–meson system.

3.3.1 Flavor mixing pattern

Historically flavor mixing was "observed" first by a comparison of the decays $K \to \mu\nu$ and $\pi \to \mu\nu$. Only u, d and s flavors (isospin and strangeness) were known at that time and the approximate $SU(3)_{flavor}$ symmetry was established. This symmetry is substantially broken by mass splittings within $SU(3)$ multiplets like for the pseudoscalar mesons with $m_K \simeq 494$ MeV and $m_\pi \simeq 140$ MeV. Hadronic transition matrix-elements however satisfy $SU(3)$ relations quite well. Denoting the matrix element between the pseudoscalar meson P and the weak hadronic current $h^\mu(x)$ by

$$< 0|h^\mu(0)|P(p) >= ip^\mu f_P$$

one obtains for the ratio of the decay widths

$$\frac{\Gamma(K \to \mu\nu)}{\Gamma(\pi \to \mu\nu)} = \frac{m_K}{m_\pi}\left(\frac{1 - m_\mu^2/m_K^2}{1 - m_\mu^2/m_\pi^2}\right)^2 \left(\frac{f_K}{f_\pi}\right)^2 \simeq 1.3$$

and thus

$$\left(\frac{f_K}{f_\pi}\right)^2 \simeq 0.075$$

and not $O(1)$ as suggested by approximate $SU(3)$ symmetry! Cabibbo solved this puzzle by noting that the strangeness conserving $\Delta S = 0$ part and the strangeness changing $\Delta S = 1$ part

[13] Present results may be summarized as follows: **a)** Solar neutrinos: $\Delta m_{12}^2 \approx (7)\times 10^{-5}$ eV^2, $\tan^2\Theta_{12} \approx 0.4$ (large $\nu_e \leftrightarrow \nu_\mu$ mixing), $\sin^2 2\Theta_{13} < 0.067$. **b)** Atmospheric neutrinos: $\Delta m_{23}^2 \approx (1.3 - 3.0) \times 10^{-3}$ eV^2, $\sin^2 2\Theta_{23} > 0.9$ (large angle $\nu_\mu \leftrightarrow \nu_\tau$ mixing). Main features are:
• smallness of ν masses: $m_\nu < 1 - 2$ eV, at least for one mass $m_\nu > \sqrt{\Delta m_{23}^2} > 0.04$ eV,
• hierarchy of Δm^2's : $|\Delta m_{12}^2/\Delta m_{23}^2| = 0.01 - 0.15$,
• no strong hierarchy of masses: $|m_2/m_3| > |\Delta m_{12}/\Delta m_{23}| = 0.18^{+0.22}_{-0.08}$,
• bi–large or maximal mixing between neighboring families (1-2) and (2-3),
• small mixing between remote families (1-3),
in any case $m_\nu \ll m_\ell, m_q$.

of the hadronic current mix in a specific way, described by a rotation:

$$h^\mu = h^\mu_{\Delta S=0} \cos\theta_c + h^\mu_{\Delta S=1} \sin\theta_c$$

such that the effective couplings for the two processes are

$$\begin{aligned} K \to \mu\nu \quad &: \quad G_F \sin\theta_c \\ \pi \to \mu\nu \quad &: \quad G_F \cos\theta_c \end{aligned}$$

with $\sin\theta_c \simeq 0.22$ and thus

$$\left(\frac{f_K}{f_\pi}\right)^2 \to \tan^2\theta_c \left(\frac{f_K}{f_\pi}\right)^2 \simeq 0.0795 \left(\frac{f_K}{f_\pi}\right)^2$$

such that the $SU(3)$ relation $f_K = f_\pi$ is satisfied quite well. Of course if one uses the above ratio to fix the Cabibbo angle one has to consider other processes in order to see whether the above hypothesis makes sense or not. In fact it turned out to be a very successful ansatz, as it turned out to be universal (Cabibbo universality). All weak processes in the $SU(3)$ sector are described by one coupling G_F up to a rotation: $G_\mu = G_F, G_{\pi,n} = G_F \cos\theta_c$ and $G_{K,\Lambda} = G_F \sin\theta_c$ where G_μ, G_n and G_Λ are the effective couplings in μ–decay, neutron β–decay and Λ–decay, respectively.

As a next step Glashow, Iliopoulos and Maiani [25] introduces the c quark in order to explain the absence of FCNC's and thus for the first time considered a 2 family world:

$$\mathcal{L}^{CC} = \frac{g}{2\sqrt{2}} (\bar{u}, \bar{c})\, \gamma_\mu\, (1-\gamma_5)\, U_{\rm CKM} \begin{pmatrix} d \\ s \end{pmatrix} W^\mu$$

with the unitary 2×2 matrix

$$U = \begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix} .$$

For N=2 the quark mixing matrix is automatically real and given by a simple rotation, the Cabibbo rotation matrix

$$\begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$$

In the 2 family world the hadronic currents are:

$$\begin{aligned} \text{CC:} \quad J^+_\mu &= \bar{u}\gamma_\mu\,(1-\gamma_5)\,\underbrace{(\,d\cos\theta_c + s\sin\theta_c)}_{\tilde{d}} && \text{Cabibbo} \\ &+ \bar{c}\gamma_\mu\,(1-\gamma_5)\,\underbrace{(-d\sin\theta_c + s\cos\theta_c)}_{\tilde{s}} && \text{GIM piece} \\ \text{NC:} \quad J^Z_\mu &= \bar{u}\gamma_\mu\,(v_u - a_u\gamma_5)\,u + \underbrace{\bar{\tilde{d}}\gamma_\mu\,(v_d - a_d\gamma_5)\,\tilde{d}}_{\text{FCNC}} \\ &+ \bar{c}\gamma_\mu\,(v_u - a_u\gamma_5)\,c + \bar{\tilde{s}}\gamma_\mu\,(v_d - a_d\gamma_5)\,\tilde{s} && \text{GIM piece} \\ &= \bar{u}\gamma_\mu\,(v_u - a_u\gamma_5)\,u + \bar{d}\gamma_\mu\,(v_d - a_d\gamma_5)\,d \\ &+ \bar{c}\gamma_\mu\,(v_u - a_u\gamma_5)\,c + \bar{s}\gamma_\mu\,(v_d - a_d\gamma_5)\,s \quad . \end{aligned}$$

Without the c quark $\tilde{s}$ would be absent in the CC and if one assumes that in the NC only the fields already present in the CC enter one ends up with a flavor changing NC. Although NC's had not been observed at all (before 1973) such FCNC's would have had observable consequences. The N=2 mixing scheme sometimes is called *Cabibbo universality*. Due to the existence of a third family Cabibbo universality is violated, because the 2 by 2 sub-matrix of the CKM-matrix is not unitary. A comparison of the N=2 and the N=3 mixing schemes in the 2 family world yields:

U	N=2	N=3
V_{ud}	$\cos\theta_c$	c_{12}
V_{us}	$\sin\theta_c$	s_{12}
V_{cd}	$-\sin\theta_c$	$-s_{12}c_{23}$
V_{cs}	$\cos\theta_c$	$c_{12}c_{23}$

where we used the excellent approximation $c_{13} = 1$, as c_{13} is known to deviate from unity only in the fifth decimal place.

The N=2 mixing scheme was extended to N=3 by Kobayashi and Maskawa in 1973 in order to incorporate CP-violation in a natural way.

Empirically the CKM matrix elements have the approximate moduli

$$|V| \simeq \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

with $\lambda \simeq \sin\theta_c \simeq 0.22$ given by the sine of the Cabibbo angle. This suggests the Wolfenstein parametrization (by unitarization up to higher order terms)

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4)$$

where $A \sim 1$ and $\rho^2 + \eta^2 < 1$. The corresponding quark decay pattern is illustrated in the following diagram:

Note: the u quark is stable, the s and b quarks are metastable. Flavor changing neutral current transitions are allowed only as second (or higher) order transitions: e.g. $b \to s$ is in fact $b \to (t^*, c^*, u^*) \to s$, where the asterisk indicates "virtual transition".

Exercises: Section 3.3

① The left (L)- and right (R)-handed fields are given by

$$\psi_L = \frac{1-\gamma_5}{2}\psi\ , \quad \psi_R = \frac{1+\gamma_5}{2}\psi\ .$$

Show that a mass term is given by

$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\ .$$

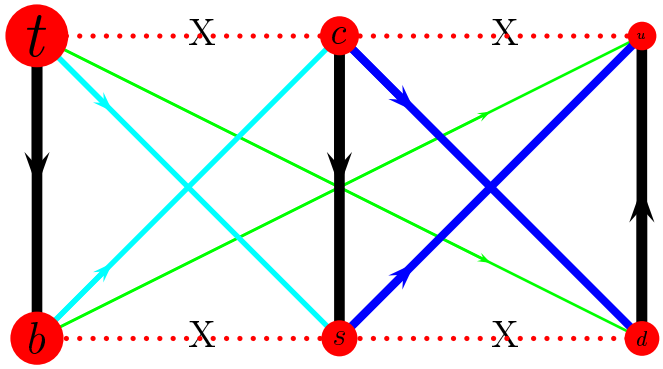


Figure 3.2: The CKM mixing hierarchy (3.15). FCNCs at tree level are forbidden [X].

Discuss helicity mixing for the terms

$$S = \bar{\psi}\psi \ , \quad P = \bar{\psi}\gamma_5\psi \ , \quad V^\mu = \bar{\psi}\gamma^\mu\psi \ , \quad A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \ , \quad T^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$$

② Consider two free fields ψ_1 and ψ_2 with different masses m_1 and m_2 . Calculate the divergences of the vector and axial vector currents

$$V^\mu = \bar{\psi}_1\gamma^\mu\psi_2 \quad \text{and} \quad A^\mu = \bar{\psi}_1\gamma^\mu\gamma_5\psi_2$$

$$\partial_\mu V^\mu = ? \qquad \partial_\mu A^\mu = ?$$

What is the consequence of this result for the weak currents ? (Hint: Use the Dirac equations). Comment on the hadronic charged weak current responsible for β–decay by comparing the nucleon current with the quark current.

③ Find the form of the quark currents which couple to W and Z and the photon. As a starting point use the $SU(2)_L \otimes U(1)_Y$ covariant derivative form

$$\begin{aligned} \mathcal{L}_q \ &= \ \bar{b}_R i\gamma^\mu \left(\partial_\mu + i\frac{1}{3}g' B_\mu\right) b_R + \bar{t}_R i\gamma^\mu \left(\partial_\mu - i\frac{2}{3}g' B_\mu\right) t_R \\ &+ \bar{L}_q i\gamma^\mu \left(\partial_\mu - i\frac{1}{3}\frac{g'}{2}B_\mu - ig\frac{\tau_a}{2}W_{\mu a}\right) L_q \end{aligned}$$

Check the correctness of the covariant derivatives with the quantum numbers assigned to the quarks. We have denoted by $L_q = \binom{t}{b}_L$ the left–handed doublet. t_R and b_R are the corresponding right–handed singlets. Quark mixing should be ignored.

④ Give numerical values for the Higgs couplings to the various particles. Calculate the decay width of the Higgs into a fermion pair and estimate the branching ratios for the different flavors. Discuss the dependence of the Higgs mass for the range $0 < m_H < 2M_W$. Compare the Higgs width with the width of known particles.

3.4 The Chiral Structure of Low Energy Effective QCD

The flavor symmetries seen in the hadron spectrum (multiplet classification of hadrons) are the approximate flavor groups $SU(N_f)$: $SU(2)$ isospin, $SU(3)$ isospin+strangeness etc., just vector symmetries and not approximate chiral symmetry $SU(N_f)_{\rm V} \otimes SU(N_f)_{\rm A}$ as suggested by the QCD Lagrangian. Nambu in 1960 proposed a fundamental solution of this symmetry mismatch: spontaneous breaking of the chiral symmetry

$$SU(N_f)_{\rm V} \otimes SU(N_f)_{\rm A} \to SU(N_f)_{\rm V}$$

by a non symmetric vacuum. In the symmetry limit (conserved chiral currents), this necessarily implies the existence of a set of massless pseudoscalar Nambu-Goldstone bosons: in $SU(2)$ the pions, in $SU(3)$ the pions, Kaons and the eta meson. Spontaneous symmetry breaking is a non-perturbative phenomenon, e.g. in perturbative QCD one never would get out pions. Fortunately, a firm low energy effective theory of QCD exists and is very well developed: chiral perturbation theory (CHPT) [Weinberg, Gasser, Leutwyler] an expansion for low momenta p and in the light current quark masses as chiral symmetry breaking parameters [5, 6]. CHPT is based on the chiral flavor structure $SU(3)_L \otimes SU(3)_R$ of the low lying hadron spectrum (u, d, s quark bound states). The $SU(3)_V$ vector currents $j^\mu_k = \sum_{ij} \bar\psi_i (T_k)_{ij}\, \gamma^\mu \psi_j$ as well as the $SU(3)_A$ axial currents $j^\mu_{5k} = \sum_{ij} \bar\psi_i (T_k)_{ij}\, \gamma^\mu\gamma_5 \psi_j$[14] are partially conserved in the $SU(3)$ sector of the (u, d, s) quark flavors, and strictly conserved in the chiral limit of vanishing quark masses $m_u, m_d, m_s \to 0$, modulo the axial anomaly in the axial singlet current. The partial conservation of the chiral currents[15] derives from $\partial_\mu(\bar\psi_1\gamma^\mu\psi_2) = {\rm i}(m_1 - m_2)\, \bar\psi_1\psi_2$ (CVC in the isospin limit $m_u = m_d$) and $\partial_\mu(\bar\psi_1\gamma^\mu\gamma_5\psi_2) = {\rm i}(m_1 + m_2)\, \bar\psi_1\gamma_5\psi_2$ (PCAC) and the setup of a perturbative scheme is based on the phenomenologically observed smallness of the current quark masses.

Chiral perturbation theory is the low energy effective form of QCD which essentially exploits the symmetries of the theory. It has been developed to a high level of sophistication by Gasser and Leutwyler [5]. In the $SU(2)$ flavor sector of u and d quarks, discarding the baryons and 2nd and 3rd family quarks, CHPT involves as the basic fields the Goldstone pion fields. We consider the chiral Lagrangian for two flavors in the isospin limit in the following. The chiral expansion is an expansion in $\hbar$

$$\mathcal{L}_{\rm eff} = \mathcal{L}_2 + \hbar\mathcal{L}_4 + \hbar^2\mathcal{L}_6 + \cdots$$

and is expanded in powers of derivatives and quark masses. In standard chiral counting one power of quark mass counts as two powers of derivatives, or momentum p in momentum space. The pion fields are encoded in the unitary 2×2 matrix

$$\begin{aligned} U &= \sigma + {\rm i}\frac{\phi}{F^2}\ , \quad \sigma^2 + \frac{\phi^2}{F^2} = \mathbf{1} \\ \phi &= \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \end{aligned}$$

where $\mathbf{1}$ is the unit 2×2 matrix and the auxiliary field σ, as indicated, is a nonlinear function of the pion fields π^i fixed by the unitarity of U[16]. The leading order Lagrangian starts at $O(p^2)$

[14] T_k ($k = 1, \ldots, 8$) are the generators of the global $SU(3)$ transformations and $i, j = u, d, s$ flavor indices.

[15] Especially in the $SU(2)$ isospin subspace, the small mass splitting $|m_1 - m_2| \ll m_1 + m_2$ motivates the terminology: conserved vector current (CVC) and partially conserved axial vector current (PCAC).

[16] In $SU(3)$ there appears an octet of massless pseodoscalar particles (π, K, η). The corresponding U field is the

and is given by the nonlinear σ–model Lagrangian in the presence of external fields, which are commonly represented in terms of matrix fields $v_\mu = v_\mu^i \sigma^i$, $a_\mu = a_\mu^i \sigma^i$:

$$\mathcal{L}_2 \;=\; \frac{F^2}{4}\langle D^\mu U D_\mu U^\dagger + M^2 \left(U + U^\dagger\right)\rangle$$

with

$$D_\mu U \;=\; \partial_\mu U - \mathrm{i}\,(v_\mu + a_\mu)\,U + \mathrm{i}\,U\,(v_\mu - a_\mu) \tag{3.17}$$

the covariant derivative in the presence of the external fields and

$$M^2 \;=\; 2B\hat{m} \tag{3.18}$$

where B is proportional to the quark condensate $\langle 0|\bar{u}u|0\rangle$. The parameters M and F are the leading order versions of the pion mass and the pion decay constant, respectively:

$$m_\pi^2 = M^2\,[1 + O(\hat{m})]\;, \quad F_\pi = F\,[1 + O(\hat{m})]\;. \tag{3.19}$$

The external fields v_μ^i and a_μ^i have to be understood as the external sources for the low energy representations of the quark currents V_μ^i and A_μ^i, respectively. These low energy effective currents again are nonlinear in the pion fields and in CHPT again appear expanded in the derivatives of U and the quark masses. For the axial vector current, for example, one obtains

$$\begin{aligned} V_\mu^i &= \frac{\mathrm{i}F^2}{4}\langle\sigma^i(U^\dagger D_\mu U + U D_\mu U^\dagger)\rangle + O(p^3) = \left[\varepsilon^{ijk}\phi^j\partial_\mu\phi^k + O(\phi^3)\right] + O(p^3)\;, \\ A_\mu^i &= \frac{\mathrm{i}F^2}{4}\langle\sigma^i(U^\dagger D_\mu U - U D_\mu U^\dagger)\rangle + O(p^3) = \left[-F\partial_\mu\phi^i + O(\phi^3)\right] + O(p^3)\;. \end{aligned}$$

which implies the conserved vector current (CVC) and the partially conserved axial vector current (PCAC) relations. Despite the fact that this Lagrangian is non-renormalizable, one can use it to calculate matrix elements like in standard perturbation theory. However, unlike in renormalizable theories where only terms already present in the original bare Lagrangian get reshuffled by renormalization, in non-renormalizable theories order by order in the expansion new vertices of increasing dimensions and associated new low energy constants show up and limit the predictive power of the effective theory. In fact the expansion (3.15) in $\hbar$ automatically produces an expansion of the matrix elements in powers of momenta and quark masses. For the matrix elements we are looking for, tree diagrams from $\mathcal{L}_2$ generate leading order contributions, while one–loop diagrams yield terms at next-to-leading order and so on. The occurring divergences in the one–loop contributions (in $d = 4$ dimensions) can be absorbed by introducing the effective

---

unitary 3×3 matrix

$$U(\Phi) = \exp\left(-\mathrm{i}\sqrt{2}\,\frac{\Phi(x)}{F}\right) \tag{3.15}$$

with (T_i the SU(3) generators)

$$\Phi(x) \;=\; \sum_i T_i\Phi_i = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -2\,\frac{\eta}{\sqrt{6}} \end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix} \eta' & & \\ & \eta' & \\ & & \eta' \end{pmatrix} \tag{3.16}$$

where the second term is the diagonal singlet contribution by the η' meson. The latter is not a Goldstone boson, however it is of leading order in $1/N_c$.

Lagrangian at $O(p^4)$

$$\begin{aligned}\mathcal{L}_4 &= \frac{1}{4}l_1\langle D^\mu U D_\mu U^\dagger\rangle^2 + \frac{1}{4}l_2\langle D_\mu U D_\nu U^\dagger\rangle\langle D^\mu U D^\nu U^\dagger\rangle \\ &+\frac{1}{16}l_3 M^4\langle U+U^\dagger\rangle^2 + \frac{\mathrm{i}}{2}l_4 M^2\langle a_\mu\left(D^\mu U - D^\mu U^\dagger\right)\rangle \\ &+l_5\langle F_R^{\mu\nu}UF_{L\,\mu\nu}U^\dagger\rangle + \frac{\mathrm{i}}{2}l_6\langle F_R^{\mu\nu}D_\mu U D_\nu U^\dagger + F_L^{\mu\nu}D_\mu U^\dagger D_\nu U\rangle + \ldots\end{aligned} \tag{3.20}$$

with

$$F_{R,L}^{\mu\nu} = \partial^\mu(v^\nu \pm a^\nu) - \partial^\nu(v^\mu \pm a^\mu) - \mathrm{i}[v^\mu \pm a^\mu, v^\mu \pm a^\mu]\ . \tag{3.21}$$

We omitted terms which contain external fields only. Since the current considered here are non-singlet currents, $\langle v_\mu\rangle = \langle a_\mu\rangle = 0$, there is no contribution from the ABJ anomaly at $O(p^4)$.

The coupling constants l_i may be split into a divergent and a finite term

$$l_i = \gamma_i\lambda + l_i^r(\mu) \tag{3.22}$$

where

$$\lambda = \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}[\ln 4\pi + \Gamma'(1) + 1]\right\} \tag{3.23}$$

represents the UV singular factor, and the finite terms are scale-independent by definition

$$\mu\frac{d}{d\mu}l_i = \gamma_i\frac{\mu^{d-4}}{16\pi^2} + \mu\frac{d}{d\mu}l_i^r(\mu) + O(d-4) = 0\ . \tag{3.24}$$

The finite parts have been determined phenomenologically for the first time by Gasser and Leutwyler [5]. We adopt their notation and use instead of the $l_i^r(\mu)$, the finite and scale-independent quantities $\bar{l}_i$,

$$\bar{l}_i = \left(\frac{\gamma_i}{32\pi^2}\right)^{-1} l_i^r(\mu)\ -\ \ln\frac{M^2}{\mu^2} \tag{3.25}$$

The up to date values for the relevant $\bar{l}_i$ and the corresponding γ_i are listed in Tab. 2.

Table 2: The set of $\bar{l}_i$ and corresponding γ_i which we need for the calculation of the matrix elements in question. The values are taken from [25] for $i = 1, 2$, and from [5] for all the others. The γ_i determine the relation between the $\bar{l}_i$ and the $l_i^r(\mu)$

$i =$	1	2	3	4	6
$\bar{l}_i$	-1.7 ± 1.0	6.1 ± 0.5	2.9 ± 2.4	4.3 ± 0.9	16.5 ± 1.1
γ_i	$1/3$	$2/3$	$-1/2$	2	$-1/3$

For completeness we give the expressions for the pion decay constant and the pion mass up to and including $O(p^4)$,

$$m_\pi^2 = M^2\left[1 - \frac{M^2}{32\pi^2F^2}\bar{l}_3 + O(M^4)\right]\ ,\quad F_\pi = F\left[1 + \frac{M^2}{16\pi^2F^2}\bar{l}_4 + O(M^4)\right] \tag{3.26}$$

The mass splitting $m_{\pi^\pm}^2 - m_{\pi^0}^2$ is proportional to $(m_u - m_d)^2$ and thus may be neglected. In the numerical evaluation, we will use $m_\pi = 139.57\,\mathrm{MeV}$ and[17] $F_\pi = 93.1\,\mathrm{MeV}$.

[17] F_π is extracted from the decay width $\Gamma(\pi \to \mu\nu_\mu)$

If one wants to go beyond the next-to-leading order, one has to calculate two–loop diagrams with $\mathcal{L}_2$, and one–loop diagrams with one vertex from $\mathcal{L}_4$. Again these diagrams will be divergent, but this is not a problem since at the same order one has contributions from tree diagrams with the $\mathcal{L}_6$ Lagrangian (which has been constructed in the case of three lights flavors [26]). By defining appropriately the new coupling constants occurring in this Lagrangian, one is able to remove the divergences at the next-to-next-to-leading order, and get finite matrix elements.

On remark shuld be made here: in CHPT the low energy constants in first place are free parameters, which are fixed as input parameters by confronting predictions with measured quantities, i.e., by phenomenological constraints. In fact CHPT represents QCD only for specific values of the low energy constants. The latter can be predicted from non-perturbative QCD, as realized in discretized version by lattice QCD. In future we can expect precise predictions on may of these constants, such that CHPT will gain dramatically in its predictive power.

At physical quark masses the value of the condensate is estimated to be $\langle m_q \bar{q} q \rangle \sim -(0.098 \text{ GeV})^4$ for $q = u, d$. The key relation to identify the quark condensates in terms of physical quantities is the Gell-Mann, Oakes and Renner (GOR) relation. In the chiral limit the mass operators $\bar{q}_R u_L$, or $\bar{q}_L u_R$ transform under $(3^*, 3)$ of the chiral group $SU(3)_L \otimes SU(3)_R$. Hence the quark condensates would have to vanish identically in case of an exact chirally symmetric world. In fact the symmetry is spontaneously broken and the vacuum of the real world is not chirally symmetric, and the quark condensates do not have to vanish. In order to determine the quark condensates, consider the charged axial currents and the related pseudoscalar density

$$\begin{aligned} A_\mu &= \bar{d} \gamma_\mu \gamma_5 u \\ P &= \bar{d}\, \mathrm{i} \gamma_5 u \end{aligned} \tag{3.27}$$

and the OPE of the product

$$A_\mu(x)\, P^+(y) = \sum_i C^i_\mu(x-y)\, \mathcal{O}^i(\frac{x+y}{2}) \ . \tag{3.28}$$

In QCD we may inspect the short distance expansion and study its consequences. One observation is that taking the VEV only the scalar operators contribute and one obtains the exact relation

$$\langle 0 | A_\mu(x)\, P^+(y) | 0 \rangle = \frac{(x-y)_\mu}{2\pi^2\, (x-y)^4}\, \langle 0 | \bar{u} u + \bar{d} d | 0 \rangle \ . \tag{3.29}$$

The spectral representation for the two–point function on the l.h.s. is of the form $p_\mu\, \rho(p^2)$ and current conservation requires $p^2\, \rho(p^2) = 0$ such that only the Goldstone modes, the massless pions, contribute, such that with

$$\begin{aligned} \langle 0 | A_\mu(0) | \pi^+ \rangle &= \mathrm{i}\, F_\pi\, p_\mu \\ \langle 0 | P^+(0) | \pi^+ \rangle &= g_\pi \end{aligned} \tag{3.30}$$

we get

$$F_\pi g_\pi = -\langle 0 | \bar{u} u + \bar{d} d | 0 \rangle \ . \tag{3.31}$$

For non-vanishing quark masses the PCAC relation $\partial^\mu A_\mu = (m_u + m_d)\, P$ then implies the exact relation

$$F_\pi m^2_{\pi^+} = (m_u + m_d)\, g_\pi \tag{3.32}$$

and the famous GOR relation

$$F^2_\pi m^2_{\pi^+} = -(m_u + m_d)\, \langle 0 | \bar{u} u + \bar{d} d | 0 \rangle \tag{3.33}$$

follows from the last two relations. Note that the quark condensates must be negative! They are a measure for the asymmetry of the vacuum in the chiral limit, and thus are true order parameters. If both F_π and $\langle 0|\bar{u}u + \bar{d}d|0\rangle$ have finite limits as $m_q \to 0$ the pion mass square must go to zero linear with the quark masses

$$m_{\pi^+}^2 = B\,(m_u + m_d)\;;\quad B \equiv -\frac{1}{F_\pi^2}\,\langle 0|\bar{u}u + \bar{d}d|0\rangle\;;\quad B > 0\;. \tag{3.34}$$

The deviation from the chiral limit is controlled by CHPT. The quark masses as well as the quark condensates depend on the renormalization scale μ, however, the product $\langle 0|m_q\bar{q}q|0\rangle$ is RG invariant as is inferred by the GOR relation.

Exercises: Section 3.4

① Write the PCAC relation in terms of physical parameters (masses and decay constants of hadrons) and try to understand the Goldberger-Treiman (GT) relation [Phys. Rev. 110 (1958) 1178, ibid. 111 (1958) 354]

$$g_A\,(m_p + m_n)/2 = F_\pi\,g_{\pi pn}$$

where m_p and m_n are the proton and neutron mass, respectively, and $g_{\pi pn}$ the effective pion nucleon coupling. The other couplings show up in the matrix elements for neutron β–decay

$$\mathcal{M}(n \to p + e^- + \bar{\nu}_e) = \frac{G_F}{\sqrt{2}}\,\Big[\bar{u}_p\,(f_V\,\gamma_\lambda - g_A\gamma_\lambda\gamma_5)\,u_n\,L^\lambda\Big]$$

and π–decay

$$\mathcal{M}(\pi^- \to \mu^- + \bar{\nu}_\mu) = \frac{G_F}{\sqrt{2}}\,\Big(-\mathrm{i}\,F_\pi p_\lambda\,L^\lambda\Big)\;.$$

By $L^\lambda = \bar{u}_\ell\gamma^\lambda\,(1-\gamma_5)\,v_{\nu_\ell}$ we denoted the leptonic "current", p is the pion momentum and G_F is the Fermi constant.

② Calculate the decay rate for $\pi^0 \to \gamma\gamma$ using the effective $O(p^4)$ coupling

$$\mathcal{L}_{\rm WZW} = \frac{\alpha}{\pi}\frac{N_c}{12F_\pi}\,\left(\pi^0 + \frac{1}{\sqrt{3}}\,\eta_8 + 2\sqrt{\frac{2}{3}}\,\eta_0\right)\,\tilde{F}_{\mu\nu}F^{\mu\nu}\;,$$

which is the Wess-Zumino-Witten [9, 10] Lagrangian. The latter reproduces the ABJ anomaly on the level of the hadrons. π^0 is the neutral pion field, F_π the pion decay constant ($F_\pi = 92.4\,\mathrm{MeV}$). The pseudoscalars η_8, η_0 are mixing into the physical states η, η'.

References

[1] S. L. Adler, Phys. Rev. **177** (1969) 2426

[2] J. S. Bell, R. Jackiw, Nuovo Cim. **60A** (1969) 47

[3] C. Bouchiat, J. Iliopoulos, P. Meyer, Phys. Lett. **38**B (1972) 519;
D. Gross, R. Jackiw, Phys. Rev. D **6** (1972) 477;
C. P. Korthals Altes, M. Perrottet, Phys. Lett. **39**B (1972) 546

[4] S. L. Adler, W. A. Bardeen, Phys. Rev. **182** (1969) 1517

[5] J. Gasser, H. Leutwyler, Ann. of Phys. (NY) **158** (1984) 142

[6] J. Gasser, H. Leutwyler, Nucl. Phys. B **250** (1985) 517

[7] J. Bijnens, G. Colangelo, J. Gasser, Nucl. Phys. B427 (1994) 427

[8] H.W. Fearing, S. Scherer, Phys. Rev. D53 (1996) 315

[9] J. Wess and B. Zumino, Phys. Lett. B **37** (1971) 95

[10] E. Witten, Nucl. Phys. B **223** (1983) 422

[11] G. 't Hooft, In: *Recent Developments in Gauge Theories*, Proceedings of the Summer-Institute, Cargese, France, 1979, ed by G. 't Hooft et al., NATO Advanced Study Institute Series B: Physics Vol. 59 (Plenum Press, New York 1980)

[12] H. Albrecht *et al.* [ARGUS COLLABORATION Collaboration], Phys. Lett. B **192** (1987) 245.

[13] F. Abe, et al. (CDF collaboration), Phys. Rev. Lett. 74, 2626 (1995);
S. Abachi, et al. (D collaboration), Phys. Rev. Lett. 74, 2632 (1995).

[14] [Tevatron Electroweak Working Group and CDF Collaboration and D0 Collab.], arXiv:0803.1683 [hep-ex]

[15] LEP Electroweak Working Group (LEP EWWG),
`http://lepewwg.web.cern.ch/LEPEWWG/plots/summer2006`
[ALEPH, DELPHI, L3, OPAL, SLD Collaborations], *Precision electroweak measurements on the Z resonance*, Phys. Rept. **427** (2006) 257;
`http://lepewwg.web.cern.ch/LEPEWWG/Welcome.html` (March 2008 update); (see also: Tevatron Electroweak Working Group, arXiv:0808.0147 [hep-ex])

[16] LEP Electroweak Working Group (LEP EWWG),
`http://lepewwg.web.cern.ch/LEPEWWG/lepww/tgc/`

[17] K. Abe *et al.* [Belle Collaboration], Phys. Rev. Lett. **87** (2001) 091802, Phys. Rev. Lett. **89** (2002) 071801. B. Aubert *et al.* [BABAR Collaboration], Phys. Rev. Lett. **87** (2001) 091801, Phys. Rev. Lett. **89** (2002) 201802.

[18] K. Nakamura, in *Proceedings of the 25th International Conference on High Energy Physics*, Singapore, 2-8 August 1990;
D. C. Kennedy, University of Pennsylvania Report No. UPR-0442T, 1990.

[19] K. Eguchi *et al.*, KamLAND Collaboration, Phys. Rev. Lett. **90** (2003) 021802.

[20] Q. R. Ahmad *et al.*, SNO Collaboration, Phys. Rev. Lett. **89**, 011301 (2002) 011302.

[21] For a recent review, see for example C. K. Jung, C. McGrew, T. Kajita, and T. Mann, Ann. Rev. Nucl. Part. Sci. **51** (2001) 451.

[22] M. Apollonio *et al.*, Phys. Lett. B 466 (1999) 415; F. Boehm *et al.*, Phys. Rev. D 64 (2001) 112001.

[23] N. Cabibbo, Phys. Rev. Lett. **10** (1963) 531.
M. Kobayashi, K. Maskawa, *Prog. Theor. Phys.* **49** (1973) 652.

[24] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28** (1962) 870.
B. Pontecorvo, Sov. Phys. JETP **6** (1957) 429 [Zh. Eksp. Teor. Fiz. **33** (1957) 549], Sov. Phys. JETP **26** (1968) 984 [Zh. Eksp. Teor. Fiz. **53** (1967) 1717].
V. N. Gribov and B. Pontecorvo, Phys. Lett. B **28** (1969) 493.
S. M. Bilenky and B. Pontecorvo, Phys. Rept. **41** (1978) 225.

[25] S. L. Glashow, J. Iliopoulos, L. Maiani, Phys. Rev. D **2** (1970) 1285

[26] Heavy Flavor Averaging Group (HFAG),
`http://www.slac.stanford.edu/xorg/hfag/`
`http://www-cdf.fnal.gov/physics/new/bottom/bottom.html`

4 Gauge principles and gauge invariance

4.1 Abelian gauge theory: Quantum Electrodynamics

Quarks and leptons, the constituents of matter, interact via gauge fields. Quantum electrodynamics (QED) describes the interaction of charged particles with the photon, which is described by an **Abelian gauge field** $A_\mu(x)$. As we shall see the form of the interaction may be understood as a consequence of a local symmetry: **local gauge invariance** (Weyl 1929). QED has been tested with extreme accuracy at its quantum effects level (Lamb shift, anomalous magnetic moments) and is the prototype of a very successful quantum field theory.

We assume the reader to be familiar with QED. Here we give a short account of its basic features only. The aim is to remind the reader of some basic problems which one encounters with massless spin 1 fields and how they are solved. Similar problems will show up in non-Abelian gauge theories which we will considered at a later stage.

We first consider the free photon field $A_\mu(x)$ and a free electron field $\psi_\alpha(x)$. The independent free Lagrangian densities read:

$$\begin{aligned} \mathcal{L}_{0A} &= \mathcal{L}_{\text{free photon}} = -\tfrac{1}{4} : F_{\mu\nu}F^{\mu\nu} : \\ \mathcal{L}_{0\psi} &= \mathcal{L}_{\text{free electron}} = : \bar{\psi}_\alpha \left(i\gamma^\mu \partial_\mu - m\right)_{\alpha\beta} \psi_\beta : \end{aligned}$$

The "$: \cdots :$" prescription means normal ordering, i.e. represent the fields in terms of annihilation and creation operators and commute (anticommute) all creation operators to the left of the annihilation operators. The c-number commutator (anticommutator) terms are omitted (subtracted)[18]. By this prescription we have subtracted the vacuum density such that now

$$< 0 \mid \mathcal{L}_0^{A,\psi}(x) \mid 0 >= 0 \ .$$

Notice that the action $i \int d^4x \mathcal{L}(x)$, for the infinite space-time volume, only may exist after subtraction of the vacuum density.

The photon field determines the antisymmetric electromagnetic field strength tensor

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{4.1}$$

which is gauge invariant – i.e. an Abelian gauge transformation

$$A_\mu \to A_\mu - \partial_\mu \alpha(x) \ , \tag{4.2}$$

where $\alpha(x)$ is an arbitrary scalar function, leaves $F_{\mu\nu}(x)$ invariant. As we know, if we represent $F_{\mu\nu}(x)$ as a curl of a vector-potential A_μ, Eq. 4.1, the homogeneous Maxwell equation

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad \text{with} \ \ \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{4.3}$$

is automatically satisfied. The pseudo-tensor (parity odd) $\tilde{F}^{\mu\nu}$ is the dual of $F^{\mu\nu}$.

We now consider the free field equations.

[18] For operators which are bilinear in the (free) fields, the normal ordering prescription is equivalent to the subtraction of the vacuum expectation value. For higher powers in the fields the relationship between ordinary products and normal products is discussed in my Lausanne lectures.

- **Field equation for** $A^\mu(x)$

The free photon is a solution of the free Maxwell equation, which is the Euler Lagrange equation for the free photon Lagrangian:

$$\partial_\mu \frac{\partial \mathcal{L}_{0A}}{\partial \partial_\mu A_\nu} = \frac{\partial \mathcal{L}_{0A}}{\partial A_\nu} \quad \Rightarrow \quad \partial_\mu F^{\mu\nu}(x) = 0$$

or, in terms of the vector-potential,

$$\Box A^\mu(x) - \partial^\mu \left(\partial_\nu A^\nu(x)\right) = \left(\Box g^{\mu\nu} - \partial^\mu \partial^\nu\right) A_\nu = 0$$

If we try to solve this latter equation a fundamental problem shows up. This equation of motion does not determine the field A_μ. The differential operator $\Box g^{\mu\nu} - \partial^\mu \partial^\nu$ has no inverse, since $\phi_\mu = \partial_\mu \alpha(x)$ is a solution with eigenvalue zero.

There is a simple reason for the problem. The field A^μ is not an observable and therefore has unphysical properties. A^μ is supposed to describe a massless spin 1 particle, which has two physical degrees of freedom only, the two transversely polarized states. Therefore two components of A^μ must be redundant. In particular, A^μ has a scalar component $\partial_\mu A^\mu = \phi$ which cannot be physical and must be required to vanish or to decouple from the physical degrees of freedom.

An idea of how to cure the problem we get if we notice that

$$\mathcal{L}_{0A} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

is a degenerate quadratic form in A^μ, which means that a change of A^μ does not necessarily change $\mathcal{L}_{0A}$. In particular we know that a gauge transformation of A^μ leaves $\mathcal{L}_{0A}$ unchanged. Obviously, in order to obtain an equation of motion which determines A^μ uniquely we have to break this degeneracy. This forces us to **add a gauge dependent term** to the invariant Lagrangian. Doing so, we fix a particular gauge and loose manifest gauge invariance.

In order to get an idea of what kind of term we may add in order to break the gauge symmetry of the Lagrangian without affecting the physics, let us consider the problem on the level of the field equation. This is a second order linear partial differential equation. Since it is not sufficient to determine A^μ uniquely we need some supplementary condition $C(A) = 0$. The latter should be linear, in order to keep the problem linear, and covariant, both requirements are not mandatory, however. A possibility, actually the only covariant and linear choice, is the use of the Lorentz gauge condition

$$\partial_\mu A^\mu(x) = 0$$

as a subsidiary condition. Strictly speaking this condition does not determine uniquely a gauge, because we still can perform a gauge transformation

$$\partial_\mu A^\mu(x) = 0 \rightarrow \partial_\mu A^\mu(x) - \Box\alpha(x) = 0 \quad \text{if} \quad \Box\alpha(x) = 0$$

which respects the Lorentz condition if we choose a restricted class of gauge functions $\alpha(x)$ which are solutions of $\Box\alpha(x) = 0$. In practice we need not bother about this problem further because we will see that we get a well defined perturbation expansion if we use the Lorentz condition for gauge fixing.

Geometrically the gauge condition picks a hyper-surface in the A^μ field space. Each point on the hyper-surface corresponds to a physically distinct field. Gauge transformations move the field orthogonal to the surface. Fields connected by gauge transformations are called **gauge copies** of each other. They are physically equivalent and form so called **gauge orbits**. The gauge condition selects the cut point of the gauge orbit with the hyper-surface as one particular representative field from each gauge orbit.

The gauge condition $C(A) = 0$ may be imposed by adding a Lagrange-multiplier term $\lambda\frac{1}{2}C(A)^2$ to $\mathcal{L}_{0A}$ where λ is an arbitrary constant. We obtain the gauge dependent Lagrangian

$$\mathcal{L}_{0A}^{\xi} = \mathcal{L}_{0A} + \mathcal{L}_{GF} \;\; ; \quad \mathcal{L}_{GF} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu(x)\right)^2 \quad ,$$

where $\mathcal{L}_{GF}$ is the **gauge fixing term** which lifts the degeneracy of $\mathcal{L}_{0A}$. ξ is called the **gauge parameter** and ξ^{-1} corresponds to the Lagrange multiplier. If we can show that physical predictions, like scattering matrix-elements, for example, are independent of ξ we also have shown that $\partial_\mu A^\mu = 0$ for what concerns physics. The gauge fixed Lagrangian $\mathcal{L}_{0A}^{\xi}$ is now suitable as a starting point for the quantization of the vector-potential while the original invariant Lagrangian was not.

The modified equation of motion following from the Lagrangian

$$\mathcal{L}_{0A}^{\xi} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi^{-1}\left(\partial_\mu A^\mu(x)\right)^2$$

is

$$\left(\Box g^{\mu\nu} - (1-\xi^{-1})\partial^\mu\partial^\nu\right)A_\nu(x) = 0$$

obtained by adding the extra term

$$\partial_\mu\frac{\partial\mathcal{L}_{GF}}{\partial\partial_\mu A_\nu} = -\xi^{-1}\partial^\nu\left(\partial_\mu A^\mu(x)\right)$$

to the previous form of the equation of motion.

Now the free Maxwell equation

$$\partial_\mu F^{\mu\nu} = \xi^{-1}\partial^\nu\left(\partial_\rho A^\rho(x)\right) \neq 0 \quad !$$

has no longer its classical form unless $\partial_\mu A^\mu(x) = 0$ in some sense. This is in contradistinction to the Proca field (massive spin 1 field) for which the Proca equation

$$\left((\Box + m^2)g^{\mu\nu} - \partial_\mu\partial_\nu\right)A_\nu = 0$$

automatically implies $\partial_\mu A^\mu(x) \equiv 0$.

Notice that the "vacuum" $|0>$ also makes troubles. Under a gauge transformation

$$\begin{aligned} <0 \mid A_\mu(x) \mid 0> \;\; \rightarrow \;\; & <0 \mid A_\mu(x) \mid 0> -\partial_\mu\alpha(x) <0 \mid 0> \\ & =<0 \mid A_\mu(x) \mid 0> -\partial_\mu\alpha(x) \end{aligned}$$

which would be a contradiction if $A_\mu(x)$ and $\mid 0>$ are supposed to have the naive properties, we except them to have. It turns out that the "vacuum" of the photon states must be considered

as an equivalence class $\{|\,0>\}_{A^\mu}$ and each gauge representative of a gauge orbit has a different formal vacuum.

The only clean covariant way to treat the problem is the Gupta-Bleuler formalism. In this formalism one can show that the physical state space $\mathcal{H}_{\rm phys}$ is characterized by

$$\partial_\mu A^{\mu(+)}(x)\mathcal{H}_{\rm phys} = 0$$

where $A^{\mu(+)}(x)$ denotes the positive frequency part (annihilation term) of the covariant photon field. The formal Hilbert space, before imposing this Gupta-Bleuler condition, includes unphysical states. Only on the physical subspace the physical laws have the "classical" form. For example, the Maxwell equation

$$\partial_\mu F^{\mu\nu} = -e j^\nu_{em} \tag{4.4}$$

is only true in the sense

$$\partial_\mu F^{\mu\nu(+)}\mathcal{H}_{\rm phys} = -e j^{\nu(+)}_{em}\mathcal{H}_{\rm phys}\ .$$

The reason why we need not worry to much about these problems is the fact that we have a simple check of whether or not the physics is gauge invariant: Physical matrix elements must turn out to be independent of the gauge parameters ξ ! In this case $\mathcal{L}_{GF}$ does not affect physical predictions and it looks as if $\partial_\mu A^\mu(x) = 0$. Thus gauge invariance is the "instrument" which allows to single out physics from technical artifacts.

- **Field equation for** $\psi_\alpha(x)$

The Euler-Lagrange equation for the free electron Lagrangian is the Dirac equation:

$$\partial_\mu \frac{\partial \mathcal{L}_{0\psi}}{\partial \partial_\mu \psi} = \frac{\partial \mathcal{L}_{0\psi}}{\partial \psi} \quad \Rightarrow \quad (i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0\ .$$

The electromagnetic current of the electron can be constructed from $\mathcal{L}_{0\psi}$ as follows:

$\mathcal{L}_{0\psi}$ has a **global** $U(1)$ symmetry: $\psi \to e^{-i\alpha}\psi$ where α is an arbitrary **constant**. Then by the **Noether theorem** there exists a conserved current.

$$\delta\mathcal{L}_{0\psi} \;=\; 0 \quad \text{under} \quad \begin{cases} \psi \to \psi + \delta\psi & ; \quad \delta\psi = -i\alpha\psi \\ \bar\psi \to \bar\psi + \delta\bar\psi & ; \quad \delta\bar\psi = \;\; i\alpha\bar\psi \end{cases}$$

where

$$\delta\mathcal{L}_{0\psi} = \delta\bar\psi\,\frac{\partial\mathcal{L}}{\partial\bar\psi} + \delta(\partial_\mu\bar\psi)\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)} + \frac{\partial\mathcal{L}}{\partial\psi}\,\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta(\partial_\mu\psi)\ .$$

For **global** transformations we have $\delta\partial_\mu\psi = \partial_\mu\delta\psi$ and the equation of motion tells us that

$$\frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial_\mu\psi} \quad ; \quad \frac{\partial\mathcal{L}}{\partial\bar\psi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar\psi}$$

and hence

$$\begin{aligned}\delta\mathcal{L}_{0\psi} &= \partial_\mu\left(\delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}}+\frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi\right)\\ &= -i\alpha\partial_\mu\left(\bar{\psi}i\gamma^\mu\psi\right)=\alpha\partial_\mu j^\mu_{em}=0\end{aligned}$$

with

$$j^\mu_{em} = :\bar{\psi}_\alpha(\gamma^\mu)_{\alpha\beta}\psi_\beta: \tag{4.5}$$

the conserved electromagnetic current.

Notice: For all interactions which do not exhibit derivations of ψ the form of j^μ_{em} is determined solely by $\mathcal{L}_{0\psi}$

- **Local gauge invariance and the electromagnetic interaction**

We observed that the problems with the photon field $A_\mu(x)$ requires local gauge invariance to hold. For the free electron we have another problem which at first seems not to be related to the problem of quantization of massless spin 1 particles: We know that in quantum mechanics the phase of a wave function is not observable. For a Dirac field we would expect therefore invariance under local phase transformations

$$\psi(x)\to e^{-ie\alpha(x)}\psi(x)\ . \tag{4.6}$$

These transformations, which again are related to some redundancy in the description of a particle by a quantum field, correspond to the local gauge transformations discussed before for the photon field and therefore are also called gauge transformations. However, $\mathcal{L}_{0\psi}$ is not locally gauge invariant because

$$\partial_\mu\psi(x)\to e^{-ie\alpha(x)}\partial_\mu\psi(x)-ie\,e^{-ie\alpha(x)}\psi(x)\partial_\mu\alpha(x)$$

and thus

$$\begin{aligned}\delta\mathcal{L}_{0\psi} &= e\,\bar{\psi}(x)i\gamma^\mu\left(-i\partial_\mu\alpha(x)\right)\psi(x)\\ &= e\,\bar{\psi}(x)\gamma^\mu\psi\partial_\mu\alpha(x)=e\,j^\mu_{em}\partial_\mu\alpha(x)\ .\end{aligned}$$

A free electron cannot be described in a locally gauge invariant way! The requirement of local gauge invariance implies that "electrons must couple to photons via minimal substitution". Which means that we have to replace the troublesome derivative

$$\partial_\mu\psi\to D_\mu\psi$$

by a **covariant derivative** $D_\mu\psi$ defined in such a way that it transforms in the same way as ψ:

$$D_\mu\psi\to e^{-ie\alpha(x)}D_\mu\psi \tag{4.7}$$

under a local gauge transformation of the electron-photon system[19]

$$\begin{aligned}\psi(x) &\to e^{-ie\alpha(x)}\psi(x)\\ A_\mu(x) &\to A_\mu(x)-\partial_\mu\alpha(x)\ .\end{aligned} \tag{4.8}$$

[19]Note that (4.8) must be a **simultaneous** transformation of the electron and the photon field with identical local gauge function $\alpha(x)$. With separate local functions (4.7) [in conjunction with (4.9)] does not hold and the QED Lagrangian is obviously not manifestly invariant. However, the apparent non–invariance is not real, because once the photon couples to the electron the non-invariant terms can always be reabsorbed by a gauge transformation of the photon field which cannot (or should not) affect the physics.

The requirement (4.7) implies the form

$$D_\mu = \partial_\mu - ieA_\mu \tag{4.9}$$

for the covariant derivative, which thus may be obtained by the so called **minimal substitution**

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu \tag{4.10}$$

to be applied to the free electron Lagrangian. As a consequence

$$\mathcal{L}_{0\psi} \to \mathcal{L}_\psi = \mathcal{L}_{0\psi} + ej^\mu_{\rm em} A_\mu(x) = \bar\psi\left(i\gamma^\mu D_\mu - m\right)\psi$$

includes automatically a certain type of electron-photon interactions. Thus the principle of local gauge invariance implies the following specific form of the interaction:

$$\mathcal{L}_{\rm int} = ej^\mu_{\rm em} A_\mu(x). \tag{4.11}$$

Obviously $F^{\mu\nu}$ and j^μ are gauge invariant objects. Thus one easily checks:

$$\begin{aligned} & \delta\left(\mathcal{L}_{0\psi} + ej^\mu_{em} A_\mu(x)\right) \\ = \;& \delta\mathcal{L}_{0\psi} - ej^\mu_{em}\partial_\mu\alpha(x) \\ = \;& ej^\mu_{\rm em}\partial_\mu\alpha(x) - ej^\mu_{\rm em}\partial_\mu\alpha(x) = 0. \end{aligned}$$

We notice that the form (4.11) of the coupling of photons to the electromagnetic current, which is prescribed by local gauge invariance, is the crux why it is **mandatory** to describe the photon, interacting with the charged particles, by the gauge dependent **four–potential** A_μ (**gauge potential**). Note that the two physical photon states with the fixed helicities ± 1 transform separately as an irreducible representation of the Lorentz group. The reducible representation combining the two photon states into one field would have two independent components. By the above construction, however, we are forced to describe the photon by a four component field, which necessarily has two superfluous components. This causes a lot of technical complications part of which have been addressed above when discussing the free photon field $A_\mu(x)$ (for more details see my Lausanne lectures).

The result of our discussion may be summarized as follows:

> Local $U(1)$ gauge invariance implies electron-photon interaction according to minimal coupling. The electromagnetic interaction is described by
>
> $$\begin{aligned} \mathcal{L}_{\rm QED} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\xi^{-1}\left(\partial_\mu A^\mu\right)^2 + \bar\psi\left(i\gamma^\mu D_\mu - m\right)\psi \\ &= \mathcal{L}^\xi_{0A} + \mathcal{L}_{0\psi} + ej^\mu_{em}(x)A_\mu(x) \\ \mathcal{L}_{\rm int} &= ej^\mu_{em}(x)A_\mu(x)\ . \end{aligned} \tag{4.12}$$
>
> Correspondingly, the field equations for QED read
>
> $$\left\| \begin{array}{l} \left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = -e : A_\mu(x)\gamma^\mu\psi(x) : \\ \left(\Box g^{\mu\nu} - \left(1-\xi^{-1}\right)\partial^\mu\partial^\nu\right)A_\nu(x) = -e : \bar\psi(x)\gamma^\mu\psi(x) : \end{array} \right. \tag{4.13}$$
>
> The minimally coupled electron-photon system has more symmetry than the free electron system, namely, a local one instead of a global one only. Due to the particular form of the interaction, resulting from the minimal substitution, the global gauge symmetry is promoted to a local gauge symmetry.

Let us add here a remark on how the field strength tensor in QED (Abelian Gauge Theory) may be obtained once we know the covariant derivative: $D_\mu = \partial_\mu - ieA_\mu$. Note that both $F_{\mu\nu}$ and $[D_\mu, D_\nu]$ are covariant antisymmetric tensor linear in the gauge field, in fact they are proportional:

$$\begin{aligned}
[D_\mu, D_\nu]\Phi(x) &= (\partial_\mu - ieA_\mu)\,(\partial_\nu - ieA_\nu)\;\Phi(x) \\
&- (\partial_\nu - ieA_\nu)\,(\partial_\mu - ieA_\mu)\;\Phi(x) \\
&= \partial_\mu\partial_\nu\;\Phi - ie\,(\partial_\mu(A_\nu\Phi) + A_\mu\partial_\nu\Phi) - e^2 A_\mu A_\nu\;\Phi \\
&- \mu \leftrightarrow \nu \\
&= -ie\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\,\Phi(x) \\
&= -ieF_{\mu\nu}\;\Phi(x)
\end{aligned}$$

holds for an arbitrary field $\Phi(x)$; hence

$$[D_\mu, D_\nu] \;=\; -ieF_{\mu\nu}\;,$$

with

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\;.$$

This trick will help us to find the non-Abelian field strength tensor in a simple way, below.

Let us come back to the simultaneous local gauge transformation (4.8) under which the classical QED Lagrangian (i.e., discarding the gauge fixing term in first place) is manifestly invariant: it is important to note that we may relax from the manifest invariance requirement. If we transform the electron field only, for example, the non-invariant term which shows up may be always eliminated by a gauge transformation of the photon field. The latter is required to leave the physics unchanged, which has to be proven of course. Thus the formal non–invariance of the Lagrangian not necessarily implies the non–invariance of the physics. We may thus precise the meaning of (4.8) as follows: *to each local gauge transformation of the electron field $\psi(x) \to e^{-ie\alpha(x)}\psi(x)$ there exists a gauge transformation of the photon field, namely, $A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x)$ such that under the combined transformation the classical part of the Lagrangian is manifestly invariant.*

As we shall see nature frequently makes use of the possibility that particles (the electrons) conspire with other particles (the photon) in order to enhance the symmetry. In this sense local gauge symmetries are **conspirative symmetries** which are only possible by conspiracy of particles of different kind.

Empirical fact: Nature makes use of the principle of local gauge invariance which is similar to the equivalence principle known from general relativity. Known fundamental elementary particle interactions are minimal couplings with respect to an invariance principle:

interaction	gauge group	quantum numbers
QED	$U(1)_{em}$	electric charge
QFD	$SU(2)_L \otimes U(1)_Y \overset{\text{broken}}{\to} U(1)_{em}$	weak isospin and weak hypercharge
QCD	$SU(3)_c$	color

Quantum Flavordynamics QFD we call the combination of the **electroweak theory**, also called electroweak **Standard Model**, and QCD, the theory of strong interactions which describes

the strong forces between nucleons and other hadrons. In the sections to follow the basic ideas behind the construction of these theories will be developed.

Exercises: Section 4.1

① Show that the Maxwell equation $\partial_\mu F^{\mu\nu} = 0$ as a field equation for the vector potential takes the form

$$\Box A^\nu(x) - \partial^\nu \left(\partial_\mu A^\mu(x)\right) = 0 \ .$$

Show that $A^\nu(x)$ is not determined by this equation because the operator $\Box g^{\mu\nu} - \partial^\mu\partial^\nu$ has no inverse. Hint.: $\varphi_\mu = \partial_\mu \alpha(x)$, $\alpha(x)$ an arbitrary scalar function, is a solution of the above equation with eigenvalue 0.

② Show that for a massive spin 1 field the Proca equation

$$\left(\Box + m^2\right) A^\nu(x) - \partial^\nu \left(\partial_\mu A^\mu(x)\right) = 0$$

implies $\partial_\mu A^\mu(x) \equiv 0$ automatically. Comment on the number of degrees of freedom ! Show that the Proca equation is the Euler-Lagrange equation of the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \ \ ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

Discuss the invariance properties of $\mathcal{L}$ under gauge transformations.

③ Prove that under local gauge transformations

$$\psi \to e^{-ie\alpha(x)}\psi \ , \quad A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha(x)$$

the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ has the property: $D_\mu\psi$ transforms identical to ψ and $\bar{\psi}\Gamma D_\mu \psi$ is gauge invariant provided D_μ commutes with the 4 by 4 matrix Γ.

4.2 Non-Abelian gauge theories

Quantum Chromodynamics may be considered as a straight forward non-Abelian generalization of Quantum Electrodynamics, from a local $U(1)_{\rm em}$ to local $SU(3)_c$ symmetry.

4.2.1 Global non-Abelian symmetry

Instead of a single field electron $\psi(x)$ we now consider a multiplet (e.g. a quark triplet) of fields

$$\Phi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ . \\ . \\ \\ \psi_n \end{pmatrix} (x) \; ; \quad \Phi(x) \to \Phi'(x) = U\,\Phi(x) \; ; \quad U \in SU(n)$$

which transforms under $SU(n)$. For a global internal symmetry the Lagrangian, like the free one

$$\mathcal{L}_{0\Phi} = \bar{\Psi}(x)\,(\mathrm{i}\,\gamma^\mu \partial_\mu - M)\,\Psi(x) \;\; ,$$

is invariant under global $SU(n)$ transformations, in particular we may consider infinitesimal transformations

$$\delta\Psi = \mathrm{i}g \sum_i T_i \Psi(x) \delta\omega_i \; ; \quad \delta\bar{\Psi} = -\mathrm{i}g \sum_i bar\Psi(x) T_i \delta\omega_i \;\; .$$

The invariance of $\mathcal{L}$ according to Noether implies conserved currents:

$$\delta\mathcal{L}_0^\Psi = \delta\bar{\Psi}\,\frac{\partial\mathcal{L}}{\partial\bar{\Psi}} + \delta\left(\partial_\mu\bar{\Psi}\right)\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\Psi}} + \frac{\partial\mathcal{L}}{\partial\Psi}\,\delta\Psi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi}\,\delta\left(\partial_\mu\Psi\right) \;\; .$$

Since the transformation is global (x-independent) $\delta\partial_\mu\Psi = \partial_\mu\delta\Psi$ and by the Euler-Lagrange equation $\frac{\partial\mathcal{L}}{\partial\Psi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi}$ we find

$$\begin{aligned} \delta\mathcal{L}_0^\Psi &= \partial_\mu\left\{\delta\bar{\Psi}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\Psi}} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi}\delta\Psi\right\} \\ &= -g\sum_i\left(\partial_\mu j_i^{\mu\Psi}\right)\delta\omega_i \end{aligned}$$

where

$$j_i^{\mu\Psi} = \bar{\Psi}(x)\gamma^\mu T_i\Psi(x) \equiv \bar{\Psi}_{\alpha a}(x)\left(\gamma^\mu\right)_{\alpha\beta}\left(T_i\right)_{ab}\Psi_{\beta b}(x)$$

are the fermionic $SU(n)$ Noether currents.

Since $\delta\mathcal{L}_0^\Psi = 0$ for arbitrary $\delta\omega_i$, we indeed must have the currents being conserved

$$\partial_\mu j_i^{\mu\Psi}(x) = 0 \; ; \quad i = 1, \ldots, n^2 - 1 \;\; .$$

4.2.2 Local non-Abelian symmetry

While global symmetries give raise to a classification of states according to irreducible representations of the symmetry group and to conserved quantum numbers, local symmetries have a rather different physical implication. The requirement of local symmetries is a **dynamical principle** which implies that matter fields **must** be in interaction with massless spin 1 gauge fields in a specific way. The property of local gauge invariance corresponds to the validity of an **equivalence principle:** the n degrees of freedom of the local field $\Psi_a(x)$ are locally indistinguishable in the sense that $\Psi(x)$ and $U(x)\Psi(x)$ describe the same physics and

$$\Psi(x) \to \Psi'(x) = U(x)\Psi(x) \ , \ \ U(x) \in SU(n)$$

is a symmetry of the dynamics. This means that an observer at space-time point x may choose a coordinate frame for the internal degrees of freedom in the multiplet $\Psi_a(x)$ independently from an observer at a different space-time point x'. At first sight this requirement might look quite natural and harmless. It has dramatic consequences, however. Essentially, it dictates the form of the dynamics once the local transformation laws of the matter fields are known. The equivalence principle for internal symmetries is very similar to the classical equivalence principle of gravity, which implies that gravity emerges as the geometry of space-time.

The freedom to have associated with each space-time point an independent frame for the internal symmetry space only makes sense if we are able to "synchronize" local frames at different space-time points. In order to be able to actually perform such a synchronization we need carriers of physical signals traveling at the universal speed of light, namely, $n^2 - 1$ massless spin 1 bosons described by a matrix $\boldsymbol{V}_\mu(x) = \sum_i T_i V_{\mu i}(x)$ which is an element of the Lie-algebra. A change of local frames between space-time points x and $x + dx$ must be correlated with a local gauge transformation of $\boldsymbol{V}_\mu(x)$. What we need is a statement saying when "a field $\Psi(x)$ does not change between x and $x+dx$". With other words, we need a definition of **parallel displacement**. When there are no internal symmetries and $\Psi(x)$ is a real field we would say that the field does not change between x and $x + dx$ if

$$\Psi(x + dx) = \Psi(x) + \partial_\mu \Psi(x)\, dx^\mu = \Psi(x)$$

or $\partial_\mu \Psi(x) = 0$. In QED the complex electron field $\psi(x)$ is coupled to the photon field $A_\mu(x)$, a $U(1)$ gauge field, and in the coupled electron-photon system the phase $\psi(x) = e^{ie\alpha(x)} \mid \psi(x) \mid$ has no physical significance. However, the statement that $\mid \psi(x) \mid$ is constant over dx:

$$\partial_\mu \mid \psi(x) \mid = \partial_\mu \left(e^{-ie\alpha(x)} \psi(x) \right) = 0$$

or $(\partial_\mu - ie\partial_\mu \alpha(x))\psi(x) = 0$ is not gauge invariant and must be replaced by the condition that the **covariant derivative**

$$D_\mu \psi(x) = (\partial_\mu - ieA_\mu(x))\, \psi(x) = 0$$

vanishes. For the non-Abelian $SU(n)$ gauge symmetry this generalizes to

$$(D_\mu \Psi(x))_a = (\partial_\mu - ig\boldsymbol{V}_\mu(x))_{ab}\, \Psi_b(x) = 0$$

In Fig. 4.1 we have illustrated the geometrical meaning of the covariant derivative. For infinitesimal dx we compare the fields $\Psi(x)$ at point x and $\Psi(x + dx)$ at point $x + dx$. Consider the covariant expansion $\Psi(x + dx) = \Psi(x) + D_\mu \Psi(x)\, dx^\mu$. If $\Psi(x)$ satisfies $D_\mu \Psi(x) = 0$ the field is "parallel". We denote by $\Psi^{||}_{(dx)}(x)$ the field $\Psi(x+dx)$ which has been shifted parallel from $x+dx$ to x along the path dx. Then $D_\mu \Psi(x)\, dx^\mu = \Psi^{||}_{(dx)}(x) - \Psi(x)$.

We now discuss in more detail how locally gauge invariant field theories can be constructed.

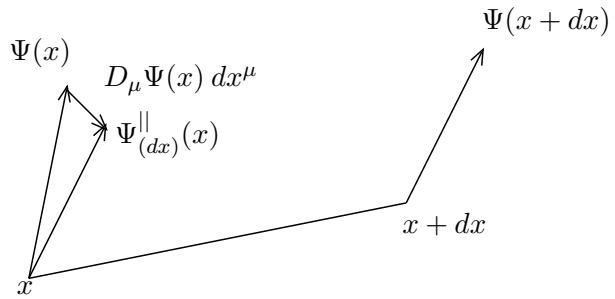


Figure 4.1: Geometrical interpretation of the parallel displacement.

4.2.3 Minimal couplings of the matter fields

Global G ($= SU(n)$) invariance of $\mathcal{L}_0^\Psi$ follows from the fact that with $\Psi(x)$ also $\partial_\mu\Psi(x)$ transforms as a vector:

$$\begin{aligned}\Psi(x) &\rightarrow \Psi'(x) = U\Psi(x)\\ \partial_\mu\Psi(x) &\rightarrow \partial_\mu\Psi'(x) = U\partial_\mu\Psi(x)\end{aligned}$$

when $\partial_\mu U = 0$. Under local transformations $\partial_\mu U \neq 0$

$$\begin{aligned}\partial_\mu\Psi(x) \rightarrow \partial_\mu\Psi'(x) &= U(x)\partial_\mu\Psi(x) + (\partial_\mu U(x))\,\Psi(x)\\ &= U(x)\left(\partial_\mu + U^{-1}(x)\,(\partial_\mu U(x))\right)\Psi(x)\\ &\neq U(x)\partial_\mu\Psi(x)\end{aligned}$$

does no longer transform vector-like because of the extra term

$$U^{-1}(x)\,(\partial_\mu U(x)) = ig\sum_i T_i\tilde{V}_{i\mu} \;\in\; G'.$$

As indicated, this term is an element of the Lie-algebra G' of the symmetry group G and a four-vector under Lorentz transformation. Neglecting higher order terms, for infinitesimal transformations $U = 1 + ig\sum_i T_i\omega_i$ we easily calculate

$$U^{-1}(x)\,(\partial_\mu U(x)) = ig\sum_i T_i\partial_\mu\omega_i(x)$$

such that $\tilde{V}_{i\mu}(x) = \partial_\mu\omega_i(x)$. For finite transformations we may write $\tilde{V}_{i\mu}(x) = \sum_l \Lambda_{il}(\omega)\partial_\mu\omega_l(x)$ where the matrix $\Lambda_{il}(\omega)$ is given in the Appendix.

When applied to $\mathcal{L}_0^\Psi$ local gauge transformations induce a non-invariant term:

$$\begin{aligned}\mathcal{L}_0^\Psi &\rightarrow \mathcal{L}_0^\Psi + \bar{\Psi}i\gamma^\mu U^{-1}\,(\partial_\mu U)\,\Psi\\ &= \mathcal{L}_0^\Psi - g\sum_i \bar{\Psi}\gamma^\mu T_i\Psi\tilde{V}_{i\mu}(x)\\ &= \mathcal{L}_0^\Psi - g\sum_i j_i^\mu(x)\tilde{V}_{i\mu}(x)\;.\end{aligned}$$

This term describes the coupling of a set of $r = n^2 - 1$ real vector fields $\tilde{V}_{i\mu}(x)$ to the Noether currents $j_i^\mu(x)$, which are conserved under global transformations.

In order to obtain a locally gauge invariant **extension** of the free system we must introduce a set of r real vector fields $V_{i\mu}(x)$ as dynamical variables (physical degrees of freedom) which couple to the Noether currents:

$$\begin{aligned}\mathcal{L}_0^\Psi \to \mathcal{L}^\Psi &= \mathcal{L}_0^\Psi + g\sum_i j_i^\mu(x) V_{i\mu}(x) \\ &= \mathcal{L}_0^\Psi + g\sum_i \bar\Psi\gamma^\mu T_i V_{i\mu}(x)\Psi \\ &= \bar\Psi\left(i\gamma^\mu D_\mu - m\right)\Psi\ .\end{aligned}$$

The fields $V_{i\mu}(x)$ are called **Yang-Mills fields** or **non-Abelian gauge fields**.

Formally $\mathcal{L}^\Psi$ follows from $\mathcal{L}_0^\Psi$ by **minimal substitution**

$$\partial_\mu \to D_\mu = \partial_\mu - ig\ \boldsymbol{V}_\mu$$

where

$$\boldsymbol{V}_\mu = \sum_i T_i V_{i\mu}(x) \quad \in\ G'$$

is an element of the Lie-algebra.

D_μ defines the **covariant derivative**. $\mathcal{L}^\Psi$ is locally gauge invariant provided $D_\mu\Psi$ transforms as a vector

$$\begin{aligned}\Psi(x) &\to \Psi'(x) = U(x)\Psi(x) \\ D_\mu\Psi(x) &\to (D_\mu\Psi)' = U(x)D_\mu\Psi(x)\end{aligned}$$

and hence

$$D'_\mu U(x) = U(x)D_\mu\ .$$

This condition fixes the transformation law of the fields $V_{i\mu}(x)$:

$$\begin{aligned}D'_\mu &= \partial_\mu - ig\boldsymbol{V}_\mu{}' = U(x)D_\mu U^{-1}(x) \\ &= U(x)\left(\partial_\mu - ig\boldsymbol{V}_\mu\right)U^{-1}(x) \\ &= \partial_\mu - igU(x)\boldsymbol{V}_\mu U^{-1}(x) + U(x)\partial_\mu U^{-1}(x) \\ &= \partial_\mu - igU(x)\left(\boldsymbol{V}_\mu - \frac{i}{g}U^{-1}(\partial_\mu U)\right)U^{-1}(x)\ .\end{aligned}$$

Here, we have used $\partial_\mu\left(UU^{-1}\right) = \left(\partial_\mu U\right)U^{-1} + U\left(\partial_\mu U^{-1}\right) = 0$ or $U\left(\partial_\mu U^{-1}\right) = -\left(\partial_\mu U\right)U^{-1}$.

Consequently, we find

$$\boldsymbol{V}_\mu \to \boldsymbol{V}_\mu{}' = U(x)\left(\boldsymbol{V}_\mu - \frac{i}{g}U^{-1}(\partial_\mu U)\right)U^{-1}(x)\ .$$

Like $\partial_\mu\Psi(x)$, $\boldsymbol{V}_\mu(x)$ does not transform as a vector since the local transformation law is different from the global one. In fact $\boldsymbol{V}_\mu$ has been **required** to produce a compensating term for the non-covariant term obtained for $\partial_\mu\Psi$, in order that $(\partial_\mu - ig\boldsymbol{V}_\mu)\Psi$ is a vector.

For infinitesimal transformations $U = 1 + ig \sum_i T_i \omega_i(x)$ we obtain to linear order in $\omega_i(x)$:

$$\begin{aligned} \boldsymbol{V}_\mu{}' = \sum_i T_i V'_{i\mu}(x) &= \sum_i T_i \left(V_{i\mu}(x) + \partial_\mu \omega_i(x)\right) + ig \sum_{k,\,l} \left[T_k, T_l\right] \omega_k V_{l\mu} \\ &= \sum_i T_i \left(V_{i\mu}(x) - g c_{ikl} V_{l\mu} \omega_k + \partial_\mu \omega_i\right) \end{aligned}$$

or

$$V'_{i\mu} = V_{i\mu} - g c_{ikl} V_{l\mu}\, \omega_k + \partial_\mu \omega_i \; .$$

Since $(\tilde{T}_k)_{il} = i c_{ikl}$ represent the generators in the adjoint representation we have

$$\begin{aligned} \delta V_{i\mu} &= -g c_{ikl} V_{l\mu}\, \omega_k + \partial_\mu \omega_i \\ &= ig (\tilde{T}_k)_{il} V_{l\mu}\, \omega_k + \partial_\mu \omega_i \end{aligned}$$

which compares to

$$\delta \Psi_a = ig (T_k)_{ab} \Psi_b \omega_k$$

for the matter fields. We notice that $V_{i\mu}(x)$ transforms under the adjoint representation up to a divergence term[20]. Accordingly the fields $V_{i\mu}(x)$ carry $SU(n)$ charge, which is obvious also from their coupling to the charged Noether currents.

As a result, local gauge invariance requires the matter fields to interact with $n^2 - 1$ massless gauge fields via minimal coupling

$$\mathcal{L}^\Psi = \bar{\Psi} \left(i \gamma^\mu D_\mu - m\right) \Psi = \mathcal{L}_0^\Psi + g \sum_i \bar{\Psi} \gamma^\mu T_i \Psi V_{i\mu} \; .$$

The gauge coupling constant g is a free parameter. The interaction vertex is depicted in Fig. 4.2.

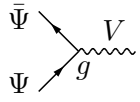


Figure 4.2: Matter field couplings of a gauge theory.

4.2.4 Non-Abelian gauge field strength tensor, Yang-Mills action

The field strength tensor in QCD (non-Abelian gauge theory) may be constructed from the covariant derivative: $(D_\mu)_{ij} = \partial_\mu\, \delta_{ij} - ig\, (T_k)_{ij} G_{k\,\mu}$. What we are looking for is a generalization of

[20] As we know from QED, massless four–component gauge fields necessarily exhibit non–physical degrees of freedom because there are only two physical states the transversal one's. Hence, the gauge potentials describe among the physics also redundant stuff. Thats why the transformation laws of the gauge fields under local gauge transformations are anomalous (by the disturbing divergence term). Attempts to describe gauge interactions directly in terms of the more physical field strength tensor (see below) seem not to be possible. By the construction presented before somehow the gauge potentials are quantities which show up in a natural way. As in QED, at the end one has to show that the physical transition matrix elements are gauge invariant and do not depend on the redundancies of the formalism

the Abelian field strength tensor, a covariant antisymmetric tensor, this time covariant not only with respect to Lorentz transformations, but also with respect to non-Abelian gauge transformations. Since D_μ is a Lorentz vector and satisfies $D'_\mu = U(x)D_\mu U^{-1}(x)$ the commutator $[D_\mu, D_\nu]$ satisfies all the properties required for $\boldsymbol{G}_{\mu\nu}$.

It is now easy to calculate $\boldsymbol{G}_{\mu\nu}$. Using

$$(D_\mu)_{ik} = \partial_\mu \delta_{ik} - ig \sum_j (T_j)_{ik} V_{j\mu}(x)$$

we have

$$\begin{aligned}(D_\mu)_{ik}(D_\nu)_{kl} &= \partial_\mu\partial_\nu\delta_{il} - ig\sum_j (T_j)_{il}\left(V_{j\mu}\partial_\nu + V_{j\nu}\partial_\mu\right) \\ &- ig\sum_j (T_j)_{il}\partial_\mu V_{j\nu} - g^2 \sum_{j,\,j'} (T_j)_{ik}(T_{j'})_{kl} V_{j\mu}V_{j'\nu}\end{aligned}$$

and hence (the symmetric terms drop)

$$\begin{aligned}[D_\mu, D_\nu]_{il} &= -ig\sum_j (T_j)_{il}\left(\partial_\mu V_{j\nu} - \partial_\nu V_{j\mu}\right) \\ &\quad -g^2 \sum_{j',\,j''} \left[T_{j'}, T_{j''}\right]_{il} V_{j'\mu}V_{j''\nu} \\ &= -ig\sum_j (T_j)_{il} G_{j\mu\nu} = -ig\ (\boldsymbol{G}_{\mu\nu})_{il}\end{aligned}$$

where, using $\left[T_{j'}, T_{j''}\right] = ic_{j'j''j}T_j$,

$$G_{i\mu\nu} = \partial_\mu V_{i\nu} - \partial_\nu V_{i\mu} + g c_{ijk} V_{j\mu} V_{k\nu}\ .$$

This is indeed the gauge covariant generalization of $\overset{\circ}{G}_{i\mu\nu}$. In absence of matter fields the Lagrangian density

$$\begin{aligned}\mathcal{L}_{YM} &= -\frac{1}{4}\ \overset{\circ}{G}_{i\mu\nu}\overset{\circ}{G}{}_i^{\mu\nu} \\ &\quad -\frac{1}{2} g c_{ikl}\left(\partial^\mu V_i^\nu - \partial^\nu V_i^\mu\right) V_{k\mu} V_{l\nu} \\ &\quad -\frac{1}{4} g^2 c_{ikl} c_{ik'l'} V_k^\mu V_l^\nu V_{k'\mu} V_{l'\nu}\end{aligned}$$

defines the so called **pure Yang-Mills theory** . It provides the kinetic term and therefore the **dynamics** of the gauge fields. However, in contrast to the Abelian fields which are not self interacting, non-Abelian gauge invariance enforces triple and quartic self-interactions of the Yang-Mills fields, of course due to the fact that they carry non-Abelian charge.

The corresponding interaction vertices are shown in Fig. 4.3.

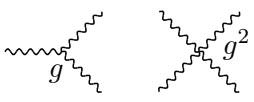


Figure 4.3: Yang-Mills couplings.

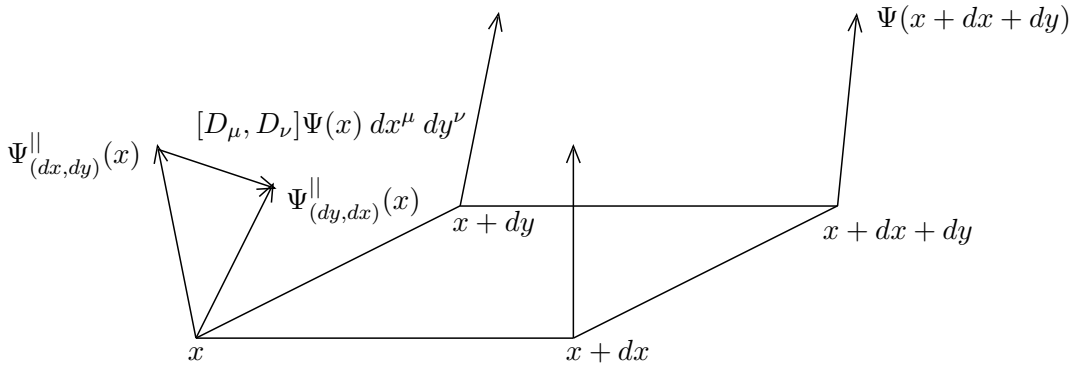


Figure 4.4: Geometrical interpretation of the field strength tensor.

If we include the matter fields we have the complete locally gauge invariant Lagrangian density

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4}\sum_i G_{i\mu\nu}G_i^{\mu\nu} + \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi$$

with **one** coupling constant g as a free parameter. The strengths of the three different interaction vertices are fixed by the same gauge coupling constant.

Let me add a remark about the geometrical interpretation of the field strength tensor which derives from the one of the covariant derivative. To this end we consider an infinitesimal parallelogram of points x, $x+dx$, $x+dy$ and $x+dx+dy$ and a field $\Psi(x_i)$ at the different points. In order to shift $\Psi(x+dx+dy)$ parallel to the point x we have two possible paths to follow along the sides of the parallelogram. We denote these paths by (dx, dy) and (dy, dx). The parallel displaced fields are $\Psi^{||}_{(dx,dy)}(x)$ and $\Psi^{||}_{(dy,dx)}(x)$. We now calculate the difference of these two fields. To this end we perform a covariant expansion along the two paths:

$$\begin{aligned}
\Psi(x+dx+dy) &= \Psi(x+dy) + D_\mu\Psi(x+dy)\, dx^\mu \\
&= \Psi(x) + D_\nu\Psi(x)\, dy^\nu + D_\mu\Psi(x)\, dx^\mu + D_\mu D_\nu\Psi(x)\, dx^\mu\, dy^\nu \\
\Psi(x+dx+dy) &= \Psi(x+dx) + D_\nu\Psi(x+dx)\, dy^\nu \\
&= \Psi(x) + D_\mu\Psi(x)\, dx^\mu + D_\nu\Psi(x)\, dy^\nu + D_\nu D_\mu\Psi(x)\, dx^\mu\, dy^\nu
\end{aligned}$$

For the difference we obtain

$$\Psi^{||}_{(dy,dx)}(x) - \Psi^{||}_{(dx,dy)}(x) = [D_\mu, D_\nu]\Psi(x)\, dx^\mu\, dy^\nu = -ig\boldsymbol{G}_{\mu\nu}\Psi(x)\, dx^\mu\, dy^\nu$$

exhibiting the field strength tensor as a **curvature tensor** . If the field strength is non-vanishing the parallel-displacements of a vector along different paths yield a different result. For infinitesimal shifts the difference vector is proportional to the original vector to the field strength and to the area of the parallelogram. The curvature is illustrated in Fig. 4.4.

4.2.5 Equations of motion and currents

Given the invariant Lagrangian

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4}\sum_i G_{i\mu\nu}G_i^{\mu\nu} + \bar{\Psi}\left(i\gamma^\mu\left(\partial_\mu - ig\sum_i T_i V_{i\mu}\right) - m\right)\Psi$$

the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\Psi}} = \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} \quad \text{and} \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu V_{i\nu}} = \frac{\partial \mathcal{L}}{\partial V_{i\nu}}$$

read

$$\left(i\gamma^\mu \partial_\mu - m\right) \Psi = -g\gamma^\mu T_i V_{i\mu} \Psi$$

and

$$\partial_\mu G_i^{\mu\nu} = -g\left(j_i^{\nu\Psi} + j_i^{\nu V}\right) = -g j_i^\nu$$

with

$$\begin{aligned} j_i^{\nu\Psi} &= \bar{\Psi}\gamma^\nu T_i \Psi \quad \text{Noether current of the matter field} \\ j_i^{\nu V} &= c_{ikl} G_k^{\nu\rho} V_{l\rho} \quad \text{current of the gauge field} \end{aligned}$$

The total current $j_i^\nu = j_i^{\nu\Psi} + j_i^{\nu V}$ is obviously conserved

$$\partial_\nu j_i^\nu = -\frac{1}{g} \partial_\nu \partial_\mu G_i^{\mu\nu} = 0$$

because $\partial_\nu \partial_\mu$ is symmetric whereas $G_i^{\mu\nu}$ is antisymmetric in $(\mu\nu)$.

The equations of motion may be written in a manifestly gauge invariant form

$$\begin{aligned} \left(i\gamma^\mu D_\mu - m\right) \Psi(x) &= 0 \\ D_\mu G_i^{\mu\nu}(x) &= -g J_i^\nu(x) \end{aligned}$$

where

$$D_\mu G_i^{\mu\nu} = \partial_\mu G_i^{\mu\nu} - g c_{ikl} V_{k\mu} G_l^{\mu\nu}.$$

The gauge invariant form is simply obtained by adding the appropriate terms on both sides of the equations of motion given before. Notice that $D_\mu G_i^{\mu\nu}$ and hence J_i^ν, given by

$$J_i^\nu = j_i^\nu - c_{ikl} V_{k\mu} G_l^{\mu\nu} = j_i^{\nu\Psi} = \bar{\Psi}\gamma^\nu T_i \Psi,$$

are vectors under local gauge transformations.

The covariant current coincides with the matter field current which is not conserved: $\partial_\nu J_i^\nu \neq 0$. On the other hand the conserved current

$$j_i^\nu = \bar{\Psi}\gamma^\nu T_i \Psi + c_{ikl} V_{k\mu} G_l^{\mu\nu} = -\frac{1}{g} \partial_\mu G_i^{\mu\nu}$$

is obviously not covariant because it is the ordinary derivative of a vector. We then arrive at the conclusion:

> In a locally gauge invariant theory a covariant conserved current with respect to the gauge symmetry does not exist.

This tells us that local symmetries are not symmetries in the usual global sense, like Poincaré invariance, Isospin invariance etc. . Local symmetries are **dynamical symmetries** and a consequence of the validity of an equivalence principle. Global symmetries describe algebraic properties of a system, only.

Summary (of subsection 4.2)

1. If we require n matter fields $(\psi_1, \dots \psi_n)$ to be locally indistinguishable, such that

$$\Psi(x) = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \rightarrow \Psi'(x) = U(x)\Psi(x) \; ; \;\; U(x) = \exp ig \sum_i T_i \omega_i(x) \;\; \in \;\; SU(n)$$

   is a local symmetry of the system, the matter fields must couple minimally to a set of $r = n^2 - 1$ massless spin 1 gauge fields $V_{i\mu}(x)$:

$$\mathcal{L}^{\Psi}_{\text{inv}} = \bar{\Psi}(x) \left(i\gamma^\mu D_\mu - m \right) \Psi(x)$$

   with covariant derivative

$$D_\mu = \partial_\mu - ig \sum_i T_i V_{i\mu}(x) \; .$$

   All matter fields $\psi_1 \dots \psi_n$ must have identical mass and spin.

2. The locally gauge invariant Lagrangian of the gauge fields must be of the form

$$\mathcal{L}_{YM} = -\frac{1}{4} \sum_i G_{i\mu\nu} G_i^{\mu\nu}$$

   with covariant field strength tensor

$$G_{i\mu\nu} = \partial_\mu V_{i\nu}(x) - \partial_\nu V_{i\mu}(x) + g c_{ikl} V_{k\mu}(x) V_{l\nu}(x)$$

   The non-Abelian gauge fields must be self-interacting in the specific way as prescribed by $\mathcal{L}_{YM}$. A mass term

$$\frac{M^2}{2} \sum_i V_{i\mu}(x) V_i^\mu(x)$$

   for the gauge bosons is not admitted.

3. Under infinitesimal gauge transformations the fields transform as

$$\delta\Psi_a = \; ig \, (T_i)_{ab} \Psi_b \delta\omega_i \; , \;\; \delta\bar{\Psi}_a = -ig \; \bar{\Psi}_b (T_i)_{ba} \delta\omega_i$$

   with $(T_i)_{ab}$ the generators of $SU(n)$ in the fundamental representation.

$$\delta V_{k\mu} = ig \left(\tilde{T}_i \right)_{kl} V_{l\mu} \delta\omega_i + \partial_\mu \delta\omega_i \; , \;\; \delta G_{k\mu\nu} = ig \left(\tilde{T}_i \right)_{kl} G_{l\mu\nu} \delta\omega_i$$

   with $\left(\tilde{T}_i \right)_{kl} = -i c_{ikl}$ the generators of $SU(n)$ in the adjoint representation.

A final remark concerns the generalization of the Yang-Mills construction to other gauge groups G. The necessary and sufficient conditions for the Yang-Mills construction to be possible are that the structure constants c_{ikl}

i) satisfy the Jacobi-identity

ii) are fully antisymmetric.

Whereas i) is automatic for finite matrices (finite dimensional representations) but may not hold for infinite matrices, the condition ii) is more restrictive. It holds for compact semi-simple Lie groups. Compact mean that the parameter space has a finite volume $\int \prod_i d\omega_i = V_\omega < \infty$. Semi-simple means G is a product of simple Lie-groups e.g. $G = SU(3) \otimes SU(2) \otimes U(1)$. Simple Lie-groups are those which cannot be decomposed into invariant subgroups. For each simple subgroup of a non-simple group G there would be a set of non-Abelian or Abelian ($U(1)$) gauge fields and an independent coupling constant.

Notice that the conditions i) and ii) are really non-trivial. As an example the Poincaré group is non-compact and has no finite dimensional unitary representation.

Outlook: For a long time, after Yang and Mills had proposed to extend local gauge invariance from Abelian to non–Abelian symmetry groups, non–Abelian gauge theories were considered to be unphysical, because they required the existence of multiplets of massless spin 1 bosons, which were known not to exist in Nature. At that time it was not known that there are two ways out of the dilemma. One is the Higgs mechanism where the gauge bosons acquire a mass by "spontaneous symmetry breaking of the local gauge symmetry". Today we know that the electroweak gauge group $SU(2)_L \otimes U(1)_Y$, of weak isospin and weak hypercharge Y, is broken down to the Abelian electromagnetic gauge group $U(1)_{em}$. Three out of the 3+1 gauge bosons acquire a mass and the remaining massless state is the photon. The weak gauge bosons $W^\pm$ and Z in fact turned out to be very heavy (about 80 GeV and 91 GeV, respectively) and were discovered in 1983 at the CERN $p\bar{p}$ collider. The LEP e^+e^- storage ring at CERN, in operation since 1989, is a Z factory and produces millions of Z's.

The other "solution" is **confinement**. Unbroken non-Abelian gauge theories are asymptotically free (Politzer, Gross and Wilzcek 1973), which means that they have small effective coupling at high energies (short distances) but strong effective coupling at low energies (large distances). We know that the strong interactions of hadrons are described by an unbroken color $SU(3)_c$ local gauge theory, called quantum chromodynamics (QCD) (Gell-Mann, Fritzsch and Leutwyler 1973). The matter fields are the colored quarks triplets which interact through the octet of massless gauge fields, called gluons. Quarks and gluons are permanently confined inside of the hadrons. This is another mechanism which hides massless gauge bosons from the physical spectrum.

Obviously the earlier conclusion that Yang-Mills theories are not of relevance for physics was premature.

Exercises: Section 4.2

① If $\boldsymbol{V}_\mu = \frac{\mathrm{i}}{g} U^{-1}(x)\,(\partial_\mu U(x))$ the field is called a " pure gauge" field. Show that in this case $\boldsymbol{G}_{\mu\nu} \equiv 0$.

② Show that a mass term $\frac{M^2}{2} \sum_i V_{i\mu} V_i^\mu$ cannot be locally gauge invariant.

③ Show that $\mathcal{L}_{\mathrm{YM}}$ can be written in the form of a trace

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2} Tr\left(\boldsymbol{G}_{\mu\nu}\boldsymbol{G}^{\mu\nu}\right)$$

④ Prove the validity of the "homogeneous Maxwell equation" or "Bianchi-identity":

$$D_\rho \boldsymbol{G}_{\mu\nu} + \quad \text{terms} \;\; \text{cyclic} \;\; \text{in} \;\; (\rho\mu\nu) \equiv 0$$

5 Quantization of gauge theories

Quantum chromodynamics, the modern theory of the strong interactions, is a non–Abelian gauge theory with gauge group $SU(3)_c$ consisting of unitary 3×3 matrices of determinant unity. The corresponding internal degrees of freedom are called *color*. The generators are given by the basis of hermitian traceless 3×3 matrices $T_i,\ i = 1, \cdots 8$. Quarks transform under the fundamental 3–dimensional representation 3 (quark triplets) antiquarks under the complex conjugate 3^* (antiquark anti–triplets). The requirement of local gauge invariance with respect to $SU(3)_c$ transformations implies that quark fields $\psi_i(x)$ must couple to an octet of gauge fields, the gluon fields $G_{\mu j},\ j = 1, \cdots, 8$, and together with the requirement of renormalizability this fixes the form of the interactions of the quarks completely: in the free quark Dirac–Lagrangian we have to replace the derivative by the *covariant derivative*

$$\partial_\mu \psi(x) \to D_\mu \psi(x)\ \ , (D_\mu)_{ik} = \partial_\mu \delta_{ik} - \mathrm{i}\, g_s \sum_j (T_j)_{ik} G_{\mu j}(x)$$

where g_s is the $SU(3)_c$ gauge coupling constant. The dynamics of the gluon fields is controlled by the non–Abelian *field strength tensor*

$$G_{\mu\nu i} = \partial_\mu G_{\nu i} - \partial_\nu G_{\mu i} + g_s c_{ijk} G_{\mu j} G_{\nu k}$$

where c_{ijk} are the $SU(3)$ structure constants obtained from the commutator of the generators $[T_i, T_j] = \mathrm{i}\, c_{ijk}\, T_k$. The locally gauge invariant Lagrangian density is then given by

$$\mathcal{L}_{\mathrm{inv}} = -\frac{1}{4} \sum_i G_{i\mu\nu}\, G_i^{\mu\nu} + \bar{\psi}\, (i\gamma^\mu D_\mu - m)\, \psi\ \ .$$

We split $\mathcal{L}_{\mathrm{inv}}$ into a free part $\mathcal{L}_0$ and an interaction part $\mathcal{L}_{\mathrm{int}}$ which is taken into account as a formal power series expansion in the gauge coupling g_s. The perturbation expansion is an expansion in terms of the free fields described by $\mathcal{L}_0$. The basic problem of quantizing massless spin 1 fields is familiar from QED. Since $\mathcal{L}_{\mathrm{YM}}$ is gauge invariant, the gauge potentials $G_{i\mu}$ cannot be uniquely determined from the gauge invariant field equations. Again one has to break the gauge invariance, now, for a $SU(n)$ gauge group, by a sum of $r = n^2 - 1$ gauge fixing conditions

$$C_i(G) = 0\ ,\ \ i = 1, \cdots, r\ \ . \tag{5.1}$$

It is known from QED that the only relativistically invariant condition linear in the gauge potential which we can write is the Lorentz condition. Correspondingly we require

$$C_i(G) = -\partial_\mu\, G_i^\mu\,(x)\ = 0\ ,\ \ i = 1, \cdots, r\ \ .$$

It should be stressed that a covariant formulation is mandatory for calculations beyond the tree level. We are thus lead to break the gauge invariance of the Lagrangian by adding the **gauge fixing** term

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi} \sum_i\, (\partial_\mu\, G_i^\mu\,(x))^2$$

with ξ a free gauge parameter. Together with the term $\mathcal{L}_0^G$ from $\mathcal{L}_{\mathrm{inv}}$ we obtain for the bilinear gauge field part

$$\mathcal{L}_{0,i}^{G,\xi} = -\frac{1}{4}\, (\partial_\mu\, G_{i\,\nu} - \partial_\nu\, G_{i\,\mu})^2 - \frac{1}{2\xi}\, (\partial_\mu\, G_i^\mu\,(x))^2$$

which now uniquely determines a free gauge field propagator. Unlike in QED, however, $\mathcal{L}_{\rm GF}$ breaks local gauge invariance explicitly and one has to restore gauge invariance by a compensating **Faddeev-Popov term** (Faddeev and Popov 1967). The Faddeev-Popov trick consists in adding further charged ghost fields $\bar{\eta}_i(x)$ and $\eta_i(x)$, the so called **Faddeev-Popov ghosts**, which conspire with the other ghosts in such a way that physical matrix elements remain gauge invariant. Unitarity and renormalizability are then restored. The FP–ghosts must be **massless spin 0 fermions**. For the unphysical ghosts this wrong spin–statistics assignment is no obstacle. The Faddeev-Popov term must be of the form

$$\mathcal{L}_{\rm FP} = \bar{\eta}_i(x) M_{ik} \eta_k(x) \tag{5.2}$$

where

$$\begin{aligned} M_{ik} &= \frac{\partial C_i(G)}{\partial G_{j\mu}(x)} \left(D_\mu\right)_{jk} = -\partial^\mu \left(\partial_\mu \delta_{ik} - g c_{ikj} G_{j\mu}(x)\right) & (5.3)\\ &= -\Box \delta_{ik} + g c_{ikj} G_{j\mu}(x)\, \partial^\mu + g c_{ikj} \left(\partial^\mu G_{j\mu}(x)\right) \quad . & (5.4) \end{aligned}$$

By partial integration of $S_{FP} = \int \mathrm{d}^4x\, \mathcal{L}_{\rm FP}(x)$ we may write

$$\mathcal{L}_{\rm FP} = \partial_\mu \bar{\eta}_i \partial^\mu \eta_i - g c_{ikj} \left(\partial^\mu \bar{\eta}_i\right) C_{j\mu} \eta_k$$

which describes massless scalar fermions in interaction with the gauge fields.

5.1 The QCD Feynman rules

The complete Lagrangian for a quantized Yang-Mills theory is

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm inv} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP} \quad .$$

The free (bilinear) part

$$\mathcal{L}_0 = \mathcal{L}_0(G) + \mathcal{L}_0(\psi) + \mathcal{L}_0(\eta) \tag{5.5}$$

with

$$\begin{aligned} \mathcal{L}_0(G) &= \frac{1}{2} G_{i\mu} \left[\left(\Box g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right) \delta_{ik} \right] G_{k\nu} & (5.6)\\ \mathcal{L}_0(\psi) &= \bar{\psi}_{\alpha a} \left[\left(\left(i\gamma^\mu\right)_{\alpha\beta} \partial_\mu - m \delta_{\alpha\beta} \right) \delta_{ab} \right] \psi_{\beta b} & (5.7)\\ \mathcal{L}_0(\eta) &= \bar{\eta}_i \left[(-\Box)\ \delta_{ik} \right] \eta_k & (5.8) \end{aligned}$$

determines the free propagators, the differential operators in the square brackets being the inverses of the propagators. By Fourier transformation the free propagators are obtained in algebraic form (i.e. the differential operators are represented by c–numbers) in momentum space. Inverting these c–number matrices we obtain the results depicted in Fig. 5.1.

The interaction part of the Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{\rm int} &= g_s \bar{\psi} \gamma^\mu T_i \psi G_{i\mu} - \frac{1}{2} g_s c_{ikl} \left(\partial^\mu G_i^\nu - \partial^\nu G_i^\mu\right)\ G_{k\mu} G_{l\nu} \\ &- \frac{1}{4} g_s^2 c_{ikl} c_{ik'l'} G_k^\mu G_l^\nu G_{k'\mu} G_{l'\nu} - g_s c_{ikj} \left(\partial^\mu \bar{\eta}_i\right)\ G_{j\mu} \eta_k \end{aligned}$$

with a single coupling constant g_s for the four different types of vertices.

While the formal argumentation which leads to the construction of local gauge theories looks not too different for Abelian and non–Abelian gauge groups, the physical consequences are very different and could not be more dramatic: in contrast to Abelian theories where the gauge field is neutral and exhibits no self–interaction, non–Abelian gauge fields necessarily carry non–Abelian charge and must be self–interacting. These Yang-Mills self–interactions are responsible for the *anti–screening* of the non–Abelian charge, known as *asymptotic freedom* (AF) (see end of section). It implies that the strong interaction force gets weaker the higher the energy, or equivalently, the shorter the distance. While it appears most natural to us that particles interact the less the farther apart they are, non–Abelian forces share the opposite property, the forces get the stronger the farer away we try to separate the quarks. In QCD this leads to the *confinement* of the constituents within hadrons. The latter being quark bound states which can never be broken up into free constituents. This makes QCD an intrinsically non–perturbative theory, the fields in the Lagrangian, quarks and gluons, never appear in scattering states, which define the physical state space and the S–matrix. QED is very different, it has an perturbative S-matrix, its proper definition being complicated by the existence of the long range Coulomb forces. Nevertheless, the fields in the QED Lagrangian as interpolating fields are closely related to the physical states, the photons and leptons. This extends to the electroweak SM, where the weak non–Abelian gauge bosons, the $W^{\pm}$ and the Z particles, become massive as a consequence of the breakdown of the $SU(2)_L$ gauge symmetry by the Higgs mechanism. Also the weak gauge bosons cannot be seen as scattering states in a detector, but this time because of their very short lifetime. Due to its non–perturbative nature, precise predictions in strong interaction physics are often difficult, if not impossible. Fortunately, besides perturbative QCD which applies to hard subprocesses, non–perturbative methods have been developed to a high level of sophistication, like *chiral perturbation theory* (CHPT) and QCD on a Euclidean space–time lattice (*lattice QCD*). Chiral perturbation theory is based on the low energy structure of QCD: in the limit of vanishing quark masses QCD has a global $SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V$ symmetry (*chiral symmetry*). Thereby the $SU(N_f)_A$ subgroup turns out broken spontaneously, which, in the isospin limit $N_f = 2$, $m_u = m_d = 0$, implies the existence of a triplet of massless pions (Goldstone bosons). $U(1)_V$ is responsible for *baryon number conservation*, whereas in contrast $U(1)_A$ is broken by the *Adler-Bell-Jackiw anomaly*.

The derivation of the proper quantization of QCD has been very formal and needs better justification. In fact the products of field operators, which are used in particular in the Lagrangian are not well defined on the quantum level. This becomes immediately clear when we try calculations of leading quantum corrections. What one has to do first is a proper definition of the theory, means the theory has to be regularized, such that singularities are absent at least order by order in the perturbative expansion. Our aim is to discuss dimensional regularization before we are going to calculate one–loop counterterms and leading renormalization group coefficients, which play a key role in any phenomenological application of QCD.

Exercises: Section 5

① Spin 1 propagators are symmetric second rank Lorentz tensors proportional to $g^{\mu\nu}$ and $k^\mu k^\nu$, where k is the four momentum of the spin 1 boson. Show that

$$T^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \; ; \;\; L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}$$

are projection tensors to the **transversal** and the **longitudinal** field components. Verify

the projector property of T and L. Use the decomposition

$$K^{\mu\nu} = A\,T^{\mu\nu} + B\,L^{\mu\nu}$$

for the bilinear part of the $\mathcal{L}_{0G} = G_\mu K^{\mu\nu} G_\nu$ of the free part of the Lagrangian (what are A and B in case of the gluon), calculate the propagator (inverse of the kernel K) using the projection technique. Formulate the condition under which the inverse exists.

② Given the Yang-Yills Lagrangian in coordinate space derive the triple gluon coupling

$$-\mathrm{i}g_s c_{ijk}\,\{g^{\mu\nu}\,(p_2-p_1)^\rho \;+\; g^{\mu\rho}\,(p_1-p_3)^\nu \;+\; g^{\nu\rho}\,(p_3-p_2)^\mu\}$$

and the quartic gluon coupling

$$-g_s^2\,\{c_{nij}c_{nkl}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + c_{nik}c_{njl}\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + c_{nil}c_{njk}\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\}$$

in momentum space.

③ According to Faddeev and Popov the fatal violation of gauge invariance by the gauge fixing term can be avoided by taking into account the functional determinant obtained in the functional integral under a gauge transformation of the fields (integration variables). If we define the functional integral as follows, with a Faddeev-Popov determinant,

$$\int \mathcal{D}G_{\mu a}\mathrm{Det}\left(\frac{\delta C_a}{\delta\omega_b}\right)\mathrm{e}^{\mathrm{i}\int\left(\mathcal{L}_{\mathrm{inv}}-\frac{1}{2\xi}C_a^2\right)\mathrm{d}^4x}$$

one easily checks that now the functional integral is independent on the specific choice of the gauge function C_a. Check this!

By introducing anticommuting scalars, the FP ghost fields $\bar\eta_a$ and η_a, we may represent the FP-determinant as a Berezin integral over Grassmann variables (algebra of anticommuting c-numbers)

$$\mathrm{Det}\left(\frac{\delta C_a}{\delta\omega_b}\right) = \int \mathcal{D}\eta\mathcal{D}\bar\eta\mathrm{e}^{\mathrm{i}\int\mathcal{L}_{\mathrm{FP}}\mathrm{d}^4x}$$

with

$$\mathcal{L}_{\mathrm{FP}} = \bar\eta_a M_{ab}\eta_b\;;\quad M_{ab} \doteq \frac{\delta C_a}{\delta\omega_b}\quad .$$

Check that this agrees with the expression adopted above.

④ **Becchi-Rouet-Stora (BRS) symmetry:** given the "quasi invariant" effective Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{inv}} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}}\;,$$

the local gauge invariance of the functional integral (path integral quantization)

$$\int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar\eta\,\mathrm{e}^{\mathrm{i}\int\mathcal{L}_{\mathrm{eff}}\mathrm{d}^4x} \tag{5.9}$$

yields relations between Green functions, the Slavnov-Taylor (ST) identities. They generalize the Ward-Takahashi (WT) identities, relations between Green functions which derive from global symmetries. In contrast to canonical quantization, which relies on the non gauge-invariant splitting into a free and an interacting part of the Lagrangian, the path integral formulation work with the complete effective Lagrangian.

The ST-identities provide the tool needed for proofs of

i) gauge invariance

ii) unitarity

iii) renormalizability

of the S-matrix. ST-identities may be obtained from the BRS-symmetry of $\mathcal{L}_{\text{eff}}$.

Convince yourself that the BRS procedure (described in the following) to control gauge invariance indeed works. The idea behind BRS-symmetry is to dispose of the as yet undefined transformation properties of the FP-ghost fields $\bar{\eta}$ and η such that

$$\delta^{\text{BRS}}\mathcal{L}_{\text{eff}} = 0 \ . \tag{5.10}$$

In order to achieve this it is natural to demand the following relations to hold:

$$\begin{aligned}
&i) \quad \delta\mathcal{L}_{\text{inv}} &&= 0 \\
&ii) \quad \delta\mathcal{L}_{\text{GF}} &&= -\tfrac{1}{\xi}C_a M_{ab}\omega_b \\
&iii) \quad \delta\mathcal{L}_{\text{FP}} &&= \delta\bar{\eta}M_{ab}\eta_b + \bar{\eta}_a\delta(M_{ab}\eta_b) \\
&\qquad\qquad &&= -\delta\mathcal{L}_{\text{GF}} \\
&iv) \quad \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta} &&\text{ invariant .}
\end{aligned}$$

A solution for this set of conditions may be obtained as follows:
1) Introduce anticommuting global $c-$number variables $\delta\lambda$, $\delta\lambda$ anticommuting with η and $\bar{\eta}$, and identify $\omega_b = \eta_b\delta\lambda$. Thus

- $\delta G_a = D_{ab}\eta_b\delta\lambda$

where G_a can be a gauge field, a scalar or a Fermi field with $D_{ab}\eta_b$ given by

$$\frac{\delta G_{a\mu}}{\delta\omega_b}\eta_b \doteq D_{\mu ab}\eta_b$$

in case of the gauge field.

2) Assume η to transform according to the regular representation, thus

- $\delta\eta_a = -\frac{1}{2}gf_{abc}\eta_b\eta_c\delta\lambda$

where a permutation symmetry factor 1/2 (antisymmetry of f_{abc} and anticommutativity of the η's) has been taken into account. 1) and 2) imply $\delta(M_{ab}\eta_b) = 0$. We thus take the freedom to choose: 3) The field $\bar{\eta}$ transforms as

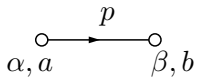
- $\delta\bar{\eta}_a = -\frac{1}{\xi}C_a\delta\lambda$

such that conditions i) to iii) are satisfied. One can show iv) to be true for the above set of transformations which define the BRS-transformation.

ST-identities:

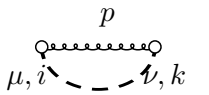
The *BRS* invariance of $\mathcal{L}_{\text{eff}}$ allows a simple derivation of the ST-identities. Performing a change of integration variables in the functional integral does not change the value of the integral. See my TASI Lectures *Renormalizing the Standard Model* for more details.

a). Quark propagator



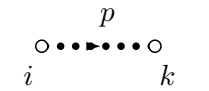
$$: \qquad \tilde{\Delta}_F^{\psi}(p)_{\alpha\beta,\,ab} = \left(\frac{1}{\not{p}-m+\mathrm{i}\varepsilon}\right)_{\alpha\beta}\delta_{ab}$$

b). Massless gluon propagator



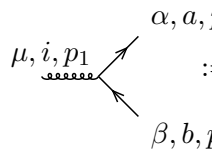
$$: \qquad \tilde{\Delta}_F^{G}(p,\xi)_{ik}^{\mu\nu} = -\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2+\mathrm{i}\varepsilon}\,\delta_{ik}$$

c). Massless FP–ghost propagator



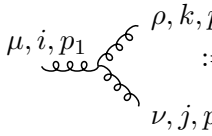
$$: \qquad \tilde{\Delta}_F^{\eta}(p)_{ik} = \frac{1}{p^2+\mathrm{i}\varepsilon}\,\delta_{ik}$$

d). Quark–gluon coupling



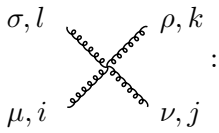
$$:= \qquad g_s\,(\gamma^\mu)_{\alpha\beta}\,(T_i)_{ab}$$

e). Triple gluon coupling



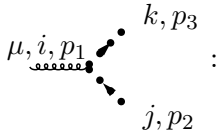
$$:= \quad -\mathrm{i}g_s c_{ijk}\left\{g^{\mu\nu}\,(p_2-p_1)^\rho \;+\; g^{\mu\rho}\,(p_1-p_3)^\nu \;+\; g^{\nu\rho}\,(p_3-p_2)^\mu\right\}$$

f). Quartic gluon coupling



$$:= \qquad -g_s^2\left\{\begin{array}{r} c_{nij}c_{nkl}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ +c_{nik}c_{njl}\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ +c_{nil}c_{njk}\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}) \end{array}\right.$$

g). FP–ghost gluon coupling



$$:= \qquad -\mathrm{i}g_s c_{ijk}\,(p_3)^\mu$$

Figure 5.1: Feynman rules for QCD. Momenta at vertices are chosen ingoing

6 Dimensional regularization

In local QFT field products (monomials in the fields) which show up at the classical Lagrangian level (usually the starting point for the quantization) are ill defined due to short distance singularities showing up when going to coinciding space–time points: $\lim_{x\to y}\varphi(x)\varphi(y)$.

When calculating perturbative contributions, to a Green function say, associated with a particular Feynman diagram Γ one has to consider an expression like

$$\Pi_{V\in\Gamma}\, P_V(\partial)\,\Pi_{\ell\in\Gamma}\, P_\ell(\partial)\,\Delta_{\rm F}(x_{i\ell}-x_{f\ell}) \tag{6.1}$$

to be integrated over all interaction vertices $V\in\Gamma$. The lines labeled by $\ell\in\Gamma$ represent the Feynman propagators $P_\ell(\partial)\,\Delta_{\rm F}(x_{i\ell}-x_{f\ell})$. The polynomial P_ℓ depends of the spin of the particle and the polynomial P_V represents the possible derivative couplings at the vertex V (in momentum space $P_\ell(\partial)$ and $P_V(\partial)$ are polynomials of the appropriate momenta). All singularities in coordinate space are at coinciding points (representing the vertices), which means that the problem concerns a local distribution

$$\Pi_\ell P_\omega(\partial)\,\delta(x_{i\ell}-x_{f\ell})\ , \tag{6.2}$$

where $P_\omega(\partial)$ is a polynomial in the derivatives with respect to the arguments of the delta functions. For a renormalizable QFT the degree of the polynomial is not higher than 2 (for a proof see *Introduction to the theory of quantized fields* by N.N. Bogolyubov, D.V. Shirkov. 1980.) and subtracting (6.2) from (6.1) with appropriate values for the coefficients of the polynomial, removes all singularities and determines the renormalized QFT order by order in perturbation theory. His means that the bare theory is well defined almost everywhere, except from a low dimensional subspace of coinciding vertex points. Renormalization we call the procedure to make the theory well defined everywhere, i.e. to perform a continuous extension to the subspace.

In momentum space the short distance singularities show up as ultraviolet (UV) singularities, which manifest themselves as a polynomial in the momenta (at most quadratic in a renormalizable theory), however with singular coefficients. In order to get the perturbative expansion well defined, the theory has to be regularized such that one has a well defined staring point. For non-Abelian gauge theories one needs a regularization which preserves in particular the gauge symmetry as much as possible. Dimensional regularization makes use of the fact that a QFT becomes less singular as the dimension of space–time gets lower (than 4), without changing basic symmetry properties (identical Feynman rules) of the theory. In the following we describe in short the basic mechanism how it works.

Dimensional regularization of theories with spin is defined in three steps.

1. Start with Feynman rules formally derived in $d=4$.

2. Generalize to $d=2n>4$. This intermediate step is necessary in order to treat the vector and spinor indices appropriately. Of course it means that the UV behavior of Feynman integrals at first gets worse.

1) For fermions we need the $d=2n$–dimensional Dirac algebra:

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}\mathbf{1}\ ;\ \ \{\gamma^\mu,\gamma_5\}=0$$

where γ_5 must satisfy $\gamma_5^2=\mathbf{1}$ and $\gamma_5^+=\gamma_5$ such that $\frac{1}{2}(\mathbf{1}\pm\gamma_5)$ are the chiral projection matrices.

The metric has dimension d

$$g^{\mu\nu}g_{\mu\nu} = g^{\mu}_{\mu} = d \;\; ; \;\; g_{\mu\nu} = \begin{pmatrix} 1 & 0 & \cdots & \\ 0 & -1 & & \\ \vdots & & \ddots & \\ & & & -1 \end{pmatrix} \quad . \tag{6.3}$$

By $\mathbf{1}$ we denote the unit matrix in spinor space. In order to have the usual relation for the adjoint spinors we furthermore require

$$\gamma^{\mu +} = \gamma^0\gamma^\mu\gamma^0 \quad .$$

Simple consequences of this d–dimensional algebra are:

$$\begin{aligned} \gamma_\alpha\gamma^\alpha &= d\,\mathbf{1} \\ \gamma_\alpha\gamma^\mu\gamma^\alpha &= (2-d)\,\gamma^\mu \\ \gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\alpha &= 4g^{\mu\nu}\,\mathbf{1} + (d-4)\,\gamma^\mu\gamma^\nu \\ \gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\alpha &= -2\gamma^\rho\gamma^\nu\gamma^\mu + \,(4-d)\gamma^\mu\gamma^\nu\gamma^\rho \text{ etc.} \end{aligned} \tag{6.4}$$

Traces of strings of γ–matrices are very similar to the ones in 4–dimensions. In $d = 2n$ dimensions one can easily write down $2^{d/2}$–dimensional representations of the Dirac algebra. Then

$$\begin{aligned} \mathrm{Tr}(\mathbf{1}) &= f(d) = 2^{d/2} \\ \mathrm{Tr}\prod_{i=1}^{2n-1}\gamma^{\mu_i}(\gamma^5) &= 0 \\ \mathrm{Tr}\,\gamma^\mu\gamma^\nu &= f(d)\,g^{\mu\nu} \\ \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= f(d)\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \;\text{ etc.} \end{aligned} \tag{6.5}$$

One can show that for *renormalized quantities* the only relevant property of $f(d)$ is $f(d) \to 4$ for $d \to 4$. Very often the convention $f(d) = 4$ (for any d) is adopted. Bare quantities and the related *minimally subtracted* MS or modified minimally subtracted $\overline{\text{MS}}$ quantities (see below for the precise definition) depend upon this convention (by terms proportional to $\ln 2$).

In anomaly free theories we can assume γ_5 to be fully anticommuting! But then

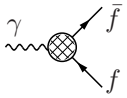
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 = 0 \;\text{ for all }\; d \neq 4! \tag{6.6}$$

The 4–dimensional object

$$4\mathrm{i}\varepsilon^{\mu\nu\rho\sigma} = \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 \;\text{ for }\; d = 4 \tag{6.7}$$

cannot be obtained by dimensional continuation if we use an anticommuting γ_5 (see Jegerlehner 2001 and references therein).

Since fermions do not have self interactions they only appear as closed fermion loops, which yield a trace of γ–matrices, or as a fermion string connecting an external $\psi\cdots\bar{\psi}$ pair of fermion fields. In a transition amplitude $|T|^2 = \mathrm{Tr}\,(\cdots)$ we again get a trace. Consequently, in principle, we have eliminated all γ's! Commonly one writes a covariant tensor decomposition into invariant amplitudes, like, for example,



$$= \mathrm{i}\Gamma^\mu = -\mathrm{i}e\left\{\gamma^\mu A_1 + \mathrm{i}\sigma^{\mu\nu}\frac{q_\nu}{2m}A_2 + \gamma^\mu\gamma_5 A_3 + \cdots\right\}$$

where μ is an external index, q^μ the photon momentum and $A_i(q^2)$ are scalar form factors.

2) External momenta (and external indices) must be taken $d = 4$ dimensional, because the number of independent "form factors" in covariant decompositions depends on the dimension, with a fewer number of independent functions in lower dimensions. Since four functions cannot be analytic continuation of three etc. we have to keep the external structure of the theory in $d = 4$. The reason for possible problems here is the non–trivial spin structure of the theory of interest. The following rules apply:

$$\begin{array}{lll} \text{External momenta :} & p^\mu = (p^0, p^1, p^2, p^3, 0, \cdots, 0) & 4 - \text{dimensional} \\ \text{Loop momenta :} & k^\mu = (k^0, \cdots k^{d-1}) & d - \text{dimensional} \\ & k^2 = (k^0)^2 - (k^1)^2 - \cdots - (k^{d-1})^2 & \\ & pk = p^0 k^0 - \vec{p} \cdot \vec{k} & 4 - \text{dimensional etc.} \end{array}$$

3. Interpolation in d to complex values and extrapolation to $d < 4$.

Loop integrals now read

$$\mu^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \cdots$$

with μ an arbitrary scale parameter. The crucial properties valid in DR independent of d are: (F.P. = finite part)

a) $\int \mathrm{d}^d k k_\mu f(k^2) = 0$

b) $\int \mathrm{d}^d k f(k+p) = \int \mathrm{d}^d k f(k)$
which is not true with UV cut − off′s

c) If $f(k) = f(\mid k \mid)$:
$\int \mathrm{d}^d k f(k) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty \mathrm{d}r r^{d-1} f(r)$

d) For divergent integrals, by analytic subtraction :
F.P. $\int_0^\infty \mathrm{d}r r^{d-1+\alpha} \equiv 0$ for arbitrary α
so called *minimal subtraction* (MS). Consequently

$$\text{F.P.} \int \mathrm{d}^d k f(k) = \text{F.P.} \int \mathrm{d}^d k f(k+p) = \text{F.P.} \int \mathrm{d}^d(\lambda k) f(\lambda k) \ .$$

This implies that
dimensionally regularized integrals behave like convergent integrals
and formal manipulations are justified. Starting with d sufficiently small, by partial integration, one can always find a representation for the integral which converges for $d = 4 - \epsilon$, $\epsilon > 0$ small.

In order to elaborate in more detail how DR works in practice, let us consider a generic one–loop Feynman integral

$$I_\Gamma^{\mu_1 \cdots \mu_m}(p_1, \cdots, p_n) = \int \mathrm{d}^d k \frac{\prod_{j=1}^m k^{\mu_j}}{\prod_{i=1}^n ((k+p_i)^2 - m_i^2 + \mathrm{i}\varepsilon)} \tag{6.8}$$

which has superficial degree of divergence

$$d(\Gamma) = d + m - 2n \leq d - 2 \tag{6.9}$$

where the bound holds for two– or more–point functions in renormalizable theories and for $d \leq 4$. Since the physical tensor and spin structure has to be kept in $d = 4$, by contraction with external momenta or with the metric tensor $g_{\mu_i\mu_j}$ it is always possible to write the above integral as a sum of integrals of the form

$$I_\Gamma^{\hat{\mu}_1 \cdots \hat{\mu}_{m'}}(\hat{p}_1, \cdots, \hat{p}_{n'}) = \int \mathrm{d}^d k \frac{\prod_{j=1}^{m'} \hat{k}^{\mu_j}}{\prod_{i=1}^{n'}((k+\hat{p}_i)^2 - m_i^2 + \mathrm{i}\varepsilon)} \tag{6.10}$$

where now $\hat{\mu}_j$ and $\hat{p}_i$ are $d = 4$–dimensional objects and

$$\mathrm{d}^d k = \mathrm{d}^4\hat{k}\, \mathrm{d}^{d-4}\bar{k} = \mathrm{d}^4\hat{k}\, \omega^{d-5}\, \mathrm{d}\omega\, \mathrm{d}\Omega_{d-4} \ . \tag{6.11}$$

In the $d-4$–dimensional complement the integrand depends on ω only! The angular integration over $\mathrm{d}\Omega_{d-4}$ yields

$$\int \mathrm{d}\Omega_{d-4} = S_{d-4} = \frac{2\pi^{\epsilon/2}}{\Gamma(\epsilon/2)} \ ; \ \ \epsilon = d-4 \ , \tag{6.12}$$

which is the surface of the $d-4$–dimensional sphere. Using this result we get (discarding the four–dimensional tensor indices)

$$I_\Gamma(\{\hat{p}_i\}) = \int \mathrm{d}^4\hat{k} J_\Gamma(d, \hat{p}, \hat{k}) \tag{6.13}$$

where

$$J_\Gamma(d, \hat{p}, \hat{k}) = S_{d-4} \int_0^\infty \mathrm{d}\omega \omega^{d-5} f(\hat{p}, \hat{k}, \omega) \ . \tag{6.14}$$

Now this integral can be analytically continued to complex values of d. For the ω–integration we have

$$d^\omega(\Gamma) = d - 4 - 2n \tag{6.15}$$

i.e. the ω–integral converges if

$$d < 4 + 2n \ . \tag{6.16}$$

In order to avoid infrared singularities in the ω–integration one has to analytically continue by appropriate partial integration. After p–fold partial integration we have

$$I_\Gamma(\{\hat{p}_i\}) = \frac{2\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d-4}{2}+p)} \int \mathrm{d}^4\hat{k} \int_0^\infty \mathrm{d}\omega \omega^{d-5+2p} \left(-\frac{\partial}{\partial\omega^2}\right)^p f(\hat{p}, \hat{k}, \omega) \tag{6.17}$$

where the integral is convergent in $4 - 2p < \mathrm{Re}\ d < 2n - m = 4 - d^{(4)}(\Gamma) \geq 2$.
For a renormalizable theory at most 2 partial integrations are necessary to define the theory.

6.1 Tools for the Evaluation of Feynman Integrals

6.1.1 $\epsilon = 4 - d$ Expansion, $\epsilon \to +0$

For the expansion of integrals near $d = 4$ we need some asymptotic expansions of Γ–functions:

$$\Gamma(1+x) \ = \ \exp\left[-\gamma\, x + \sum_{n=2}^\infty \frac{(-1)^n}{n} \zeta(n) x^n\right] \qquad |x| \leq 1 \tag{6.18}$$

$$\psi(1+x) \ = \ \frac{d}{\mathrm{d}x} \ln\Gamma(1+x) = \frac{\Gamma'(1+x)}{\Gamma(1+x)} \stackrel{|x|<1}{=} -\gamma + \sum_{n=2}^\infty (-1)^n \zeta(n) x^{n-1} \tag{6.19}$$

where $\zeta(n)$ denotes Riemann's Zeta function. The defining functional relation is

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \ , \tag{6.20}$$

which for $n = 0, 1, 2, \cdots$ yields $\Gamma(n+1) = n!$ with $\Gamma(1) = \Gamma(2) = 1$. Furthermore we have

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{6.21}$$

$$\Gamma(\frac{1}{2}+x)\,\Gamma(\frac{1}{2}-x) = \frac{\pi}{\cos \pi x} \ . \tag{6.22}$$

Important special constants are

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \tag{6.23}$$

$$\Gamma'(1) = -\gamma \ ; \ \ \gamma = 0.577215\cdots \ \text{Euler}'\text{s constant} \tag{6.24}$$

$$\Gamma''(1) = \gamma^2 + \zeta(2) \ ; \ \ \zeta(2) = \frac{\pi^2}{6} = 1.64493\cdots \tag{6.25}$$

As a typical result of an ϵ–expansion, which we should keep in mind for later purposes, we have

$$\Gamma(1+\frac{\epsilon}{2}) = 1 - \frac{\epsilon}{2}\gamma + (\frac{\epsilon}{2})^2 \frac{1}{2}(\gamma^2 + \zeta(2)) + \cdots \tag{6.26}$$

6.1.2 Bogolubov-Schwinger Parametrization

Suppose we choose for each propagator an independent momentum and take into account momentum conservation at the vertices by δ–functions. Then, for $d = n$ integer, we use

i)

$$\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} = \int_0^\infty \mathrm{d}\alpha \, \mathrm{e}^{-\mathrm{i}\alpha(m^2 - p^2 + \mathrm{i}\varepsilon)}$$

ii)

$$\delta^{(d)}(k) = \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} \mathrm{d}^d x \, \mathrm{e}^{\mathrm{i}kx}$$

and find that all momentum integrations are of Gaussian type. The Gaussian integrals yield

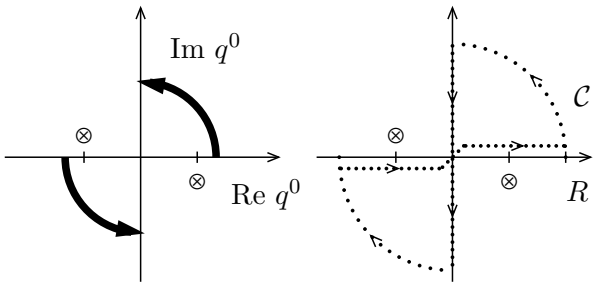
$$\int_{-\infty}^{+\infty} \mathrm{d}^d k P(k) \mathrm{e}^{\mathrm{i}(ak^2 + 2b(k\cdot p))} = P\left(\frac{-\mathrm{i}}{2b}\frac{\partial}{\partial p}\right)\left(\frac{\pi}{\mathrm{i}a}\right)^{d/2} \mathrm{e}^{-\mathrm{i}\, b^2/a\, p^2}$$

for any polynomial P. The resulting form of the Feynman integral is the so called Bogolubov-Schwinger representation.

6.1.3 Feynman Parametric Representation

Transforming pairs of α–variables in the above Bogolubov-Schwinger parametrization according to (l is denoting the pair (i,k))

$$(\alpha_i, \alpha_k) \to (\xi_l, \alpha_l) : (\alpha_i, \alpha_k) = (\xi_l \alpha_l, (1-\xi_l)\alpha_l)$$



(6.28)

Figure 6.1: Wick rotation in the complex q^0–plane. The poles of the Feynman propagator are indicated by $\otimes$'s. $\mathcal{C}$ is an integration contour, R is the radius of the arcs.

$$\int_0^\infty \int_0^\infty \mathrm{d}\alpha_i \mathrm{d}\alpha_k \cdots = \int_0^\infty \mathrm{d}\alpha_l \, \alpha_l \int_0^1 \mathrm{d}\xi_l \cdots \quad ,$$

the integrals are successively transformed into $\int_0^1 \mathrm{d}\xi \cdots$ integrals and at the end there remains <u>one</u> α–integration only which can be performed using

$$\int_0^\infty \mathrm{d}\alpha \, \alpha^a \, \mathrm{e}^{-\alpha x} = \Gamma(a+1) x^{-(a+1)} \quad .$$

The result is the Feynman parametric representation. If L is the number of lines of a diagram, the Feynman integral is $L-1$–dimensional.

6.1.4 Euclidean Region, Wick–Rotations

The basic property which allows us to perform a Wick rotation is analyticity which derives from the causality of a relativistic QFT. In momentum space the Feynman propagator

$$\begin{aligned} \frac{1}{q^2 - m^2 + \mathrm{i}\varepsilon} &= \frac{1}{q^0 - \sqrt{\vec{q}^{\,2} + m^2 - \mathrm{i}\varepsilon}} \, \frac{1}{q^0 + \sqrt{\vec{q}^{\,2} + m^2 - \mathrm{i}\varepsilon}} \\ &= \frac{1}{2\omega_p} \left\{ \frac{1}{q^0 - \omega_p + \mathrm{i}\varepsilon} - \frac{1}{q^0 + \omega_p - \mathrm{i}\varepsilon} \right\} \end{aligned}$$

is an analytic function in q^0 with poles at $q^0 = \pm(\,\omega_p - \mathrm{i}\varepsilon)$[21] where $\omega_p = \sqrt{\vec{q}^{\,2} + m^2}$. This allows us to rotate by $\frac{\pi}{2}$ the integration path in q^0, going from $-\infty$ to $+\infty$, without crossing any singularity. In doing so, we rotate from Minkowski space to Euclidean space

$$q^0 \to -\mathrm{i} q^d \Rightarrow q = (q^0, q^1, \ldots, q^{d-2}, q^{d-1}) \to \underline{q} = (q^1, q^2, \ldots, q^{d-1}, q^d) \tag{6.27}$$

and thus $q^2 \to -\underline{q}^2$. This rotation to the Euclidean region is called **Wick rotation**.

More precisely: analyticity of a function $\tilde{f}(q^0, \vec{q}\,)$ in q^0 implies that the **contour integral**

$$\oint_{\mathcal{C}(R)} \mathrm{d}q^0 \, \tilde{f}(q^0, \vec{q}\,) \, = 0$$

[21] Note that because of the positivity of $\vec{q}^{\,2} + m^2$ for any non–vacuum state, we have $\omega_p - \mathrm{i}\varepsilon = \sqrt{\vec{q}^{\,2} + m^2 - \mathrm{i}\varepsilon}$ in the limit $\lim_{\varepsilon \to 0}$, which is always understood. The symbolic parameter ε of the iε prescription, may be scaled by any fixed positive number.

for the closed path $C(R)$ in Fig. 6.1 vanishes. If the function $\tilde{f}(q^0, \vec{q}\,)$ falls off sufficiently fast at infinity, then the contribution from the two "arcs" goes to zero when the radius of the contour $R \to \infty$. In this case we obtain

$$\int\limits_{-\infty}^{\infty} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) + \int\limits_{+\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) = 0$$

or

$$\int\limits_{-\infty}^{\infty} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) = \int\limits_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}q^0\, \tilde{f}(q^0, \vec{q}\,) = -\mathrm{i} \int\limits_{-\infty}^{+\infty} \mathrm{d}q^d\, \tilde{f}(-\mathrm{i}q^d, \vec{q}\,) \ ,$$

which is the Wick rotation. At least in perturbation theory, one can prove that the conditions required to allow us to perform a Wick rotation are fulfilled.

We notice that the Euclidean Feynman propagator obtained by the Wick rotation

$$\frac{1}{q^2 - m^2 + \mathrm{i}\varepsilon} \to -\frac{1}{\underline{q}^2 + m^2} \tag{6.29}$$

has no singularities (poles) and an iε–prescription is not needed any longer.

In configuration space a Wick rotation implies going to **imaginary time** $x^0 \to \mathrm{i}x^0 = x^d$ such that $qx \to -\underline{q}\underline{x}$ and hence

$$x^0 \to -\mathrm{i}x^d \Rightarrow x^2 \to -\underline{x}^2 \ , \quad \Box_x \to -\Delta_{\underline{x}} \ , \quad \mathrm{i}\int \mathrm{d}^d x \cdots \to \int \mathrm{d}^d \underline{x} \cdots \ . \tag{6.30}$$

While in Minkowski space $x^2 = 0$ defines the light–cone $x^0 = \pm|\vec{x}|$, in the Euclidean region $\underline{x}^2 = 0$ implies $\underline{x} = 0$. Note that possible singularities on the light–cone like $1/x^2$, $\delta(x^2)$ etc. turn into singularities at the point $\underline{x} = 0$. This simplification of the singularity structure is the merit of the positive definite metric in Euclidean space.

In momentum space the Euclidean propagators are positive (discarding the overall sign) and any Feynman amplitude in Minkowski space may be obtained via

$$I_M(p) = (-\mathrm{i})^{N_{\mathrm{int}}} (-\mathrm{i})^{V-1}\, I_E(\underline{p}\,)\,\big|_{p^4 = \mathrm{i}p^0\ ;\ m^2 \to m^2 - \mathrm{i}\varepsilon} \tag{6.31}$$

from its Euclidean version. Here, N_{int} denotes the number of internal lines (propagators) and V the number of vertices if we use the substitutions (convention dependent)

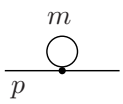
$$\frac{1}{p^2 - m^2 + \mathrm{i}\varepsilon} \to \frac{1}{\underline{p}^2 + m^2} \ ; \ \ \mathrm{i}g_i \to \mathrm{i}\,(\mathrm{i}g_i) = -g_i \ ; \ \ \int \mathrm{d}^d k \to \int \mathrm{d}^d \underline{k}$$

to define the Euclidean Feynman amplitudes. By g_i we denote the gauge couplings.

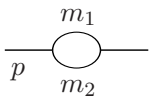
For the dimensionally regularized amplitudes, where potentially divergent integrals are defined via analytic continuation from regions in the complex d–plane where integrals are manifestly convergent, the terms from the arc segments can always be dropped. Also note that dimensional regularization and the power counting rules (superficial degree of divergence etc.) hold irrespective of whether we work in d–dimensional Minkowski space–time or in d–dimensional Euclidean space. The metric is obviously not important for the UV–behavior of the integrals.

6.2 Scalar One–Loop Integrals

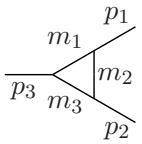
Here we apply our tools to the simplest scalar one–loop integrals (p.i.= partial integration).



$$
\begin{aligned}
&= \tfrac{\mu^{4-d}}{(2\pi)^d}\int \mathrm{d}^d\underline{k}\,\tfrac{1}{\underline{k}^2+m^2} = \mu^{4-d}(4\pi)^{-d/2}\int_0^\infty \mathrm{d}\alpha\alpha^{-d/2}\mathrm{e}^{-\alpha m^2}\\
&\qquad \text{convergent for } d<2 \quad ***\,^{1}\\
&\overset{p.i.}{=} -\tfrac{2m^2}{d-2}\mu^{4-d}(4\pi)^{-d/2}\int_0^\infty \mathrm{d}\alpha^{1-d/2}\mathrm{e}^{-\alpha m^2}\\
&\qquad \text{convergent for } d<4\\
&= -2m^2(4\pi)^{-d/2}\tfrac{\Gamma(2-d/2)}{d-2}\left(\tfrac{m^2}{\mu^2}\right)^{d/2-2}\\
&= -2m^2(4\pi)^{-2}\tfrac{2}{\epsilon}\Gamma(1+\tfrac{\epsilon}{2})\tfrac{1}{2-\epsilon}\mathrm{e}^{\frac{\epsilon}{2}(\ln 4\pi-\ln\frac{m^2}{\mu^2})}\\
&\overset{\epsilon\to+0}{\simeq} m^2(4\pi)^{-2}\left\{\tfrac{2}{\epsilon}-\gamma+1+\ln 4\pi-\ln\tfrac{m^2}{\mu^2}\right\}+O(\epsilon)
\end{aligned}
$$



$$
\begin{aligned}
&= \tfrac{\mu^{4-d}}{(2\pi)^d}\int \mathrm{d}^dk\,\tfrac{1}{\underline{k}^2+m_1^2}\,\tfrac{1}{(\underline{k}+\underline{p})^2+m_2^2}\\
&= \mu^{4-d}(4\pi)^{-d/2}\int_0^\infty \mathrm{d}\alpha_1\mathrm{d}\alpha_2(\alpha_1+\alpha_2)^{-d/2}\mathrm{e}^{-(\alpha_1m_1^2+\alpha_2m_2^2+\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\underline{p}^2)}\\
&\qquad \alpha_1=x\lambda\,;\;\; \alpha_2=(1-x)\lambda\\
&= \mu^{4-d}(4\pi)^{-d/2}\Gamma(2-\tfrac{d}{2})\int_0^1 \mathrm{d}x(xm_1^2+(1-x)m_2^2+x(1-x)\underline{p}^2))^{d/2-2}\\
&\qquad \text{convergent for } d<4\\
&= (4\pi)^{-2}\tfrac{2}{\epsilon}\Gamma(1+\tfrac{\epsilon}{2})\mathrm{e}^{\frac{\epsilon}{2}\ln 4\pi}\int_0^1 \mathrm{d}x\mathrm{e}^{-\frac{\epsilon}{2}\ln\frac{xm_1^2+(1-x)m_2^2+x(1-x)\underline{p}^2}{\mu^2}}\\
&\overset{\epsilon\to+0}{\simeq} (4\pi)^{-2}\left\{\tfrac{2}{\epsilon}-\gamma+\ln 4\pi-\int_0^1 \mathrm{d}x\ln\tfrac{xm_1^2+(1-x)m_2^2+x(1-x)\underline{p}^2}{\mu^2}\right\}+O(\epsilon)
\end{aligned}
$$



$$
\begin{aligned}
&= \tfrac{\mu^{4-d}}{(2\pi)^d}\int \mathrm{d}^d\underline{k}\,\tfrac{1}{\underline{k}^2+m_1^2}\,\tfrac{1}{(\underline{k}+\underline{p}_1)^2+m_2^2}\,\tfrac{1}{(\underline{k}+\underline{p}_1+\underline{p}_2)^2+m_3^2}\\
&\qquad \text{convergent for } d=4\\
&\overset{\epsilon\to+0}{\simeq} (4\pi)^{-2}\int_0^\infty \mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\tfrac{1}{(\alpha_1+\alpha_2+\alpha_3)^2}\mathrm{e}^{-(\alpha_1m_1^2+\alpha_2m_2^2+\alpha_3m_3^2)}\\
&\qquad \times\mathrm{e}^{-\frac{\alpha_1\alpha_2\underline{p}_1^2+\alpha_2\alpha_3\underline{p}_2^2+\alpha_3\alpha_1\underline{p}_3^2}{\alpha_1+\alpha_2+\alpha_3}}\\
&\qquad \alpha_1=xy\lambda\,;\;\; \alpha_2=x(1-y)\lambda\,;\;\; \alpha_3=(1-x)\lambda\,;\;\; \alpha_1+\alpha_2+\alpha_3=\lambda\\
&= (4\pi)^{-2}\int_0^1 \mathrm{d}y\mathrm{d}x x\tfrac{1}{N}
\end{aligned}
$$

$$N = x^2y\,(1-y)\underline{p}_1^2 + x\,(1-x)(1-y)\underline{p}_2^2 + x\,(1-x)\,y\underline{p}_3^2 + xym_1^2 + x\,(1-y)\,m_2^2 + (1-x)\,m_3^2$$

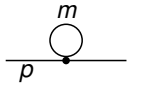
Figure 6.2: Standard scalar one–loop integrals*

*A direct integration of the one–loop tadpole yields

$$m^2(4\pi)^{-d/2}\Gamma(1-d/2)\left(\frac{m^2}{\mu^2}\right)^{d/2-2}$$

which by virtue of $\Gamma(1-d/2) = -2\Gamma(2-d/2)/(d-2)$ is the same analytic function as the one obtained via the partial integration method.

Standard Scalar One–loop Integrals ($m^2 \hat{=} m^2 - \mathrm{i}\varepsilon$).



$$= \mu_0^\epsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \doteq -\frac{\mathrm{i}}{16\pi^2} A_0(m) \ , \tag{6.32}$$

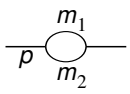
defines the standard *tadpole type integral*, where

$$A_0(m) = -m^2(\mathrm{Reg} + 1 - \ln m^2) \tag{6.33}$$

with

$$\mathrm{Reg} = \frac{2}{\epsilon} - \gamma + \ln 4\pi + \ln \mu_0^2 \equiv \ln \mu^2 \ . \tag{6.34}$$

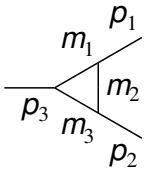
The last identification defines the $\overline{\mathrm{MS}}$ *scheme of (modified) minimal subtraction*.



$$= \mu_0^\epsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m_1^2)\left((k+p)^2 - m_2^2\right)} \doteq \frac{\mathrm{i}}{16\pi^2} B_0(m_1, m_2; p^2) \ , \tag{6.35}$$

defines the standard *propagator type integral*, where

$$B_0(m_1, m_2; s) = \mathrm{Reg} - \int_0^1 \mathrm{d}z \ln(-sz(1-z) + m_1^2(1-z) + m_2^2 z - \mathrm{i}\varepsilon) \ . \tag{6.36}$$



$$= \mu_0^\epsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m_1^2)\left((k+p_1)^2 - m_2^2\right)\left((k+p_1+p_2)^2 - m_3^2\right)}$$
$$= -\frac{\mathrm{i}}{16\pi^2} C_0(m_1, m_2, m_3; p_1^2, p_2^2, p_3^2) \ , \tag{6.37}$$

defines the standard *form factor type integral*, where

$$C_0(m_1, m_2, m_3; s_1, s_2, s_3) = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \frac{1}{ax^2 + by^2 + cxy + dx + ey + f} \tag{6.38}$$

with

$$\begin{aligned} a &= s_2, & d &= m_2^2 - m_3^2 - s_2, \\ b &= s_1, & e &= m_1^2 - m_2^2 + s_2 - s_3, \\ c &= s_3 - s_1 - s_2, & f &= m_3^2 - \mathrm{i}\varepsilon \ . \end{aligned}$$

Remark: the regulator term Reg in (6.34) denotes the UV regulated pole term $\frac{2}{\epsilon}$ supplemented with $O(1)$ terms which always accompany the pole term and result from the ϵ–expansion of the d–dimensional integrals. While in the MS scheme just the poles $\frac{2}{\epsilon}$ are subtracted, in the modified MS scheme $\overline{\mathrm{MS}}$ also the finite terms included in (6.34) are subtracted. The dependence on the UV cut–off $\frac{2}{\epsilon}$ in the $\overline{\mathrm{MS}}$ scheme defined by $\mathrm{Reg} \equiv \ln \mu^2$ is reflected in a dependence on the $\overline{\mathrm{MS}}$ *renormalization scale* μ.

The *UV*–singularities (poles in ϵ at d=4) give rise to *finite extra contributions* when they are multiplied with d (or functions of d) which arise from contractions like $g_\mu^\mu = d$, $\gamma^\mu \gamma_\mu = d$ etc. For $d \to 4$ we obtain:

$$dA_0(m) = 4A_0(m) + 2m^2 \ , \ \ dB_0 = 4B_0 - 2 \ . \tag{6.39}$$

The explicit evaluation of the scalar integrals (up to the scalar four–point function) is discussed in 't Hooft-Veltman 1979 (see also Davydychev, Kalmykov 2001, Fleischer, Jegerlehner, Tarasov 2003).

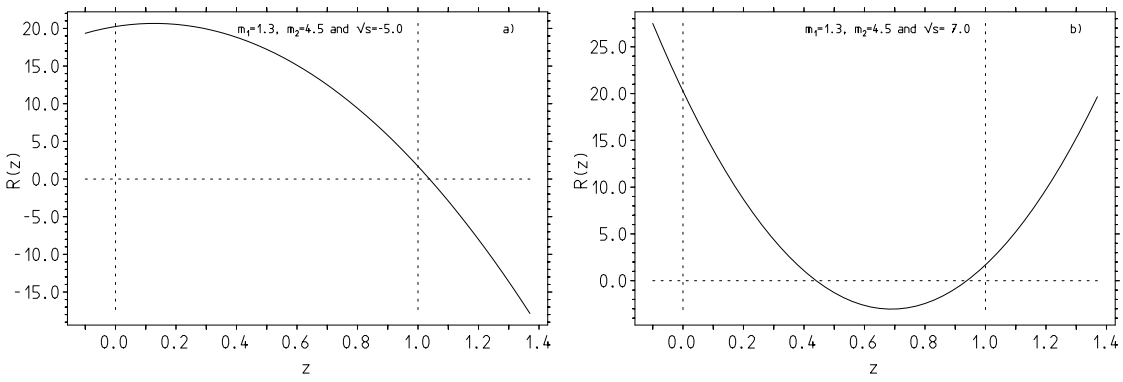


Figure 6.3: $R(z)$ as a function of z in the integration range $I:\ 0 \le z \le 1$. a) for $s < (m_1 - m_2)^2$ $R(z)$ is real and has no real zeros in I; the integral is real for real $s < (m_1 - m_2)^2$. b) for $s > (m_1 + m_2)^2$ $R(z)$ has two distinct real zeros within I; the integral is complex for real $s > (m_1 + m_2)^2$.

6.3 Selected Properties of Scalar Integrals

The general scalar two–point function is given by (see Exercises to this Section) by

$$\begin{aligned} B_0(m_1, m_2; s) &= \mathrm{Reg} + 2 - \frac{1}{2}\ln m_1^2 - \frac{1}{2}\ln m_2^2 - \frac{1}{2}\,\frac{m_1^2 - m_2^2}{s}\,\ln\frac{m_1^2}{m_2^2} \\ &+ \begin{cases} \frac{\sqrt{X}}{2s}\,\ln\frac{m_1^2+m_2^2-s+\sqrt{X}}{m_1^2+m_2^2-s-\sqrt{X}} & ; \quad s < (m_1-m_2)^2 \\ -\frac{\sqrt{-X}}{s}\,\arctan\frac{\sqrt{-X}}{m_1^2+m_2^2-s} & ; \quad (m_1-m_2)^2 < s < (m_1+m_2)^2 \\ \frac{\sqrt{X}}{2s}\left(\ln\frac{s-m_1^2-m_2^2-\sqrt{X}}{s-m_1^2-m_2^2+\sqrt{X}} + 2\pi\mathrm{i}\right) & ; \quad (m_1+m_2)^2 < s \end{cases} \end{aligned} \tag{6.40}$$

where

$$\begin{aligned} X &= s^2 - 2s\,(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2 \\ &= s^2\,(1 - y_+)\,(1 - y_-)\ ;\ \ y_\pm = \frac{(m_1 \pm m_2)^2}{s}\ . \end{aligned}$$

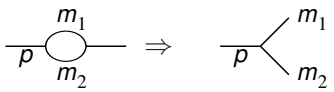
In order to understand the main properties let us look in Fig. 6.3 at the argument $R(z)$ in the defining integral

$$B_0(m_1.m_2; s) = \mathrm{Reg} - \int_0^1 \mathrm{d}z\,\ln R(z)\ .$$

We note that

$$\mathrm{Im}\,B_0(m_1, m_2; s) = 0 \quad \text{for} \quad s < (m_1 + m_2)^2\ .$$

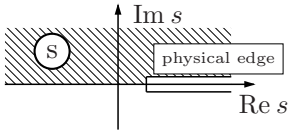
If the energy exceeds the production threshold $\sqrt{s} = m_1 + m_2$ the particles of masses m_1 and m_2 may be produced as real states, i.e. the intermediate states in the loop diagram



may exist as real physical states. B_0 then must exhibit an imaginary part (compare this with the unitarity relation discussed in Sect. **??** below)

$$\mathrm{Im}\,B_0(m_1, m_2; s) = 2\pi\,\mathrm{i}\,\frac{\sqrt{X}}{2s} \quad \text{for} \quad s > (m_1 + m_2)^2\ . \tag{6.41}$$

We note that $B_0(m_1, m_2; s)$ exhibits a cut in the complex s–plane for $s > (m_1 + m_2)^2$. The physical sheet is Im $s > 0$ i.e. $s + \mathrm{i}\varepsilon$



6.4 Tensor Integrals

In dimensional regularization also the calculation of tensor integrals is rather straight forward. Sign conventions are chosen in accordance with the Passarino-Veltman (1979) convention. Invariant amplitudes are defined by performing covariant decompositions of the tensor integrals, which then are contracted with external vectors or with the metric tensor. A factor $\mathrm{i}/16\pi^2$ is taken out for simplicity of notation, i.e.

$$\int_k \cdots = \frac{16\pi^2}{\mathrm{i}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \cdots \quad .$$

1) One point integrals:
By eventually performing a shift $k \to k + p$ of the integration variable we easily find the following results:

$$\begin{aligned} \textstyle\int_k \frac{1}{(k+p)^2-m^2} &= -A_0(m) \\ \textstyle\int_k \frac{k^\mu}{(k+p)^2-m^2} &= p^\mu A_0(m) \\ \textstyle\int_k \frac{k^\mu k^\nu}{(k+p)^2-m^2} &= -p^\mu p^\nu A_{21} + g^{\mu\nu} A_{22} \end{aligned} \tag{6.42}$$

$$\begin{aligned} A_{21} &= A_0(m) \\ A_{22} &= -\frac{m^2}{d} A_0(m) \overset{\epsilon\to 0}{\simeq} -\frac{m^2}{4} A_0(m) + \frac{m^4}{8} \end{aligned}$$

2) Two point integrals: the defining equations here are

$$\begin{aligned} \textstyle\int_k \frac{1}{(1)(2)} &= B_0(m_1, m_2; p^2) \\ \textstyle\int_k \frac{k^\mu}{(1)(2)} &= p^\mu B_1(m_1, m_2; p^2) \\ \textstyle\int_k \frac{k^\mu k^\nu}{(1)(2)} &= p^\mu p^\nu B_{21} - g^{\mu\nu} B_{22} \ , \end{aligned} \tag{6.43}$$

where we denoted scalar propagators by $(1) \equiv k^2 - m_1^2$ and $(2) \equiv (k+p)^2 - m_2^2$. The simplest non–trivial example is B_1. Multiplying the defining equation with $2p_\mu$ we have

$$2p^2 B_1 = \int_k \frac{2pk}{k^2 - m_1^2 + \mathrm{i}\varepsilon} \frac{1}{(p+k)^2 - m_2^2 + \mathrm{i}\varepsilon} \tag{6.44}$$

and we may write the numerator as a difference of the two denominators plus a remainder which does not depend on the integration variable:

$$2pk = (p+k)^2 - k^2 - p^2 = [(p+k)^2 - m_2^2] - [k^2 - m_1^2] - (p^2 + m_1^2 - m_2^2)$$

After canceling the square brackets against the appropriate denominator we obtain

$$B_1(m_1,m_2;p^2) = \frac{1}{2p^2}\left\{A_0(m_2) - A_0(m_1) - (p^2+m_1^2-m_2^2)\,B_0(m_1,m_2;p^2)\right\} \tag{6.45}$$

A further useful relation is

$$B_1(m,m;p^2) = -\frac{1}{2}B_0(m,m;p^2)\ .$$

In a similar way, by contracting the defining relation with p_ν and $g_{\mu\nu}$ we find for arbitrary dimension d

$$\begin{aligned} B_{21} &= \tfrac{1}{(d-1)\,p^2}\left\{(1-d/2)A_0(m_2) - d/2(p^2+m_1^2-m_2^2)B_1 - m_1^2B_0\right\} \\ B_{22} &= \tfrac{1}{2(d-1)}\left\{A_0(m_2) - (p^2+m_1^2-m_2^2)B_1 - 2m_1^2B_0\right\} \quad . \end{aligned}$$

Expansion in $d = 4-\epsilon, \epsilon \to 0$ yields

$$\begin{aligned} B_{21} &= \tfrac{-1}{3p^2}\left\{A_0(m_2) + 2(p^2+m_1^2-m_2^2)B_1 + m_1^2B_0 + 1/2(m_1^2+m_2^2-p^2/3)\right\} \\ B_{22} &= \tfrac{1}{6}\left\{A_0(m_2) - (p^2+m_1^2-m_2^2)B_1 - 2m_1^2B_0 - (m_1^2+m_2^2-p^2/3)\right\} \end{aligned} \tag{6.46}$$

where the arguments of the B–functions are obvious.

3) Three point integrals: for the simplest cases we define the following invariant amplitudes

$$\begin{aligned} \textstyle\int_k \frac{1}{(1)(2)(3)} &= -C_0(m_1,m_2,m_3;p_1^2,p_2^2,p_3^2) \\ \textstyle\int_k \frac{k^\mu}{(1)(2)(3)} &= -p_1^\mu C_{11} - p_2^\mu C_{12} \\ \textstyle\int_k \frac{k^\mu k^\nu}{(1)(2)(3)} &= -p_1^\mu p_1^\nu C_{21} - p_2^\mu p_2^\nu C_{22} - (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)C_{23} + g^{\mu\nu}C_{24} \end{aligned} \tag{6.47}$$

where $p_3 = -(p_1+p_2)$, $(1) \equiv k^2 - m_1^2$, $(2) \equiv (k+p_1)^2 - m_2^2$ and $(3) \equiv (k+p_1+p_2)^2 - m_3^2$.

The C_{1i}'s can be found using all possible independent contractions with $p_{1\mu,\nu}$, $p_{2\mu,\nu}$ and $g_{\mu\nu}$. This leads to the equations

$$\underbrace{\begin{pmatrix} p_1^2 & p_1p_2 \\ p_1p_2 & p_2^2 \end{pmatrix}}_{X} \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

with

$$\begin{aligned} R_1 &= \tfrac{1}{2}(B_0(m_2,m_3;p_2^2) - B_0(m_1,m_3;p_3^2) \\ &\quad - \left(p_1^2+m_1^2-m_2^2\right)C_0\big) \\ R_2 &= \tfrac{1}{2}\left(B_0(m_1,m_3;p_3^2) - B_0(m_1,m_2;p_1^2)\right. \\ &\quad \left. + (p_1^2-p_3^2-m_2^2+m_3^2)C_0\right) \quad . \end{aligned}$$

The inverse of the kinematic matrix of the equation to be solved is

$$X^{-1} = \frac{1}{DetX}\begin{pmatrix} p_2^2 & -p_1p_2 \\ -p_1p_2 & p_1^2 \end{pmatrix}\ ,\quad DetX \doteq p_1^2p_2^2 - (p_1p_2)^2$$

and the solution reads

$$\begin{aligned} C_{11} &= \frac{1}{DetX}\left\{p_2^2 R_1 - (p_1 p_2) R_2\right\} \\ C_{12} &= \frac{1}{DetX}\left\{-(p_1 p_2) R_1 + p_1^2 R_2\right\} \quad . \end{aligned} \tag{6.48}$$

The same procedure applies to the more elaborate case of the C_{2i}'s where the solution may be written in the form

$$C_{24} = -\frac{m_1^2}{2} C_0 + \frac{1}{4} B_0(2,3) - \frac{1}{4}(f_1 C_{11} + f_2 C_{12}) + \frac{1}{4} \tag{6.49}$$

$$\left(\begin{array}{c} C_{21} \\ C_{23} \end{array}\right) = X^{-1} \left(\begin{array}{c} R_3 \\ R_5 \end{array}\right); \left(\begin{array}{c} C_{23} \\ C_{22} \end{array}\right) = X^{-1} \left(\begin{array}{c} R_4 \\ R_6 \end{array}\right) \tag{6.50}$$

with

$$\begin{aligned} R_3 &= C_{24} - \tfrac{1}{2}\left(f_1 C_{11} + B_1(1,3) + B_0(2,3)\right) \\ R_5 &= -\tfrac{1}{2}\left(f_2 C_{11} + B_1(1,2) - B_1(1,3)\right) \\ R_4 &= -\tfrac{1}{2}\left(f_1 C_{12} + B_1(1,3) - B_1(2,3)\right) \\ R_6 &= C_{24} - \tfrac{1}{2}\left(f_2 C_{12} - B_1(1,3)\right) \end{aligned}$$

and

$$f_1 = p_1^2 + m_1^2 - m_2^2 \,; \quad f_2 = p_3^2 - p_1^2 + m_2^2 - m_3^2 \quad .$$

The notation used for the B–functions is as follows: $B_0(1,2)$ denotes the two point function obtained by dropping propagator $\frac{1}{(3)}$ from the form factor i.e. $\int_k \frac{1}{(1)(2)}$ and correspondingly for the other cases.

6.5 Special Integrals for massless propagators

We are again adopting the notation

$$\int_k \cdots = \frac{16\pi^2}{\mathrm{i}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \cdots$$

The base integral we consider are the two–point functions with arbitrary powers of the denominators:

$$I(r,s;m_1,m_2;p^2) = \int_k \frac{1}{(1)^r (2)^s} = \frac{1}{(r-1)!}\frac{1}{(s-1)!}\,\partial_1^{r-1}\partial_2^{s-1} \int_k \frac{1}{(1)(2)}$$

with $(1) = k^2 - m_1^2 + \mathrm{i}0$, $(2) = (k+p)^2 - m_2^2 + \mathrm{i}0$, $\partial_1 \equiv \partial/\partial m_1^2$ and $\partial_2 \equiv \partial/\partial m_2^2$. In d dimensions

$$\begin{aligned} \int_k \frac{1}{(1)(2)} &= B_0^d(m_1,m_2;p^2) \\ &= (4\pi)^{2-d/2} \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \,(\alpha_1+\alpha_2)^{-d/2}\, \mathrm{e}^{-\left(\alpha_1 m_1^2 + \alpha_2 m_2^2 - \frac{\alpha_2 \alpha_2}{\alpha_1+\alpha_2} p^2\right)} \quad . \end{aligned}$$

Utilizing this representation we may write (note the symmetry $I(s,r) = I(r,s)$ when $m_1 = m_2$)

$$\begin{aligned} I(r,s) &= \frac{(-1)^{r+s}}{(r-1)!(s-1)!}\,(4\pi)^{2-d/2}\,\int_0^\infty \mathrm{d}\alpha_1\mathrm{d}\alpha_2\,\alpha_1^{r-1}\,\alpha_2^{s-1}\,\cdots \\ &= \frac{(-1)^{r+s}}{\Gamma(r)\,\Gamma(s)}\,(4\pi)^{2-d/2}\,\int_0^\infty \mathrm{d}\lambda\,\lambda^{r+s-d/2-1} \\ &\quad\times\int_0^1 \mathrm{d}x\,x^{r-1}\,(1-x)^{s-1}\,\mathrm{e}^{-\lambda\left(x\,m_1^2+(1-x)\,m_2^2-x(1-x)\,p^2\right)} \\ &= (4\pi)^{2-d/2}\,\frac{(-1)^{r+s}}{\Gamma(r)\,\Gamma(s)}\,\Gamma(r+s-d/2) \\ &\quad\times\int_0^1 \mathrm{d}x\,x^{r-1}\,(1-x)^{s-1}\,\left(x\,m_1^2+(1-x)\,m_2^2-x(1-x)\,(p^2+\mathrm{i}0)\right)^{d/2-r-s}\quad. \end{aligned}$$

Concerning the $\mathrm{i}\varepsilon$–prescription we note that $m_i^2 \to m_i^2 - \mathrm{i}\varepsilon$ yields $x\,m_1^2+(1-x)\,m_2^2-x(1-x)\,p^2 \to x\,m_1^2+(1-x)\,m_2^2-x(1-x)\,p^2-\mathrm{i}\varepsilon$, which is fully equivalent to $x\,m_1^2+(1-x)\,m_2^2-x(1-x)\,(p^2+\mathrm{i}\varepsilon)$ as both $x \geq 0$ and $(1-x) \geq 0$ in the integration range.

We are interested in special zero mass integrals which frequently show up in QCD calculations:
1. $m_1^2 = m^2\;;\;\; m_2^2 = 0\;;\;\; p^2 = 0$: Using

$$\begin{aligned} \int_0^1 \mathrm{d}x\cdots &= (m^2)^{d/2-r-s}\int_0^1 \mathrm{d}x\,x^{\frac{d}{2}-s-1}\,(1-x)^{s-1} \\ &= (m^2)^{-\varepsilon/2}\,(m^2)^{2-r-s}\,B(\frac{d}{2}-s,s) \end{aligned}$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the well known Beta–function. We thus find

$$I(r,s;m^2,0;0) = (4\pi)^{\varepsilon/2}\,(m^2)^{-\varepsilon/2}\,(-m^2)^{2-r-s}\,\frac{\Gamma(r+s-d/2)\Gamma(d/2-s)}{\Gamma(r)\Gamma(d/2)}\quad.$$

2. $m_1^2 = m_2^2 = 0$: Using

$$\begin{aligned} \int_0^1 \mathrm{d}x\cdots &= (-(p^2+\mathrm{i}0))^{d/2-r-s}\int_0^1 \mathrm{d}x\,x^{\frac{d}{2}-s-1}\,(1-x)^{d/2-r-1} \\ &= (-(p^2+\mathrm{i}0))^{-\varepsilon/2}\,(-1)^{r+s}\,(p^2)^{2-r-s}\,B(\frac{d}{2}-s,\frac{d}{2}-r)\;, \end{aligned}$$

in this case we obtain

$$I(r,s;0,0;p^2) = (4\pi)^{\varepsilon/2}\,(-(p^2+\mathrm{i}0))^{-\varepsilon/2}\,(p^2)^{2-r-s}\,\frac{\Gamma(r+s-d/2)}{\Gamma(d-r-s)}\,\frac{\Gamma(d/2-s)}{\Gamma(s)}\,\frac{\Gamma(d/2-r)}{\Gamma(r)}\quad.$$

Here we present an application of case 1. above. Note that while

$$\int_k \frac{k^{\mu_1}\cdots k^{\mu_{2n+1}}}{(k^2-m^2)^r} = 0\;;\;\; n = 0,1,2,\cdots$$

we have

$$\int_k \frac{k^{\mu_1}\cdots k^{\mu_{2n}}}{(k^2-m^2)^r} = g^{\mu_1\mu_2}\cdots g^{\mu_{2n-1}\mu_{2n}}\,A_{2n,r}(m^2)\;.$$

From the last relation we obtain

$$d^n\,A_{2n,r} = \int_k \frac{(k^2)^n}{(k^2-m^2)^r} = (4\pi)^{\varepsilon/2}\,(m^2)^{-\varepsilon/2}\,(-m^2)^{2-r+n}\,\frac{\Gamma(r-n-d/2)\Gamma(d/2+n)}{\Gamma(r)\Gamma(d/2)}\;,$$

and, as a limit $m \to 0$, we find that

$$A_{2n,r}(0) = 0 \qquad \text{for } n \geq r \ .$$

For $n < r$, evaluating

$$\begin{aligned}
&\int_k \frac{1}{(k^2-m^2)^r} = \frac{1}{(r-1)!}\,\partial_1^{r-1} \int_k \frac{1}{k^2-m^2} \\
= \;& (4\pi)^{2-d/2}\,\frac{1}{(r-1)!}\partial_1^{r-1}\int_0^\infty \mathrm{d}\alpha\,\alpha^{-d/2}\,\mathrm{e}^{-\alpha\,m^2} = (4\pi)^{\varepsilon/2}\,\frac{(-1)^{r-1}}{\Gamma(r)}\int_0^\infty \mathrm{d}\alpha\,\alpha^{r-d/2-1}\,\mathrm{e}^{-\alpha\,m^2} \\
= \;& (4\pi)^{\varepsilon/2}\,\frac{(-1)^{r-1}}{\Gamma(r)}\,\Gamma(r-d/2)\,(m^2)^{-(r-d/2)} = -(4\pi)^{\varepsilon/2}\,(m^2)^{-\varepsilon/2}\,(-m^2)^{2-r}\,\frac{\Gamma(r-d/2)}{\Gamma(r)} \ ,
\end{aligned}$$

the limit $m \to 0$ exists, and is vanishing for all d with $\mathrm{Re}\,d > 2r$:

$$\lim_{m\to 0}\int_k \frac{1}{(k^2-m^2)^r} \equiv 0 \qquad \mathrm{Re}\,d > 2r \ .$$

Then, by analytic continuation,

$$\int_k \frac{1}{(k^2)^r} = 0 \qquad \forall d \ .$$

Together with the previous result this proves that in DR

$$\underline{\int_k \frac{k^{\mu_1}\cdots k^{\mu_n}}{(k^2)^r} = 0} \qquad \forall n, r \ .$$

Tensor integrals:
Let us denote by $(p^{\mu_1},\cdots,p^{\mu_m})$ the rank m traceless symmetric tensor, i.e. any contraction with $g^{\mu_i\mu_k}$ for $i,k \in (1,\cdots,m)$ vanishes:

$$\begin{aligned}
(p^{\mu_1}) &= p^{\mu_1} \ , \\
(p^{\mu_1}p^{\mu_2}) &= p^{\mu_1}p^{\mu_2} - \frac{1}{d}\,g^{\mu_1\mu_2}\,p^2 \ , \\
&\text{etc.}
\end{aligned}$$

We define tensor integrals $\hat{I}_m(r,s)$ by

$$I^{(\mu_1,\cdots,\mu_m)}(p) = \int_k \frac{(k^{\mu_1}\cdots k^{\mu_m})}{k^{2r}(k+p)^{2s}} = (p^{\mu_1}\cdots p^{\mu_m})\,\hat{I}_m(r,s) \ , \tag{6.51}$$

which will show up in QCD one-loop calculations below. We may write them as derivatives with respect to the external momentum $[\partial^{\mu_i} \equiv \partial/\partial p_{\mu_i}]$ as follows:

$$\begin{aligned}
&\frac{(r-m-1)!}{(r-1)!}\,\frac{1}{2^m}\,(\partial^{\mu_1}\cdots\partial^{\mu_m})\int_k \frac{1}{(k)^{2r-2m}(k+p)^{2s}} \\
= \;& \frac{(r-m-1)!}{(r-1)!}\,\frac{1}{2^m}\,(\partial^{\mu_1}\cdots\partial^{\mu_m})\int_k \frac{1}{(k-p)^{2r-2m}(k)^{2s}} \\
= \;& \int_k \frac{((k-p)^{\mu_1}\cdots(k-p)^{\mu_m})}{(k-p)^{2r}(k)^{2s}} = \int_k \frac{(k^{\mu_1}\cdots k^{\mu_m})}{(k)^{2r}(k+p)^{2s}} \ .
\end{aligned}$$

Using the representation derived above for the integrals $I(r,s)$, we therefore may write

$$\begin{aligned} I^{(\mu_1,\cdots,\mu_m)}(p) &= \frac{(r-m-1)!}{(r-1)!}\,\frac{1}{2^m}\,(\partial^{\mu_1}\cdots\partial^{\mu_m})\,I(r-m,s) \\ &= \frac{(-1)^{r+s}}{\Gamma(r)\,\Gamma(s)}\,\frac{1}{2^m}\,(4\pi)^{2-d/2}\int_0^\infty \mathrm{d}\lambda\,\lambda^{r+s-m-d/2-1} \\ &\quad\times\int_0^1 \mathrm{d}x\,x^{r-m-1}\,(1-x)^{s-1}\,(\partial^{\mu_1}\cdots\partial^{\mu_m})\,\mathrm{e}^{-\lambda\left(\cdots-x(1-x)\,p^2\right)} \\ &= \frac{(-1)^{r+s}}{\Gamma(r-m)\Gamma(s)}\frac{\Gamma(r-m)}{\Gamma(r)}\,(4\pi)^{2-d/2} \\ &\quad\times\int_0^\infty \mathrm{d}\lambda\,\lambda^{r+s-d/2-1}\,(p^{\mu_1}\cdots p^{\mu_m})\int_0^1 \mathrm{d}x\,x^{r-1}\,(1-x)^{s+m-1}\,\mathrm{e}^{-\lambda\left(\cdots-x(1-x)\,p^2\right)}\;. \end{aligned}$$

Where we have used $(\partial^{\mu_1}\cdots\partial^{\mu_m})\,\mathrm{e}^{-\lambda\,(\cdots-x(1-x)\,p^2)} = 2^m\,\lambda^m\,(p^{\mu_1}\cdots p^{\mu_m})\,x^m\,(1-x)^m\,\mathrm{e}^{-\lambda(\cdots-\,x(1-x)\,p^2)}$. After performing the λ–integration, we may read off the representation of the tensor integral

$$\begin{aligned} \hat{I}_m(r,s) &= \frac{(-1)^{r+s}}{\Gamma(r-m)\Gamma(s)}\frac{\Gamma(r-m)}{\Gamma(r)}\,(4\pi)^{2-d/2}\,\Gamma(r+s-d/2) \\ &\quad\times\int_0^1 \mathrm{d}x\,x^{r-1}\,(1-x)^{s+m-1}\,(x\,m_1^2+(1-x)\,m_2^2-x\,(1-x)\,p^2)^{d/2-r-1}\;, \end{aligned}$$

and for $m_1^2 = m_2^2 = 0$ using $\int_0^1 \mathrm{d}x\,x^{d/2-s-1}\,(1-x)^{d/2+m-r-1} = B(d/2-s,d/2+m-r)$ the x–integral simplifies to

$$(-(p^2+\mathrm{i}0))^{d/2-r-s}\;\times\;B(d/2-s,d/2+m-r)\,.$$

The massless tensor integrals then are given by

$$\hat{I}_m(r,s) = (4\pi)^{\varepsilon/2}\,(-(p^2+\mathrm{i}0))^{-\varepsilon/2}\,(p^2)^{2-r-s}\,\frac{\Gamma(r+s-\frac{d}{2})}{\Gamma(d+m-r-s)}\,\frac{\Gamma(\frac{d}{2}-s)}{\Gamma(s)}\,\frac{\Gamma(\frac{d}{2}+m-r)}{\Gamma(r)}\;.$$

For $\varepsilon\to 0$, the singular plus finite parts in any case can be represented in terms of massless B_0's and constants. We collect them in the following

Table for massless integrals: (use with care [see below])

$$I^{\mu\nu\cdots}(p,r,s) \doteq \int_k \frac{k^\mu k^\nu\cdots}{(k^2+\mathrm{i}\varepsilon)^r((k+p)^2+\mathrm{i}\varepsilon)^s}$$

a): $I(p,r,s) = \frac{\mathrm{i}}{16\pi^2}\,(p^2)^{-(r+s-2)}\,\hat{I}_0(r,s)$
b): $I^\mu(p,r,s) = \frac{\mathrm{i}}{16\pi^2}\,(p^2)^{-(r+s-2)}\,p^\mu\,\hat{I}_1(r,s)$
c): $I^{\mu\nu}(p,r,s) = \frac{\mathrm{i}}{16\pi^2}\,(p^2)^{-(r+s-2)}\,\Big(p^\mu p^\nu\,\hat{I}_{21}(r,s) + g^{\mu\nu}\,p^2\,\hat{I}_{22}(r,s)\Big)$

(r,s)	$\hat{I}_0(r,s)$	$\hat{I}_1(r,s)$	$\hat{I}_{21}(r,s)$	$\hat{I}_{22}(r,s)$
$(1,1)$	$B_0^{[UV]}$	$-\frac{1}{2}B_0^{[UV]}$	$\frac{1}{3}B_0^{[UV]}+\frac{1}{18}$	$-\frac{1}{12}B_0^{[UV]}-\frac{1}{18}$
$(2,1)$	$-B_0^{[IR]}+2$	-1	$\frac{1}{2}$	$\frac{1}{4}B_0^{[UV]}$
$(1,2)$	$-B_0^{[IR]}+2$	$B_0^{[IR]}-1$	$-B_0+\frac{1}{2}$	$\frac{1}{4}B_0$
$(3,1)$	-1	$\frac{1}{2}B_0^{[IR]}-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}B_0^{[IR]}+\frac{1}{4}$
$(2,2)$	$-2B_0^{[IR]}+2$	$B_0^{[IR]}-1$	$-B_0^{[IR]}$	$\frac{1}{2}$
$(1,3)$	-1	$-\frac{1}{2}B_0^{[IR]}+\frac{3}{2}$	$B_0^{[IR]}-\frac{3}{2}$	$-\frac{1}{4}B_0^{[IR]}+\frac{1}{4}$

$$
\begin{aligned}
B_0 = B_0(0,0;p^2) &= \mathrm{Reg} + 2 - \ln\left(-(p^2+\mathrm{i}0)\right) \\
&= \mathrm{Reg} + 2 - \ln p^2 + \mathrm{i}\pi \quad (p^2 > 0) \\
\mathrm{Reg} &= \frac{2}{\varepsilon} - \gamma + \ln 4\pi + \ln \mu_0^2 \; ; \;\; \varepsilon = 4 - d \quad .
\end{aligned}
$$

Using the massless integrals above some care is in necessary. In fact the ε–poles are not necessarily UV poles, in most cases they turn out to be IR poles. For the integrals (6.51) considered above, the correct interpretation of the ε–poles follows from the respective IR and UV convergence regions of the integrals in d. $I^{(\mu_1,\cdots,\mu_m)}(p)$ is UV convergent for $d + m - 2r - 2s < 0$ and IR convergent for $d + m - 2r > 0$ **and** $d - 2s > 0$. Thus the convergence region is

$$
\underline{2s, 2r - m < d < 2r + 2s - m} \quad .
$$

If $d_{\max} = 2r + 2s - m < 4$ $\hat{I}_m(r,s)$ has an UV pole: $\left(\frac{1}{\varepsilon}\right)_{\mathrm{UV}}$; if $d_{\min} = max(2s, 2r-m) > 4$ $\hat{I}_m(r,s)$ has an IR pole: $\left(\frac{1}{\varepsilon}\right)_{\mathrm{IR}}$. In boundary cases we have $d_{\min} = d_{\max} = 4$ the integral has both types of poles, and it must be defined via an extra regularization as discussed before.

- For $2s \geq 4$ or $2r - m \geq 4$ the ε–pole is IR–pole provided $2r + 2s - m > 4$.
- For $2s, 2r - m < 0$ the ε–pole is UV–pole.
- Boundary cases are: $2r = m$, $s = 2$ and $s = 0, 2r = m + 4$.

A typical boundary case is

$$
\begin{aligned}
\int_k \frac{1}{k^4} &= \frac{16\pi^2}{\mathrm{i}} \frac{2\pi^{d/2}}{(2\pi)^d \, \Gamma(d/2)} \times -\mathrm{i} \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega \, \omega^{d-5} \\
&= -2 \, \frac{(4\pi)^{-\varepsilon/2}}{\Gamma(d/2)} \;\; \frac{\omega^{d-4}}{d-4}\Big|_{\min}^{\max} \\
&= -2 \, \left(1 + \varepsilon/2 \, (\gamma - \ln 4\pi + 1)\right) \;\; \left(-\frac{1}{\varepsilon} + \ln \omega\right)\Big|_{\min}^{\max} \\
&= \left\{\frac{2}{\varepsilon} - \gamma + \ln 4\pi - 1 - \ln \omega^2\right\}\Big|_{\min}^{\max} \\
&= (\mathrm{Reg})_{\mathrm{UV}} - (\mathrm{Reg})_{\mathrm{IR}} = \text{"} \ln \frac{\omega_{\min}^2}{\omega_{\max}^2} \text{"} \quad .
\end{aligned}
$$

The regulator terms are defined by

$$
\begin{aligned}
(\mathrm{Reg})_{\mathrm{UV}} &= \frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln \mu_{\mathrm{UV}}^2 \\
(\mathrm{Reg})_{\mathrm{IR}} &= \frac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln \mu_{\mathrm{IR}}^2
\end{aligned}
$$

where $\mu_{\mathrm{UV}}^2 \gg \mu_{\mathrm{IR}}^2$. Thus strictly speaking we have

$$
\int_k \frac{1}{k^4} = (\mathrm{Reg})_{\mathrm{UV}} - (\mathrm{Reg})_{\mathrm{IR}} = \text{"} \; B_0^{[UV]} - B_0^{[IR]} \; \text{"} \quad ,
$$

while usually in DR we assign zero [which corresponds to the arbitrary choice $\mu_{\mathrm{UV}}^2 = \mu_{\mathrm{IR}}^2$] to such scale-less integrals.

An example:

$$\begin{aligned}
\int_k \frac{2kp}{k^4(k+p)^2} &= \int_k \frac{1}{k^4} - \int_k \frac{1}{k^2(k+p)^2} - k^2 \int_k \frac{1}{k^4(k+p)^2} \\
-2 &= \text{" } 0 \text{ "} - B_0^{[UV]} - \left(-B_0^{[IR]} + 2\right) \\
-2 &= \text{" } B_0^{[UV]} - B_0^{[IR]} \text{ "} - B_0^{[UV]} - \left(-B_0^{[IR]} + 2\right)
\end{aligned}$$

Thus the "trick" to distinguish $B_0^{[UV]}$ and $B_0^{[IR]}$ allows for a separate check of UV and IR singular terms. This, in principle, always should be done.

Exercises: Section 6

① Prove the relations (6.4) and (6.5) for the d dimensional Dirac algebra.

② Show that the γ_5–odd traces are not cyclic in the d dimensional Dirac algebra. Note that for a mathematically well defined set of matrices a trace trivially is cyclic. In particular prove the claim (6.6). Hint: calculate $\mathrm{Tr}\,\gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma_5$ and join the connected γ's such that one may use $\gamma_\alpha\gamma^\alpha = d$. Note that there are two ways to join these γ's.

③ Prove the relations (6.39) and derive the corresponding relations for the tensor integrals B_1, B_{21}, B_{22} as well as for the three point amplitudes C_{1i} and C_{2i}. Verify (6.46).

④ Calculate $B_0(m_1, m_2; p^2)$ performing the 1-dimensional integral (6.36) in terms of elementary functions.

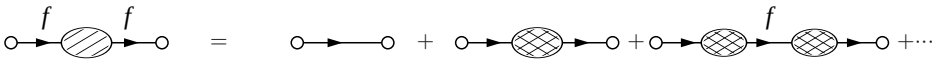
7 One–Loop Renormalization

7.1 The quark self–energy

Next we study the full propagator of a Dirac fermion f

$$\mathrm{i}S'_f(x-y) = \langle 0|T\left\{\psi_f(x)\bar{\psi}_f(y)\right\}|0\rangle$$

in momentum space, where the free Dirac propagator is given by $\mathrm{i}\,S_f(p) = \mathrm{i}/(\not{p} - m_f)$. The propagator has the structure of a repeated insertion of the 1PI self–energy $-\mathrm{i}\Sigma_f(p)$ [diagonal in color space, 4 x 4 matrix in spinor space]

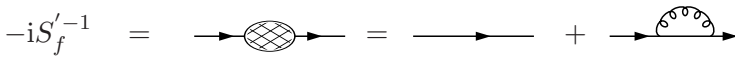


$$\begin{aligned}
\mathrm{i}\,S'_f(p) &\equiv \frac{\mathrm{i}}{\not{p}-m_f} + \frac{\mathrm{i}}{\not{p}-m_f}\,(-\mathrm{i}\Sigma_f)\,\frac{\mathrm{i}}{\not{p}-m_f} \\
&\quad + \frac{\mathrm{i}}{\not{p}-m_f}\,(-\mathrm{i}\Sigma_f)\,\frac{\mathrm{i}}{\not{p}-m_f}\,(-\mathrm{i}\Sigma_f)\,\frac{\mathrm{i}}{\not{p}-m_f} + \cdots \\
&= \frac{\mathrm{i}}{\not{p}-m_f}\left\{1 + (\Sigma_f S_f) + (\Sigma_f S_f)^2 + \cdots\right\} \\
&= \frac{\mathrm{i}}{\not{p}-m_f}\left\{\frac{1}{1-\Sigma_f S_f}\right\} = \frac{\mathrm{i}}{\not{p}-m_f-\Sigma_f}\ .
\end{aligned}$$

The last relation follows by taking the inverse:

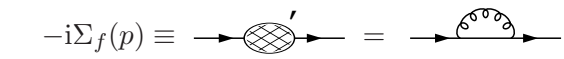
$$-\mathrm{i}S_f^{\prime\,-1} = -\mathrm{i}\,(1-\Sigma_f S_f)\;S_f^{-1} = -\mathrm{i}\left(S_f^{-1} - \Sigma_f\right)\ .$$

This demonstrates that the Dyson series is a geometric progression of matrix insertions which can be summed in closed form and the inverse full fermion propagator reads



$$= -\mathrm{i}\,\{\not{p} - m_f - \Sigma_f(p)\}\ .$$

The self–energy is given by an expansion in a series of 1PI diagrams



(7.1)

the prime ($'$) on the "blob" indicates the non-trivial part. The covariant decomposition of $\Sigma_f(p)$ for a massive fermion takes the form

$$\Sigma(p) = m_f\,\Sigma_1(p^2) + (\not{p} - m_f)\,\Sigma_2(p^2) = \not{p}\,\left(A(p^2, m_f, \cdots)\right) + m_f\,\left(B(p^2, m_f, \cdots)\right)\quad ,$$

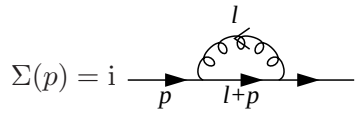
where A and B, as well as $\Sigma_1 = A + B$ and $\Sigma_2 = A$, are Lorentz scalar functions which depend on p^2 and on all parameters (indicated by the dots) of a given theory. In vector–like theories, like QED and QCD, no parity violating γ_5 terms are present, and the pole of the propagator, or, equivalently, the zero of the inverse propagator, is given by a multiple of the unit matrix in spinor space:

$$\not{p} = \tilde{m}_f\ \text{, where}\quad \tilde{m}_f^2 = s_P$$

defines the "pole mass" of the fermion in the p^2–plane

$$\not{p} - m_f - \Sigma_f(p)|_{\not{p}=\tilde{m}_f} = 0\ .$$

We are ready now to calculate the quark self–energy in the one–loop approximation. We drop the index f for the mass in the following. We have to calculate[22]

$\Sigma(p) = \mathrm{i}$ 

$$= \mathrm{i}\,\mathrm{i}^4 g^2 \mu^\varepsilon \, (T_i T_k)_{ab} \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \gamma^\rho \frac{\not{p} + \not{\ell} + m}{(p+\ell)^2 - m^2 + \mathrm{i}\varepsilon} \gamma^\sigma \left(-g_{\rho\sigma} + (1-\xi) \frac{\ell_\rho \ell_\sigma}{\ell^2} \right) \frac{1}{\ell^2 + \mathrm{i}\varepsilon} \, \delta_{ik} \ ,$$

where $\delta_{ik} \, (T_i T_k)_{ab} = (T_i T_i)_{ab} = \delta_{ab} \, C_2(R)$ is a $SU(3)$ Casimir operator, R denotes the representation of the T_i's which here is the fundamental one, for which we easily obtain $C_2(R) = \frac{4}{3}$. In the following we separate out a common factor N:

$$N = -\mathrm{i} \, g^2 \mu^\varepsilon \, \delta_{ab} \, C_2(R) \, \frac{\mathrm{i}}{16\pi^2} \ ,$$

where the last factor is a normalization of the integrals used below (see our definition of the standard integrals (6.33), (6.35)). The gluon propagator has two terms: the
1st term: has an integrand

$$\frac{1}{\ell^2} \, \frac{1}{(\ell+p)^2 - m^2} \, \underbrace{(\gamma^\alpha (\not{\ell} + \not{p}) \gamma_\alpha + m \, d)}_{(2-d)\,(\not{\ell}+\not{p}) + m\,d} \ ,$$

such that, in terms of the standard integrals introduced in the preceding section, the integration yields the term

$$T_1 = ((2-d)\not{p} + m\,d) \ B_0(0, m^2; p^2) + (2-d)\not{p} \, B_1(0, m^2; p^2)$$

with

$$B_1(m_1, m_2; p^2) = \frac{1}{2p^2} \, \left\{ -A_0(m_1) + A_0(m_2) - (p^2 + m_1^2 - m_2^2) \, B_0(m_1, m_2; p^2) \right\} \ .$$

Expanding in $\varepsilon = 4 - d$ we have to take into account (see (6.39))

$$dB_0 = 4\,B_0 - 2 \ ; \ \ d\,B_1 = 4\,B_1 + 1 \ .$$

The first term then is given by

$$\begin{aligned} T_1 &= m \ (4\,B_0 - 2) + \not{p} \ (1 - 2\,(B_0 + B_1)) \\ &= m \ (4\,B_0 - 2) + \not{p} \left(1 - \frac{A_0(m)}{p^2} - \frac{p^2 + m^2}{p^2} \, B_0 \right) \end{aligned}$$

with

$$B_0 = \mathrm{Reg} + 2 - \ln m^2 + \frac{m^2 - p^2}{p^2} \, \ln \left(1 - \frac{p^2 + \mathrm{i}\,0}{m^2} \right) \ .$$

[22] In case infrared singularities show up we use a tiny (smaller than any other mass scale) gluon mass as an IR regulator and then work with a gluon propagator of the form ('t Hooft gauge for massive spin 1 bosons)

$$D_{\rho\sigma}(k) = -\left(g_{\rho\sigma} - (1-\xi) \, \frac{k_\rho k_\sigma}{k^2 - \xi m_g^2} \right) \, \frac{1}{k^2 - m_g^2 + \mathrm{i}\varepsilon} \ .$$

2nd term: has the integrand

$$-(1-\xi)\,\frac{1}{\ell^4}\,\frac{1}{(\ell+p)^2-m^2}\,(\underbrace{\not\ell\,(\not\ell+\not p+m)\,\not\ell}_{X})\ .$$

Looking at

$$X=\not\ell\,(\not\ell+\not p+m)\,\not\ell=\ell^2\not\ell+\ell^2\,m-\ell^2\not p+(2p\ell)\,\not\ell$$

and writing $(2p\ell)$ as a difference of the denominators $(1)=\ell^2$ and $(2)=(\ell+p)^2-m^2$ plus a remainder $(2p\ell)=(2)-(1)-(p^2-m^2)$ we have

$$X=\ell^2\not\ell-\not\ell\ell^2-(\not p-m)\,(1)+\not\ell\,(2)-\not\ell\,(p^2-m^2)\ .$$

Herewith the integrand of the 2nd term reads

$$-(1-\xi)\,\left\{-(\not p-m)\,\frac{1}{(1)(2)}+\frac{\not\ell}{\ell^4}-(p^2-m^2)\,\frac{\not\ell}{(1)^2(2)}\right\}\ ,$$

and by integration we obtain as a 2nd term

$$T_2=(1-\xi)\,\left\{(\not p-m)\,B_0+(p^2-m^2)\,I\right\}\ ,$$

where (see Sect. 6.5)

$$p^\mu I\doteq\int_\ell\frac{\ell^\mu}{\ell^4\,((\ell+p)^2-m^2)}$$

with

$$I=-\frac{1}{p^2}\,\left(1+\frac{m^2}{p^2}\,\ln\left(1-\frac{p^2+\mathrm{i}0}{m^2}\right)\right)\ .$$

Note that the second term in the integrand is odd in the integration momentum and hence does not contribute to the integral. So far we have calculated the bare regularized one–loop quark self-energy

$$\Sigma(p)=m\Sigma_1+(\not p-m)\,\Sigma_2\ ,$$

where

$$\left\|\begin{array}{rcl}\Sigma_1 &=& 4\,B_0-1-\frac{A_0(m)}{p^2}-\frac{p^2+m^2}{p^2}\,B_0+(1-\xi)\,(p^2-m)\,I\ ,\\ \Sigma_2 &=& 1-\frac{A_0(m)}{p^2}-\frac{p^2+m^2}{p^2}\,B_0+(1-\xi)\,(B_0+(p^2-m)\,I)\ ,\end{array}\right. \tag{7.2}$$

with a common factor

$$\frac{g^2}{16\pi^2}\,\delta_{ab}\,C_2(R)\ .$$

In the bare result the mass m and the coupling g actually have to be identified with the bare parameters m_0 and g_0. By m and g we will denote the renormalized parameters in the next paragraph. The next step is **renormalization**.

Renormalization

After regularization and ε–expansion in $d=4-\varepsilon$ we are left with regularized bare amplitudes which, at the one–loop level, exhibit simple poles $\frac{1}{\varepsilon}$ in ε as $\varepsilon\to 0$, i.e., if we want to go to the

physical theory in $d = 4$ dimensions. Note that a QFT is defined in terms of the bare Lagrangian, which is the true Lagrangian of the theory. Renormalization is a reparametrization of the bare theory, in terms of renormalized parameters and fields. In fact a multiplicative renormalization of parameters and fields, in a renormalizable theory like QCD, is able to absorb all singularities and to obtain a renormalized parametrization which has a finite limit $\varepsilon \to 0$. In the limit the regularization is removed and we and the true $d = 4$ physical QFT is "recovered".

The renormalized full quark propagator is given by

$$\begin{aligned}
\tilde{S}'^{-1}_{\mathrm{F\,ren}} &= Z_F\, \tilde{S}'^{-1}_{\mathrm{F0}} = Z_F\, (\not{p} - m_0 - \Sigma(p)) \\
&= (\not{p} - m) + (Z_F - 1)\,(\not{p} - m) - Z_F\, (\delta m + \Sigma) \\
&= \quad \overset{-1}{\longrightarrow\!\!\!\!\!\!\blacktriangleright} \quad + \quad \longrightarrow\!\!\!\!\!\!\blacktriangleright\!\!\!\!\!\!\otimes\!\!\!\!\!\!\longrightarrow \quad + \quad \longrightarrow\!\!\!\!\!\!\blacktriangleright\!\!\!\!\!\!\boxed{}'\!\!\!\!\!\!\longrightarrow \\
&= (\not{p} - m) + \left(\delta Z_F^{(n)}\,(\not{p} - m) - \delta m^{(n)}\right) - \Sigma'^{(n)}
\end{aligned}$$

where $\delta Z_F = Z_F - 1$ and $\delta m = m_0 - m$, $m = m_{\mathrm{ren}}$ the renormalized mass. The n-th order counterterms are fixed by the following renormalization conditions at that order after the calculation of $\Sigma'^{(n)}$ which includes all loop diagrams at order n including loop diagrams with counterterm insertions from counterterms up to order $n - 1$.

1) <u>Mass renormalization.</u> By m we denote the "physical mass" in the following. Because of confinement and the corresponding non-existence of free quarks, a quark mass is not itself an observable quantity. However, it is a symmetry breaking parameter, breaking chiral $SU(N_f)_V \otimes SU(N_f)_A$ in a particular way. An example how quark masses can be given a precise physical meaning we have discussed at the end of Sect. 3.4 . Different possible mass definitions corresponding to different renormalization schemes. Since one can calculate one version of mass from the other, one usually starts with a convenient convention. At least for the light quarks one usually adopts so called *modified minimal subtraction* scheme (renormalized running mass in $\overline{\mathrm{MS}}$ scheme), which we will introduce below. First we treat the quark mass as **pole mass**, like a lepton mass, defined by the location of the pole of the propagator. Therefore, we look at

$$\begin{aligned}
\lim_{\not{p} \to m} \quad & \delta m + \Sigma(p) = \delta m + m\Sigma_1 + (\not{p} - m)\,\Sigma_2 \\
\to \quad & \delta m + m\Sigma_1 = 0
\end{aligned}$$

which yield the mass counterterm

$$\frac{\delta m}{m} = -\Sigma_1(p^2 = m^2) = 3\,\frac{A_0(m)}{m^2} - 1 \ .$$

We have used

$$B_0(0, m; m^2) = 1 - \frac{A_0(m)}{m^2} \ ,$$

and we may write

$$\frac{\delta m}{m} = -3\,\mathrm{Reg} + 3\,\ln m^2 - 4 \ .$$

Note that $\not{p} \to m$ implies $p^2 \to m^2$. We furthermore note that,

- first, the mass counterterm is gauge invariant (it is the on–shell value of an amplitude and hence a "quasi S–matrix element", albeit only the perturbative one, which is far from any true QCD S–matrix element describing real processes between hadrons),

- second, the term T_2 does not contribute at all.

Later we will represent the mass–shift, which looks like an additive renormalization, as a multiplicative renormalization factor

$$m_0 \;=\; Z_m\, m_{\rm ren} = m + \delta m = m\,\left(1 + \frac{\delta m}{m}\right) \;,$$

i.e.,

$$Z_m = 1 + \frac{\delta m}{m}\;;\;\; \delta Z_m = \frac{\delta m}{m} \;.$$

2) <u>Wave–function renormalization.</u> Again we try the academic exercise to make to "on–shell" residue of the propagator pole to be unity:

$$\delta Z_F\;(\not p - m) - (\delta m + \Sigma(p)) \;\to 0 \quad \text{as } \not p \to m \;.$$

This fixes the wave–function renormalization constant Z_F. To determine it we have to expand the self–energy about the mass–shell:

$$\begin{aligned}
\delta m + \Sigma(p) &= \delta m + m\Sigma_1 + (\not p - m)\,\Sigma_2 \\
&= \delta m + m\Sigma_1(p^2 = m^2) \\
&+ m\,(\not p - m)\,(\not p + m)\;\left.\frac{\partial \Sigma_1}{\partial p^2}\right|_{p^2=m^2} \\
&+ (\not p - m)\;\left.\Sigma_2\right|_{p^2=m^2} + O(p^2 - m^2) \\
&= \left.\left(2\,m^2\,\frac{\partial \Sigma_1}{\partial p^2} + \Sigma_2\right)\right|_{p^2=m^2}\;(\not p - m)\;+ O(p^2 - m^2) \;.
\end{aligned}$$

We then read off that the choice

$$\begin{aligned}
\delta Z_F \;&=\; \left.\left(\Sigma_2 + 2\,m^2\,\frac{\partial \Sigma_1}{\partial p^2}\right)\right|_{p^2=m^2} \\
&\overset{\xi=1}{=}\; -{\rm Reg} + \ln m^2 - 4 + 4\,\ln\frac{m}{m_g}
\end{aligned}$$

formally makes the residue of the renormalized propagator equal to 1. The last explicite expression is given for the Feynman gauge $\xi = 1$, for simplicity. Two points are crucial here,

- first, the wave–function renormalization factor is gauge dependent (as the field it renormalizes itself),
- second, for a charged (here color as the charge) field (particle) the wave-function renormalization is infrared (IR) singular, i.e. the residue of the pole does not exist.

We therefore have adopted an infinitesimally small (i.e. we take the limit $m_g \to 0$ whenever it exists) gluon mass as an IR regulator. This is a well known problem of QED, and the fake IR singularities must cancel in appropriately defined physical observables (Bloch-Nordsieck prescription).

Concerning the gauge dependence of the field renormalization, it is important to note that the on–shell limit of the bare Green functions (defining the S–matrix elements) is gauge invariant only

after the gauge–dependent wave–function renormalization has been performed. The on–shell limit of the bare quantities itself is not gauge–invariant, in general.

In particular in QCD, the IR problem is an artificial problem, since there exist no asymptotically free quarks in nature anyway. We therefore will adopt (modified) **minimal subtraction** in the following: just make the residue of the pole ultraviolet (UV) finite via the choice

$$\delta Z_F = \text{"}\delta Z_F\text{"}|_{\text{UV singular part}} \ .$$

We thus just collect the regulator terms Reg:

$$\begin{aligned}
(\Sigma_1)_{\rm UV} &= \left(4 + \frac{m^2}{p^2} - 1 - \frac{m^2}{p^2}\right) \mathrm{Reg} = 3\,\mathrm{Reg} \ , \\
(\Sigma_2)_{\rm UV} &= \left(\frac{m^2}{p^2} - 1 - \frac{m^2}{p^2} + (1-\xi)\right) \mathrm{Reg} = -\xi\,\mathrm{Reg} \ ,
\end{aligned}$$

and hence

$$(\delta Z_F)_{\rm MS} = -\xi\,\mathrm{Reg} \ .$$

As a result of the quark propagator renormalization we obtained

$$\left\| \begin{array}{rcl}
\delta Z_m & = & \frac{g^2}{16\pi^2}\, C_2(R)\, \left(-3\,\mathrm{Reg} + 3\,\ln m^2 - 4\right) \ , \\
(\delta Z_F)_{\rm MS} & = & \frac{g^2}{16\pi^2}\, C_2(R)\, \left(-\xi\,\mathrm{Reg}\right) \ ,
\end{array} \right. \tag{7.3}$$

with (γ here is Euler's constant)

$$\mathrm{Reg} = \frac{2}{\varepsilon} - \gamma + \ln 4\pi + \ln \mu^2 \ .$$

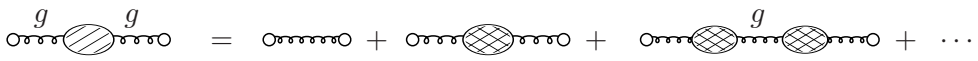
In the $\overline{\rm MS}$ renormalization scheme the renormalized quantities are obtained formally by the substitution

$$\mathrm{Reg} \to \ln \mu^2 \ .$$

This has the effect that all $\ln p^2$ or $\ln m^2$ terms only show up in the form $\ln \frac{p^2}{\mu^2}$ and $\ln \frac{m^2}{\mu^2}$, i.e., μ is the $\overline{\rm MS}$ renormalization scale (reference–scale).

7.2 The gluon self–energy

Also the full gluon propagator is given by a *Dyson series* of self–energy insertions



$$\begin{aligned}
\mathrm{i}\,D'^{\mu\nu}(k) &\equiv \mathrm{i}\,D^{\mu\nu}(k) + \mathrm{i}\,D^{\mu\rho}(k)\,(-\mathrm{i}\,\Pi_{\rho\sigma})\,\mathrm{i}\,D^{\sigma\nu}(k) \\
&\quad + \mathrm{i}\,D^{\mu\rho}(k)\,(-\mathrm{i}\,\Pi_{\rho\sigma})\,\mathrm{i}\,D^{\sigma\rho'}(k)\,\left(-\mathrm{i}\,\Pi_{\rho'\sigma'}\right)\mathrm{i}\,D^{\sigma'\nu}(k) + \cdots \\
&= \mathrm{i}\,D^{\mu\rho}(k)\left\{1 + (\Pi\,D)_\rho{}^\nu + (\Pi\,D)^2_\rho{}^\nu + \cdots\right\} \\
&= \mathrm{i}\,D^{\mu\rho}(k)\left(\frac{1}{1-\Pi\,D}\right)_\rho^\nu \ .
\end{aligned}$$

Again we get a geometrical progression which allows for a closed summation, and again this turns out to be very important. The resummed result shows that the full propagator has a **simple**

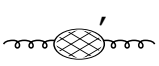
pole in k^2 only, like the free propagator, and no multi–poles as it might look like before the resummation has been performed.

If we multiply the last relation from the right with its inverse we obtain $D'\,(1-\Pi\,D)\,D^{-1}=1$ which means that

$$D^{'-1}=D^{-1}-\Pi\ .$$

The *one–particle irreducible* (1PI) self–energy function

$$-\mathrm{i}\Pi^{\mu\nu}(k)\equiv \tag{7.4}$$



is also called the vacuum polarization tensor . Its covariant decomposition reads

$$\Pi^{\mu\nu}(k)=g^{\mu\nu}\,\Pi_1(k^2)+k^\mu k^\nu\,\Pi_2(k^2)\ .$$

We have suppressed the color indices as the propagator is diagonal in the color as in the free case where with i,k color indices

$$D^{-1\,\mu\nu}_{ik}(k)=-(g^{\mu\nu}\,k^2-(1-\xi^{-1})\,k^\mu k^\nu)\,\delta_{ik}\ .$$

The Slavnov-Taylor (ST) identity, related to "$\partial_\mu G^\mu_{ia}(x)=0$", in momentum space reads

$$\begin{aligned} k^\nu\,D^{'-1}_{ik\,\mu\nu}(k) &= -k_\mu\,\left(\xi^{-1}k^2+\Pi_1(k^2)+k^2\,\Pi_2(k^2)\right)\,\delta_{ik}\\ &= -k_\mu\,\xi^{-1}k^2\,\delta_{ik}\ , \end{aligned}$$

which means that only the trivial lowest order part, stemming from the gauge fixing, prevents its vanishing. This requires the transversality of the non-trivial vacuum polarization tensor $\Pi_1=-k^2\,\Pi_2$ and hence

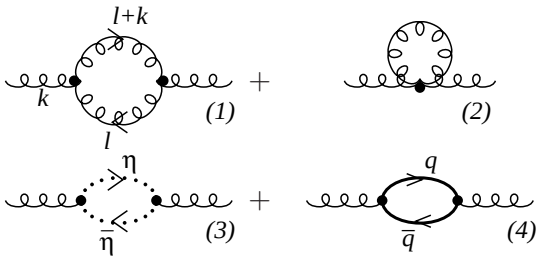
$$\Pi^{\mu\nu}_{ik}(k)=\left(k^\mu k^\nu-k^2\,g^{\mu\nu}\right)\,\Pi(k^2)\,\delta_{ik}$$

where we denoted by $\Pi_2(k^2)=\Pi(k^2)$. It means that no mass is generated by higher order corrections and hence mass renormalization is absent. We thus have the following form for the inverse full gluon propagator

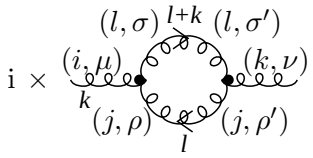
$$\boxed{\ D^{'-1\,\mu\nu}_{ik}=-\delta_{ik}\,\left\{\left(g^{\mu\nu}\,k^2-k^\mu k^\nu\right)\,\left(1-\Pi(k^2)\right)+\tfrac{1}{\xi}\,k^\mu k^\nu\right\}\ } \tag{7.5}$$

We now proceed to calculate the gluon self–energy at one-loop:

$-\mathrm{i}\,\Pi^{\mu\nu}_{ik}(k)$ =



Note that tadpole diagrams do not appear as $\mathrm{Tr}\,T_i=0$, $f_{ikl}\,\delta_{kl}=0$ and $\langle G_{\mu i}\rangle=0$. We now calculate the first diagram. Note that the first 3 diagrams have no correspondence in QED, where only the last type (fermion loops) is present. The most important preparation for a correct calculation is a proper detailed labeling of the lines and vertices:



$$
\begin{aligned}
= \;& \mathrm{i}\,\mathrm{i}^4\,\frac{1}{2}\,(\mathrm{i}\,g)^2\sum_{jl} f_{ijl}\,f_{kjl}\times\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\,\frac{1}{\ell^2+\mathrm{i}\varepsilon}\,\frac{1}{(\ell+k)^2+\mathrm{i}\varepsilon}\;\times \\
& \times\left(g_{\rho\rho'}-(1-\xi)\,\frac{\ell_\rho\ell_{\rho'}}{\ell^2}\right)\left(g_{\sigma\sigma'}-(1-\xi)\,\frac{(\ell+k)_\sigma(\ell+k)_{\sigma'}}{(\ell+k)^2}\right) \\
& \times\Big\{g^{\mu\rho}\,(\ell-k)^\sigma+g^{\mu\sigma}(\ell+2k)^\rho-g^{\rho\sigma}(2\ell+k)^\mu\Big\} \\
& \times\Big\{-g^{\nu\rho'}\,(\ell-k)^{\sigma'}-g^{\nu\sigma'}(\ell+2k)^{\rho'}+g^{\rho'\sigma'}(2\ell+k)^\nu\Big\}\;.
\end{aligned}
$$

Again we single out the common factor:

$$N_{(1)\,ik}=-\mathrm{i}\,\tfrac{g^2}{2}\,C_2(G)\,\delta_{ik}\,\tfrac{\mathrm{i}}{16\pi^2}\,,$$

where the last factor comes from the definition of the integrals (see (6.35)). The group factor is $\sum_{jl}\,f_{ijl}\,f_{kjl}=\mathrm{Tr}\,(t_it_k)=\delta_{ik}\,C_2(G)$ and one finds $C_2(G)=n$ for $SU(n)$. t_i are the generators in the adjoint representation.

1st term: with the $g_{\rho\rho'}g_{\sigma\sigma'}\,\cdots$ term of the product of the gluon propagators we have in the numerator:

$$
\begin{aligned}
& (g^{\mu\rho}\,(\ell-k)^\sigma+g^{\mu\sigma}(\ell+2k)^\rho-g^{\rho\sigma}(2\ell+k)^\mu)\;(g^{\rho\sigma}(2\ell+k)^\nu-g^{\nu\sigma}(\ell+2k)^\rho-g^{\nu\rho}(\ell-k)^\sigma) \\
= \;& (6-4d)\,\ell^\mu\ell^\nu+(3-2d)\,(\ell^\mu k^\nu+k^\mu\ell^\nu)+(6-d)\,k^\mu k^\nu-g^{\mu\nu}\,(4\,k^2+\ell^2+(\ell+k)^2)\;.
\end{aligned}
$$

Denoting propagators by $(1)=\ell^2+\mathrm{i}\varepsilon$ and $(2)=(\ell+k)^2+\mathrm{i}\varepsilon$, the integrals we need are the following:

$$\int_\ell\frac{\ell^\mu\ell^\nu}{(1)(2)}=k^\mu k^\nu\,B_{21}-g^{\mu\nu}\,B_{22}\;;\quad\int_\ell\frac{\ell^\mu}{(1)(2)}=k^\mu\,B_1\;.$$

As massless integrals these are particularly simple

$$
\begin{aligned}
B_{22}(0,0;k^2) &= k^2\left(\frac{1}{12}\,B_0+\frac{1}{18}\right)\;, \\
B_{21}(0,0;k^2) &= \frac{1}{3}\,B_0+\frac{1}{18}\;, \\
B_1(0,0;k^2) &= -\frac{1}{2}\,B_0\;, \\
B_0=B_0(0,0;k^2) &= \int_\ell\frac{1}{(1)(2)}\;.
\end{aligned}
$$

Integrating the 1st term with the numerator given above we arrive at:

$$
\begin{aligned}
T_1 \;=\; & k^\mu k^\nu\Big\{(6-4d)\,B_{21}-2\,(2d-3)\,B_1+(6-d)\,B_0\Big\} \\
& +g^{\mu\nu}\Big\{(4d-6)\,B_{22}-4k^2\,B_0\Big\}\;,
\end{aligned}
$$

where

$$\int_\ell \frac{1}{\ell^2} = 0 \; ; \quad \int_\ell \frac{1}{(\ell+k)^2} = 0$$

has been used. Using the above relations for the tensor integrals we have

$$\boxed{T_1 = k^\mu k^\nu \left\{ \tfrac{11}{3}\, B_0 + \tfrac{1}{9} \right\} - k^2\, g^{\mu\nu} \left\{ \tfrac{19}{6}\, B_0 + \tfrac{1}{9} \right\}} \; . \tag{7.6}$$

2nd term:[23] the mixed term $(\xi-1)\,\frac{\ell_\rho \ell_{\rho'}}{\ell^2}\, g_{\sigma\sigma'} \times 2$ of the gluon propagator product yields in the numerator:

$$2\,(1-\xi) \left\{ g^{\mu\nu}\,(\ell^2 + 2\,\ell k)^2 + k^\mu k^\nu\,\ell^2 - \ell^\mu \ell^\nu\,(\ell^2 + 2\,\ell k - k^2) - (\ell^\mu k^\nu + k^\mu \ell^\nu)\,(\ell^2 + 3\,\ell k) \right\} \frac{1}{\ell^2} \times \cdots$$

where we used

$$\begin{aligned} & \ell_\rho \ell_{\rho'}\, g_{\sigma\sigma'} \left\{ \cdots \right\} \left\{ \cdots \right\} \\ = \; & -\left\{ g^{\mu\sigma}\,(\ell^2 + 2\,\ell k) - \ell^\mu \ell^\sigma - \ell^\mu k^\sigma - k^\mu \ell^\sigma \right\} \left\{ g^\nu_\sigma\,(\ell^2 + 2\,\ell k) - \ell^\nu \ell_\sigma - \ell^\nu k_\sigma - k^\nu \ell_\sigma \right\} \; . \end{aligned}$$

Note that $\ell^2 + 2\,\ell k = (\ell+k)^2 - k^2$ and, in accord with dimensional regularization properties, all no-scale integrals

$$\int_\ell \frac{\ell^{\mu_1} \ldots}{(\ell^2)^r} \equiv 0 \; .$$

We then are left with the following non-trivial integrations.
a) First consider the term

$$g^{\mu\nu}\,(\ell^2 + 2\,\ell k)\,\frac{1}{\ell^4\,(\ell+k)^2} = g^{\mu\nu} \left(\frac{\ell^2 + 2\,\ell k}{\ell^4} - k^2\,\frac{1}{\ell^4} + k^4\,\frac{1}{\ell^4\,(\ell+k)^2} \right) \; .$$

While the first two terms in DR vanish upon integration, the third yields the integral

$$\hat{I}_0(2,1) = k^2 \int_\ell \frac{1}{\ell^4\,(\ell+k)^2} = -B_0 + 2$$

(see Table of Sect. 6.5). The complete term a) then yields

$$\underline{T_2^{(a)} = g^{\mu\nu}\;(-B_0 + 2)} \; . \tag{7.7}$$

b) The next term is

$$k^\mu k^\nu\,\frac{\ell^2}{\ell^4\,(\ell+k)^2} = k^\mu k^\nu\,\frac{1}{\ell^2\,(\ell+k)^2} \; ,$$

which yields the integral

$$\underline{T_2^{(b)} = k^\mu k^\nu\, B_0} \; . \tag{7.8}$$

[23] The symmetric term $(\xi-1)\,g_{\rho\rho'}\,\frac{(\ell+k)_\sigma (\ell+k)_{\sigma'}}{(\ell+k)^2}$ yields the same result: check $\ell + k = \ell'$, $k = -k'$ such that $\ell - k = \ell' + 2k'$, $2\ell + k = 2\ell' + k'$ and $\ell + 2k = \ell' - k'$ $(\ell = \ell' + k')$ as well as $\rho \leftrightarrow \sigma$ and $\rho' \leftrightarrow \sigma'$.

c) The next term we have reads

$$-\ell^\mu \ell^\nu \left(\ell^2 + 2\,\ell k - k^2\right) \frac{1}{\ell^4\,(\ell+k)^2} = -\frac{\ell^\mu \ell^\nu}{\ell^4} + 2k^2\,\frac{\ell^\mu \ell^\nu}{\ell^4\,(\ell+k)^2}\ ,$$

with a vanishing integral for the first term. The second integral is

$$\begin{aligned} k^2 \int_\ell \frac{\ell^\mu \ell^\nu}{\ell^4\,(\ell+k)^2} &= k^\mu k^\nu\,\hat{I}_{21}(2,1) + g^{\mu\nu}\,k^2\,\hat{I}_{22}(2,1)\ , \\ &= k^\mu k^\nu\,\frac{1}{2} + g^{\mu\nu}\,k^2\,\frac{1}{4}\,B_0 \end{aligned}$$

and hence

$$\underline{T_2^{(c)} = k^\mu k^\nu\ + g^{\mu\nu}\,k^2\,\frac{1}{2}B_0}\ . \tag{7.9}$$

d) The last term is

$$\begin{aligned} & -(\ell^\mu k^\nu + k^\mu \ell^\nu)\left(\ell^2 + 3\,\ell k\right)\frac{1}{\ell^4\,(\ell+k)^2} \overset{\mu\leftrightarrow\nu}{=} -2k^\mu\,\frac{\ell^\nu}{\ell^2\,(\ell+k)^2} - 3k^\mu\,\frac{\ell^\nu\,2\,\ell k}{\ell^4\,(\ell+k)^2} \\ = \quad & -2k^\mu\,\frac{\ell^\nu}{\ell^2\,(\ell+k)^2} - 3k^\mu\,\frac{\ell^\nu}{\ell^4} + 3k^\mu\,\frac{\ell^\nu}{\ell^2\,(\ell+k)^2} + 3k^\mu\,k^2\,\frac{\ell^\nu}{\ell^4\,(\ell+k)^2} \end{aligned}$$

where we have used the symmetry in $\mu \leftrightarrow \nu$. The first and third term combine while the second term vanishes upon integration. One thus obtains

$$k^\mu k^\nu\,\hat{I}_1(1,1) + 3k^\mu k^\nu\,\hat{I}_1(2,1)\ ,$$

and hence

$$\underline{T_2^{(d)} = k^\mu k^\nu\,\left(-\frac{1}{2}B_0 - 3\right)}\ . \tag{7.10}$$

Collecting all terms we have

$$\Big\| \quad T_2 = 2\,(1-\xi)\,\left(k^\mu k^\nu\ - g^{\mu\nu}\,k^2\right)\,\left(\tfrac{1}{2}B_0 - 2\right)\ , \tag{7.11}$$

which is purely transversal.

3rd term: is due to the product $(1-\xi)^2\,\frac{1}{\ell^2\,(\ell+k)^2}\,\ell_\rho \ell_{\rho'}\,(\ell+k)_\sigma\,(\ell+k)_{\sigma'}$ of the second terms of the gluon propagators which yields in the numerator:

$$\begin{aligned} & \ell_\rho \ell_{\rho'}\,(\ell+k)_\sigma\,(\ell+k)_{\sigma'}\,\Big\{\cdots\Big\}\Big\{\cdots\Big\} \\ = \quad & -\Big\{k^\mu\,(\ell k) - \ell^\mu\,k^2\Big\}\Big\{k^\nu\,(\ell k) - \ell^\nu\,k^2\Big\} \\ = \quad & -k^\mu k^\nu\,(\ell k)^2 - \ell^\mu \ell^\nu\,(k^2)^2 + (k^\mu \ell^\nu + \ell^\mu k^\nu)\,k^2\,(\ell k) \\ = \quad & \left(-k^\mu k^\nu\,k^\rho k^\sigma\ - g^{\mu\rho} g^{\nu\sigma}\,(k^2)^2 + (k^\mu g^{\nu\sigma} k^\rho + g^{\mu\rho} k^\nu k^\sigma)\,k^2\right)\ \times\ \ell_\rho \ell_\sigma \end{aligned}$$

and using

$$\int_\ell \frac{\ell_\rho \ell_\sigma}{(\ell^2)^2\,((\ell+k)^2)^2} = \Big(k_\rho k_\sigma\,\hat{I}_{21}(2,2) + g_{\rho\sigma}\,k^2\,\hat{I}_{22}(2,2)\Big)\ \times\ \frac{1}{(k^2)^2}\ ,$$

we note that

$$\begin{aligned} k_\rho k_\sigma\ (\cdots) &= -k^\mu k^\nu\,(k^2)^2 - k^\mu k^\nu\,(k^2)^2 + 2\,k^\mu k^\nu\,(k^2)^2 = 0\ , \\ g_{\rho\sigma}\,k^2\ (\cdots) &= -k^\mu k^\nu\,(k^2)^2 - g^{\mu\nu}\,(k^2)^3 + 2\,k^\mu k^\nu\,(k^2)^2\ , \end{aligned}$$

such that the 3rd term reads

$$(1-\xi)^2\,\left(k^\mu k^\nu - g^{\mu\nu}\,k^2\right)\,\hat{I}_{22}(2,2)\ ,$$

or

$$\boxed{T_3 = 2\,(1-\xi)^2\,\left(k^\mu k^\nu\,-g^{\mu\nu}\,k^2\right)\,\left(\tfrac{1}{2}\right)}\ . \tag{7.12}$$

Also this term is purely transversal.
The result for diagram (1) thus reads

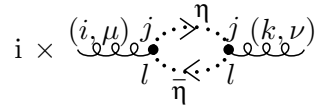
$$\begin{aligned}\Pi^{\mu\nu}_{ik}(1) &= \delta_{ik}\,\tfrac{1}{16\pi^2}\,\tfrac{g^2}{2}\,C_2(G)\,\times \\ &\quad \Big\{k^\mu k^\nu\,\left(\tfrac{11}{3}B_0+\tfrac{1}{9}+(1-\xi)\,(B_0-4)+(1-\xi)^2\,\tfrac{1}{2}\right) \\ &\quad -g^{\mu\nu}\,k^2\,\left(\tfrac{19}{6}B_0+\tfrac{1}{9}+(1-\xi)\,(B_0-4)+(1-\xi)^2\,\tfrac{1}{2}\right)\Big\}\ .\end{aligned}$$

Diagram (2) is vanishing because the self-closing line is massless, according to the features of dimensional regularization, which assigns

$$\int_\ell \left(g_{\rho\sigma}-(1-\xi)\,\frac{\ell_\rho\ell_\sigma}{\ell^2}\right)\,\frac{1}{\ell^2}\equiv 0\ .$$

The third diagram (3) is again non-trivial and was actually the missing piece before the Faddeev-Popov Lagrangian got established. We have

$$\mathrm{i}\,\times$$



$$= \ \mathrm{i}\,\mathrm{i}^4\,(-1)\,(-\mathrm{i}\,g)^2\,\sum_{jl} f_{ijl}\,f_{klj}\,\times\,\int\frac{\mathrm{d}^d\ell}{(2\pi)^d}\,\ell^\mu\,(\ell+k)^\nu\,\frac{1}{\ell^2+\mathrm{i}\varepsilon}\,\frac{1}{(\ell+k)^2+\mathrm{i}\varepsilon}\ ,$$

where the factor (-1) is due to Fermi statistics of the FP ghosts. The common factor here is:

$$N_{(3)\,ik} = -\mathrm{i}\,g^2\,C_2(G)\,\delta_{ik}\,\tfrac{\mathrm{i}}{16\pi^2}\,.$$

The integrals we need are

$$\begin{aligned}\int_\ell \frac{\ell^\mu\ell^\nu}{\ell^2\,(\ell+k)^2} &= k^\mu k^\nu\,\hat{I}_{21}(1,1)+g^{\mu\nu}\,k^2\,\hat{I}_{22}(1,1) \\ &= k^\mu k^\nu\,\left(\frac{1}{3}B_0+\frac{1}{18}\right)+g^{\mu\nu}\,k^2\,\left(-\frac{1}{12}B_0-\frac{1}{18}\right)\ , \\ \int_\ell \frac{\ell^\mu}{\ell^2\,(\ell+k)^2} &= k^\mu\,\hat{I}_1(1,1)=k^\mu\,\left(-\frac{1}{2}B_0\right)\ ,\end{aligned}$$

and we easily find

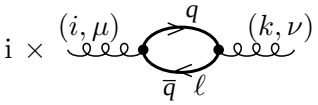
$$\Pi^{\mu\nu}_{ik}(3) \ = \ \delta_{ik}\,\tfrac{1}{16\pi^2}\,g^2\,C_2(G)\,\Big\{k^\mu k^\nu\,\left(-\tfrac{1}{6}B_0+\tfrac{1}{18}\right)-g^{\mu\nu}\,k^2\,\left(\tfrac{1}{12}B_0+\tfrac{1}{18}\right)\Big\}\ .$$

The sum of the two non-vanishing "pure gauge" diagrams yields

$$\begin{aligned}\Pi^{\mu\nu}_{ik}[(1)+(2)+(3)] &= \delta_{ik}\,\tfrac{g^2}{32\pi^2}\,C_2(G)\,\times \\ &\quad \Big\{k^\mu k^\nu\,\left(\tfrac{10}{3}B_0+\tfrac{2}{9}+(1-\xi)\,(B_0-4)+(1-\xi)^2\,\tfrac{1}{2}\right) \\ &\quad -g^{\mu\nu}\,k^2\,\left(\tfrac{10}{3}B_0+\tfrac{2}{9}+(1-\xi)\,(B_0-4)+(1-\xi)^2\,\tfrac{1}{2}\right)\Big\}\end{aligned} \tag{7.13}$$

and, as it should be, turns out to be transversal! Note that the individual diagrams do not have this property. The transversality means that the gluon self–energy insertion actually satisfies the ST–identity. The latter must hold for the pure gauge part separately, without the quarks.

We are left with the quark loop

$$\mathrm{i} \times$$ 

$$= \mathrm{i}\,\mathrm{i}^4\,(-1)\,g^2\,\mathrm{Tr}\,(T_i T_k) \times \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\,\frac{\mathrm{Tr}\,\left(\gamma^\mu\,(\not\ell + m)\,\gamma^\nu\,(\not\ell + \not k + m)\right)}{[\ell^2 - m^2 + \mathrm{i}\varepsilon]\,[(\ell+k)^2 - m^2 + \mathrm{i}\varepsilon]}\ ,$$

where the factor (-1) is due to Fermi statistics of the quarks. The spinor space trace yields

$$\begin{aligned}\mathrm{Tr}\,(\cdots) &= \mathrm{Tr}\,(\gamma^\mu \not\ell \gamma^\nu\,(\not\ell + \not k)) + m^2 \mathrm{Tr}\,\gamma^\mu\gamma^\nu \\ &= \mathrm{Tr}(\mathbf{1})\cdot\left\{\ell^\mu\,(\ell+k)^\nu - g^{\mu\nu}\,(\ell^2 + \ell k) + (\ell+k)^\mu\,\ell^\nu + m^2 g^{\mu\nu}\right\}\ .\end{aligned}$$

To proceed we may take advantage of the transversality $k_\nu \Pi^{\mu\nu} = 0$ of the fermion loop contribution, which is easy to check. We may write

$$\begin{aligned}& k_\nu \left(\gamma^\mu\,\frac{1}{\not\ell - m}\,\gamma^\nu\,\frac{1}{\not\ell + \not k - m}\right) \\ =\ & \gamma^\mu\,\frac{1}{\not\ell - m}\,\not k\,\frac{1}{\not\ell + \not k - m} = \gamma^\mu\,\frac{1}{\not\ell - m}\,[(\not\ell + \not k - m) - (\not\ell - m)]\,\frac{1}{\not\ell + \not k - m} \\ =\ & \gamma^\mu\left(\frac{1}{\not\ell - m} - \frac{1}{\not\ell + \not k - m}\right) \stackrel{\int_\ell}{=} 0\ .\end{aligned}$$

In the last step we used

$$\int_\ell \frac{1}{\not\ell - m} = \int_\ell \frac{1}{\not\ell + \not k - m}\ ,$$

which follows by a shift of the integration variable. The latter is possible in DR. This is crucial here.

As a result we have shown that the quark loop contribution is transversal

$$\Pi^{\mu\nu}_{ik}(4) = \left(k^\mu k^\nu - g^{\mu\nu}\,k^2\right)\,\Pi_{(4)\,ik}(k^2) = N_{4\,ik}\,\left(k^\mu k^\nu - g^{\mu\nu}\,k^2\right)\,\hat\Pi_{(4)}(k^2)\ ,$$

where $N_{4\,ik}$ is the common factor

$$N_{4\,ik} = -\mathrm{i}\,g^2\,T(R)\,f(d)\,\delta_{ik}\,\tfrac{\mathrm{i}}{16\pi^2}\,,$$

and we denoted $\mathrm{Tr}\,\mathbf{1} = f(d)$ (see (6.5) and comments there). The group factor follows from

$$\mathrm{Tr}\,T_i T_k = \sum_{a,b}(T_i)_{ab}\,(T_k)_{ba} = \delta_{ik}\,T(R)\ ,$$

with T_i the generators of $SU(n)$ in the fundamental representation, on finds $T(R) = \frac{1}{2}$ for $SU(n)$ per flavor, i.e. $\sum_f T(R) = \frac{N_f}{2}$.

Utilizing the transversality, we may contract the vacuum polarization tensor with $g_{\mu\nu}$ to obtain a scalar amplitude:

$$\begin{aligned}g_{\mu\nu}\,\Pi^{\mu\nu}_{ik}(4) &= N_{4\,ik}\,k^2\,(1-d)\,\hat\Pi_{(4)}(k^2) \\ &= N_{4\,ik}\,\times\int_\ell \frac{1}{(1)(2)}\left(2\,(\ell^2 + \ell k) - d\,(\ell^2 + \ell k) + m^2\,d\right)\end{aligned}$$

with $(1) = \ell^2 - m^2 + \mathrm{i}\varepsilon$ and $(2) = (\ell+k)^2 - m^2 + \mathrm{i}\varepsilon$. The numerator in the integral may be rewritten in terms of inverse scalar propagators which will cancel against the propagator denominators:

$$\begin{aligned} & (2-d)\,(\ell^2 + \ell k) + m^2\,d \\ = \;& (2-d)\left[\underbrace{(\ell^2-m^2)}_{(1)} + m^2 + \frac{1}{2}\left(\underbrace{(\ell+k)^2-m^2}_{(2)} - \underbrace{(\ell^2-m^2)}_{(1)} - k^2\right)\right] + m^2\,d \end{aligned}$$

Using this we obtain the integral

$$\begin{aligned} X &= (2-d)\left[\int_\ell \frac{1}{(2)} + m^2 \int_\ell \frac{1}{(1)(2)} + \frac{1}{2}\int_\ell \frac{1}{(1)} - \frac{1}{2}\int_\ell \frac{1}{(2)} - \frac{k^2}{2}\int_\ell \frac{1}{(1)(2)}\right] + d\,m^2 \int_\ell \frac{1}{(1)(2)} \\ &= (2-d)\left[-A_0 + m^2 B_0 - \frac{1}{2}A_0 + \frac{1}{2}A_0 - \frac{k^2}{2}B_0\right] + d\,m^2\,B_0 \end{aligned}$$

and using $dB_0 = 4B_0 - 2$, $dA_0 = 4A_0 + 2\,m^2$ we furthermore may simplify

$$\begin{aligned} X &= 2A_0 + 2\,m^2 - 2\,m^2 B_0 + 2\,m^2 + k^2 B_0 - k^2 + 4\,m^2 B_0 - 2\,m^2 \\ &= 2\,m^2 + 2\,A_0 + 2\,m^2 B_0 + k^2 B_0 - k^2 \\ &= k^2\,(1-d)\,\hat{\Pi}_{(4)}(k^2) = -3k^2\,\hat{\Pi}_{(4)} + k^2\,\varepsilon\,\hat{\Pi}_{(4)}\ . \end{aligned}$$

The last term may be worked out

$$\lim_{\varepsilon\to 0} k^2\,\varepsilon\,\hat{\Pi}_{(4)} = k^2\,\varepsilon\,\left[-2m^2 + 2m^2 + k^2\right]\,\frac{2}{\varepsilon}\,\frac{-1}{3k^2} = -\frac{2}{3}k^2\ ,$$

and we end up with

$$\hat{\Pi}_{(4)} = \frac{-1}{3k^2}\left\{2A_0 + 2\,m^2 B_0 + k^2 B_0 + 2m^2 - k^2 + \frac{2}{3}k^2\right\}$$

such that

$$\left\| \quad \Pi_{(4)\,ik}(k^2) \;=\; -\delta_{ik}\,\tfrac{g^2}{16\pi^2}\,T(R)\,f(d)\,\tfrac{1}{3k^2}\left\{2A_0 + 2m^2 B_0 + k^2 B_0 + 2m^2 - \tfrac{1}{3}k^2\right\} \right. \tag{7.14}$$

with $A_0 = A_0(m),\ B_0 = B_0(m,m;k^2)$. Using the special value

$$B_0(m,m;0) = -1 - \frac{A_0(m)}{m^2}$$

we may check that the gluon indeed remains massless:

$$\lim_{k^2\to 0} k^2\,\Pi_{(4)\,ik}(k^2) = \cdots\left\{2A_0 + 2m^2 + 2m^2\,B_0(m,m;0)\right\} = 0\ .$$

Like for the photon in QED there is no mass-renormalization for the gluon in QCD. We now have the complete one–loop self-energy for the gluon, which is given by

$$\Pi^{\mu\nu}_{ik}(k) = \left(k^\mu k^\nu - g^{\mu\nu}\,k^2\right)\,\delta_{ik}\,\Pi(k^2)\ ,$$

with scalar amplitude

$$\begin{aligned} \Pi(k^2) \;=\; & \tfrac{g^2}{32\pi^2}\left\{C_2(G)\,\left(\tfrac{10}{3}B_0(0,0;k^2) + \tfrac{2}{9} + (1-\xi)\,\left(B_0(0,0;k^2) - 4\right) + (1-\xi)^2\,\tfrac{1}{2}\right)\right. \\ & \left. -\tfrac{4\,T(R)\,f(d)}{3k^2}\sum_q\left(A_0(m_q) + \left(m_q^2 + \tfrac{k^2}{2}\right)\,B_0(m_q,m_q;k^2) + m_q^2 - \tfrac{k^2}{6}\right)\right\}\ . \end{aligned} \tag{7.15}$$

The standard one–loop integrals used here are given by

$$\begin{aligned} A_0(m_q) &= -m_q^2 \left(\mathrm{Reg} + 1 - \ln m_q^2\right) \\ B_0(0,0;k^2) &= \mathrm{Reg} + 2 - \ln\left(-(k^2 + \mathrm{i}0)\right) \\ B_0(m_q,m_q;k^2) &= \mathrm{Reg} + 2 - \ln m_q^2 + 2\left(\frac{4m_q^2}{k^2} - 1\right) G\left(\frac{4m_q^2}{k^2}\right) \end{aligned}$$

where

$$G(y) = \frac{1}{2\sqrt{1-y}} \ln \frac{\sqrt{1-y}+1}{\sqrt{1-y}-1} \ . \tag{7.16}$$

For small momenta the leading terms of the asymptotic expansion reads

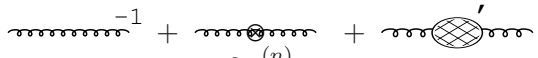
$$B_0(m_q,m_q;k^2) = \mathrm{Reg} - \ln m_q^2 + \frac{1}{6}\frac{k^2}{m_q^2} + \cdots \ ; \ \ k^2 \to 0 \ .$$

Renormalization of the gluon propagator

The structure of the gluon propagator and its renormalization may be summarized by the following relations:

$$\begin{aligned} D'^{-1\mu\nu}_{ik\,\mathrm{ren}} &= Z_3\, D'^{-1\mu\nu}_{ik\,0} = \left(k^\mu k^\nu - g^{\mu\nu}\, k^2\right) \delta_{ik}\, Z_3 \ \left(1 - \Pi(k^2)\right) - \delta_{ik}\, \frac{1}{\xi_{\mathrm{ren}}}\, k^\mu k^\nu \\ &= D^{-1\mu\nu}_{ik} + \delta_{ik} \left(k^\mu k^\nu - g^{\mu\nu}\, k^2\right) \left((Z_3 - 1) - Z_3\, \Pi\right) \\ &= D^{-1\mu\nu}_{ik} + D^{-1\mu\nu}_{ik\,\mathrm{trans}}\, \delta Z_3^{(n)} - \Pi'^{\mu\nu\,(n)}_{ik\,\mathrm{trans}} \end{aligned}$$

$$=$$



We have chosen $\xi_{\mathrm{ren}} = Z_3^{-1}\, \xi_0$. This implies that the gauge fixing Lagrangian $\mathcal{L}_{\mathrm{GF}}$ remains unrenormalized, as the field renormalization is precisely compensated by the gauge parameter renormalization. Again the non-trivial irreducible piece Π' includes diagrams with counterterm insertions of counterterms up to order $(n-1)$. The counter term $\delta Z_3^{(n)}$ at order (n) is determined such that the full (inverse) propagator is finite at this order. Once again we note that there is **no mass renormalization** since

$$\lim_{k^2 \to 0} k^2\, \Pi(k^2) = 0 \ .$$

The **wave function renormalization**

$$\delta Z_3 = Z_3 - 1 = \lim_{k^2 \to 0} \Pi(k^2) \ ,$$

for the colored gluons, like for the colored quarks, is in trouble, because the on–shell limit does not exist! Since the gluons are color charged (and hence self–interacting) massless objects, there is an infrared problem[24]. Again minimal subtraction is possible and commonly accepted. The requirement is not that the residue of the pole should by normalized to unity, but just to be finite: thus

$$\delta Z_3 = \Pi(k^2)\big|_{\mathrm{singular\ part}} \ \text{ at } k^2 = 0 \ ,$$

[24]This is in contrast to the charge neutral photon in QED, in which case the on–shell limit exists.

which yields

$$(\delta Z_3)_{\overline{\rm MS}} \;=\; \tfrac{g^2}{32\pi^2}\left\{C_2(G)\,\left(\tfrac{10}{3}+(1-\xi)\right)-\tfrac{4\,T(R)\,f(d)}{3k^2}\sum_q\left(-m_q^2+m_q^2+\tfrac{k^2}{2}\right)\right\}{\rm Reg}\ ,$$

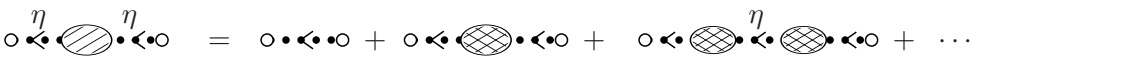
such that, adopting the scheme $f(d)=4$ (convention!),

$$\left\|\ (\delta Z_3)_{\overline{\rm MS}} \;=\; \tfrac{g^2}{32\pi^2}\left\{C_2(G)\,\left(\tfrac{13}{3}-\xi\right)-\tfrac{8}{3}\,T(R)\,N_f\right\}{\rm Reg}\right. \tag{7.17}$$

is our final result for the renormalization constant(s) related to the gluon propagator.

7.3 The Faddeev-Popov ghost self–energy

Since the FP ghosts have to obey Fermi statistics, the structure resembles the one of a quark propagator, however, it is much simpler as it is a scalar ghost. The *Dyson series* of self–energy insertions here is given by



$$\begin{aligned}
{\rm i}\,\tilde D'(k) \;&\equiv\; {\rm i}\,\tilde D(k)\,+{\rm i}\,\tilde D(k)\,\left(-{\rm i}\tilde\Pi\,{\rm i}\,\tilde D\right)(k)\,+{\rm i}\,\tilde D(k)\,\left(-{\rm i}\tilde\Pi\,{\rm i}\,\tilde D\right)^2(k)\,+\cdots\\
&=\; {\rm i}\,\tilde D(k)\,\left(\frac{1}{1-\tilde\Pi\,\tilde D}\right)\ .
\end{aligned}$$

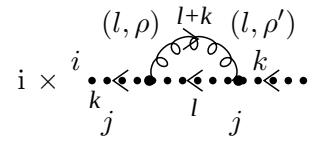
The self–energy itself is obtained by an expansion in a series of 1PI diagrams

(7.18)

The inverse full propagator is

$$\tilde D'^{-1}_{ik}=\tilde D^{-1}_{ik}-\tilde\Pi_{ik}\ \ \text{with}\ \ \tilde D^{-1}_{ik}=k^2\,\delta_{ik}\ .$$

In this case there is one diagram to be calculated at one loop:



$$\begin{aligned}
&=\ {\rm i}\,{\rm i}^4\,\frac{1}{2}\,(-{\rm i}\,g)^2\sum_{lj}f_{lij}\,f_{ljk}\times\,(-1)\,\times\\
&\quad\int\frac{{\rm d}^d\ell}{(2\pi)^d}\,\frac{\ell^\rho\,(-k)^{\rho'}}{\ell^2+{\rm i}\varepsilon}\,\left(g_{\rho\rho'}-(1-\xi)\,\frac{(\ell+k)_\rho(\ell+k)_{\rho'}}{(\ell+k)^2}\right)\,\frac{1}{(\ell+k)^2+{\rm i}\varepsilon}
\end{aligned}$$

The common factor here reads:

$$N_{ik}={\rm i}\,g^2\,C_2(G)\,\delta_{ik}\,\tfrac{\rm i}{16\pi^2}\,.$$

We again consider the two gluon propagator terms in turn.
1st term: $g_{\rho\rho'}$

$$\begin{aligned}
\int_\ell\frac{\ell k}{\ell^2\,(\ell+k)^2} \;&=\; \frac{1}{2}\int_\ell\frac{(\ell+k)^2-\ell^2-k^2}{\ell^2\,(\ell+k)^2}\\
&=\; \frac{1}{2}\int_\ell\frac{1}{\ell^2}-\frac{1}{2}\int_\ell\frac{1}{(\ell+k)^2}-\frac{k^2}{2}\int_\ell\frac{1}{\ell^2\,(\ell+k)^2}\\
&=\; -\frac{k^2}{2}\,B_0\ .
\end{aligned}$$

The first two integrals are zero in DR (no scale).
2nd term: $(1-\xi)\,(\ell+k)_\rho\,(\ell+k)_{\rho'}/(\ell+k)^2$
The integral to be considered is

$$\int_\ell \frac{(\ell^2+\ell k)\,(\ell k+k^2)}{\ell^2\,(\ell+k)^4} \ .$$

The term

$$\int_\ell \frac{\ell^2\,(\ell+k)k}{\ell^2\,(\ell+k)^4} = \int_\ell \frac{\ell k}{\ell^4} = 0$$

obviously vanishes after a shift of the integration momentum. Next we have

$$\begin{aligned}
& \int_\ell \frac{\ell k\,(k^2+\ell k)}{\ell^2\,(\ell+k)^4} = k^2 \int_\ell \frac{\ell k}{\ell^2\,(\ell+k)^4} + \frac{1}{2}\int_\ell \frac{\ell k}{\ell^2\,(\ell+k)^4}\,((\ell+k)^2-\ell^2-k^2) \\
= \ & \frac{k^2}{2}\int_\ell \frac{\ell k}{\ell^2\,(\ell+k)^4} + \frac{1}{2}\int_\ell \frac{\ell k}{\ell^2\,(\ell+k)^2} - \frac{1}{2}\int_\ell \frac{\ell k}{(\ell+k)^2} \\
= \ & \frac{k^2}{2}\,\hat{I}_1(1,2) + \frac{k^2}{2}\,\hat{I}_1(1,1) + 0 = \frac{k^2}{2}\left(B_0 - 1 - \frac{1}{2}\,B_0\right) \ ,
\end{aligned}$$

and thus

$$\tilde{\Pi}_{ik} = \frac{g^2}{32\pi^2}\,\delta_{ik}\,C_2(G)\,k^2\,\left(B_0 + (1-\xi)\,\left(\frac{1}{2}\,B_0 - 1\right)\right) \ .$$

Like the gluon, the ghost remains massless

$$\lim_{k^2\to 0}\,\tilde{\Pi}_{ik} = 0 \ .$$

Here

$$B_0 = B_0(0,0;k^2) = \mathrm{Reg} + 2 - \ln\left(-(k^2+\mathrm{i}0)\right) \ .$$

Renormalization follows according to:

$$\begin{aligned}
\tilde{D}'^{-1}_{ik\,\mathrm{ren}} &= \tilde{Z}_3\,\tilde{D}'^{-1}_{ik\,0} = \tilde{Z}_3\,\left(\delta_{ik}\,k^2 - \tilde{\Pi}_{ik}\right) \\
&= \delta_{ik}\,k^2\,\left(1+\tilde{Z}_3 - 1 - \tilde{Z}_3\tilde{b}\right) = \delta_{ik}\,k^2\,\left(1+\delta\tilde{Z}_3^{(n)} - \tilde{b}'^{(n)}\right) \\
&= \quad \text{•••}\!\!\overset{-1}{\text{•••}} \ + \ \underset{\delta\tilde{Z}_3^{(n)}}{\text{•}\!\!\otimes\!\!\text{•}} \ + \ \text{•}\!\!\overset{\prime}{\text{⊠}}\!\!\text{•}
\end{aligned}$$

with

$$\tilde{b}\ :\ \tilde{\Pi}_{ik} \doteq \delta_{ik}\,k^2\,\tilde{b}(k^2) \ .$$

While there is no mass renormalization, we need the wave function renormalization

$$\tilde{Z}_3 - 1 = \tilde{b}(k^2)\Big|_{\text{singular part}} \ .$$

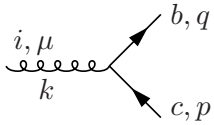
Explicitely, we have

$$\left\| \quad \left(\delta\tilde{Z}_3\right)_{\overline{\mathrm{MS}}} \ = \ \frac{g^2}{32\pi^2}\,C_2(G)\,\frac{3-\xi}{2}\,\mathrm{Reg} \right. \tag{7.19}$$

as a result for the counterterm needed for the renormalization of the FP ghost propagator.

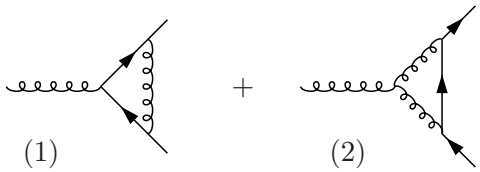
7.4 The quark-gluon vertex

At the tree level (Born approximation) we have the quark–gluon vertex

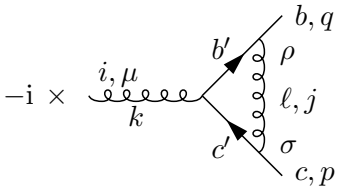


$$:= \quad \mathrm{i}\, g\, \gamma^\mu\, (T_i)_{bc}$$

and there are two diagrams contributing to the one–loop correction



We first calculate diagram (1):



$$= \quad -\mathrm{i}\,\mathrm{i}^6\, g^3 \sum_j (T_j)_{bb'}\, (T_i)_{b'c'}\, (T_j)_{c'c} \times\, (-1) \times$$

$$\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\, \frac{1}{\ell^2 + \mathrm{i}\varepsilon}\, \frac{1}{\ell_1^2 - m^2 + \mathrm{i}\varepsilon}\, \frac{1}{\ell_2^2 - m^2 + \mathrm{i}\varepsilon}\, (\gamma^\rho(\not{\ell}_2 + m)\gamma^\mu(\not{\ell}_1 + m)\gamma^\sigma) \left(g_{\rho\sigma} - (1-\xi)\, \frac{\ell_\rho \ell_\sigma}{\ell^2}\right) \quad .$$

Common factor is:

$$N_1 = -\mathrm{i}\, g^3\, \left(-\tfrac{1}{2n}\right)\, (T_i)_{bc}\, \tfrac{\mathrm{i}}{16\pi^2}\,.$$

To be evaluated

$$\begin{aligned} T_jT_iT_j = T_jT_jT_i + \mathrm{i}\, f_{ijl} T_jT_l \quad &= \quad C_2(R)\, T_i + \frac{1}{2}\, \mathrm{i}\, f_{ijl}\, (T_jT_l + T_lT_j) \\ &= \quad C_2(R)\, T_i - \frac{1}{2}\, f_{ijl} f_{jlk}\, T_k = \left(C_2(R) - \frac{1}{2}\, C_2(G)\right)\, T_i = -\frac{1}{2n}\, T_i \ , \end{aligned}$$

where we have used the $SU(n)$ relations

$$\begin{aligned} \sum_j (T_jT_j)_{bc'} \quad &= \quad C_2(R)\, \delta_{bc'}\ ; \ \ C_2(R) = \frac{n^2-1}{2n} \ , \\ \sum_{jl} f_{ijl} f_{kjl} \quad &= \quad C_2(G)\, \delta_{ik}\ ; \ \ C_2(G) = n \ . \end{aligned}$$

We consider the two terms from the gluon propagator separately.
1st term: $g_{\rho\sigma}$
The numerator reds:

$$\begin{aligned} X_1 \quad &= \quad \gamma^\alpha(\not{\ell}_2 + m)\gamma^\mu(\not{\ell}_1 + m)\gamma_\alpha = (6-d)\, \not{\ell}_2 \gamma^\mu \not{\ell}_1 \\ &+ \quad 4\, \gamma^\mu\, \ell_1\ell_2 - 4\, (\not{\ell}_2 \ell_1^\mu + \ell_2^\mu \not{\ell}_1) + m\, ((d-4)(\not{\ell}_2\gamma^\mu + \gamma^\mu \not{\ell}_1) + 4\, (\ell_1^\mu + \ell_2^\mu)) + m^2\, (2-d)\, \gamma^\mu \ . \end{aligned}$$

In order to keep the number of amplitudes small we only consider the one's which directly contribute to the on–shell matrix elements, i.e. we consider the expression above sandwiched

$$\bar{u}(q) \cdots u(p)$$

between the external spinors (=one–particle wave functions) and use the Dirac equation:

$$\not{q}u(q) = m\,u(q)\ ;\ \ \not{p}u(p) = m\,u(p)\ .$$

Doing so we obtain

$$\begin{aligned} X_1 &= (d-2)\,\gamma^\mu\ell^2 - 2\gamma^\mu k^2 + (4-2d)\,\not{\ell}\ell^\mu + 4\gamma^\mu\ell Q - 4\not{\ell}Q^\mu + 4m\ell^\mu + 4m^2\gamma^\mu \\ &= (d-6)\gamma^\mu\ell^2 + 2\gamma^\mu\,(\ell_1^2 - m^2) + 2\gamma^\mu(\ell_2^2 - m^2) + 4\gamma^\mu pq + 2(2-d)\ell^\mu\not{\ell} - 4\not{\ell}Q^\mu + 4m\ell^\mu\ , \end{aligned}$$

where we used: $Q \doteq p+q,\ p^2 = q^2 = m^2,\ 2\ell Q = (\ell_1^2 - m^2) + (\ell_2^2 - m^2) - 2\ell^2,\ 2pq = 2m^2 - k^2.$
The integral related to the 1st term then yields $[(1) = \ell_1^2 - m^2,\ (2) = \ell_2^2 - m^2,\ (3) = \ell^2]$

$$\begin{aligned} T_1 &= \int_\ell \Bigg\{ \gamma^\mu \left[(d-6)\,\frac{1}{(1)(2)} + 2\,\frac{1}{(1)(3)} + 2\,\frac{1}{(2)(3)} + 4\,pq\,\frac{1}{(1)(2)(3)} \right] \\ &\quad -4\,(Q^\mu\gamma^\alpha - mg^{\mu\alpha})\,\frac{\ell_\alpha}{(1)(2)(3)} + 2\,(2-d)\,g^{\mu\alpha}\gamma^\beta\,\frac{\ell_\alpha\ell_\beta}{(1)(2)(3)} \Bigg\} \\ &= \gamma^\mu \left[(d-6)\,B_0(m,m;k^2) + 2\,B_0(0,m;p^2) + 2\,B_0(0,m;q^2) - 4\,pq\,C_0(0,m,m;p^2,k^2,q^2) \right] \\ &\quad +4\,(Q^\mu\gamma^\alpha - mg^{\mu\alpha})\,(p_\alpha C_{11} + k_\alpha\,C_{12}) \\ &\quad +2\,(d-2)\,g^{\mu\alpha}\gamma^\beta\,(p_\alpha p_\beta\,C_{21} + k_\alpha k_\beta\,C_{22} + (p_\alpha k_\beta + k_\alpha p_\beta)\,C_{23} - g_{\alpha\beta}\,C_{24})\ , \end{aligned}$$

where we adopted Veltman's convention (sign conventions corresponds to Euclidean metric) for the tensor integrals (6.47)

$$\begin{aligned} \int_\ell \frac{\ell_\alpha}{(1)(2)(3)} &= -p_\alpha\,C_{11} - k_\alpha\,C_{12}\ , \\ \int_\ell \frac{\ell_\alpha\ell_\beta}{(1)(2)(3)} &= -p_\alpha p_\beta\,C_{21} - k_\alpha k_\beta\,C_{22} - (p_\alpha k_\beta + k_\alpha p_\beta)\,C_{23} + g_{\alpha\beta}\,C_{24}\ . \end{aligned}$$

Using once more the on–shell conditions (applying the Dirac equation), we have

$$\begin{aligned} &4\,(Q^\mu\gamma^\alpha - mg^{\mu\alpha})\,p_\alpha = 4\,(Q^\mu - p^\mu)\,m = 2\,m\,(Q+k)^\mu\ , \\ &4\,(Q^\mu\gamma^\alpha - mg^{\mu\alpha})\,k_\alpha = -4\,m\,k^\mu\ , \\ &g^{\mu\alpha}\gamma^\beta\,p_\alpha p_\beta = m\,p^\mu = m\,\frac{(Q-k)^\mu}{2}\ , \\ &g^{\mu\alpha}\gamma^\beta\,k_\alpha k_\beta = 0\ , \\ &g^{\mu\alpha}\gamma^\beta\,(p_\alpha k_\beta + k_\alpha p_\beta) = m\,k^\mu\ , \\ &g^{\mu\alpha}\gamma^\beta\,g_{\alpha\beta} = \gamma^\mu\ , \end{aligned}$$

and we obtain

$$\begin{aligned} T_1 &= \gamma^\mu \left\{ (d-6)\,B_0(m,m;k^2) + 2\,B_0(0,m;p^2) + 2\,B_0(0,m;q^2) - 4\,pq\,C_0 - 2\,(d-2)\,C_{24} \right\} \\ &\quad +m\,Q^\mu \left\{ 2\,C_{11} + (d-2)\,C_{12} \right\} + m\,k^\mu \left\{ 2\,C_{11} - 4\,C_{12} - (d-2)\,C_{12} + C_{23} \right\}\ . \end{aligned}$$

The tensor integrals may be written in terms of C_0, B_0's and A_0's according to (6.48)–(6.50). Here we only note that all invariant C_i functions are UV finite except for C_{24}, for which

$$(C_{24})_{\rm UV} = \frac{1}{4}\,{\rm Reg}\ .$$

Together with the Reg terms from the B_0's ($B_0 = \mathrm{Reg} + \cdots$) we find

$$(T_1)_{\mathrm{UV}} = \gamma^\mu \,\mathrm{Reg}\ .$$

On–shell $q^2 = p^2 = m^2$ and In the relativistic limit $m \to 0$ one obtains $C_{24} \simeq \frac{1}{4}\,(B_0(0,0;k^2)+1)$ such that

$$T_1 \simeq \gamma^\mu \left\{ 4\,B_0(0,m;m^2) - 3\,B_0(0,0;k^2) - 2 + 2\,k^2\,C_0(0,m,m;q^2,k^2,p^2) \right\}\ .$$

However, the on–shell limit does not exist! There is an infrared problem $C_0(0,m,m;q^2,k^2,p^2)$ has a logarithmic IR singularity for $q^2 \to m^2$ and $p^2 \to m^2$.
2nd term: $-(1-\xi)\,\not{\ell}\,(\not{\ell}_2 + m)\,\gamma^\mu\,(\not{\ell}_1 + m)\,\not{\ell}\,\frac{1}{\ell^2}$
Considering first the numerator: if we utilize the on–shell conditions

$$\bar{u}(q)\,(\not{q} - m)\cdots = 0 \ \text{ and } \ \cdots(\not{p} - m)\,u(p) = 0\ ,$$

respectively, we may write

$$\begin{aligned} & \not{\ell}\,(\not{\ell}_2 + m)\,\gamma^\mu\,(\not{\ell}_1 + m)\,\not{\ell} \\ = \ & (\not{\ell}_2 - m)(\not{\ell}_2 + m)\gamma^\mu(\not{\ell}_1 + m)(\not{\ell}_1 - m) = (\ell_2^2 - m^2)\,\gamma^\mu\,(\ell_1^2 - m^2) \end{aligned}$$

and the quark propagators drop out. The remaining integral is IR divergent

$$-(1-\xi)\int_\ell \frac{1}{\ell^4}\,\gamma^\mu\ .$$

In DR such integrals are zero, however, it mixes up UV and IR singularities, which cancel against each other. It should be noted that off-shell there is **no** IR problem; one thus has to study from case to case carefully how limits are taken. Alternatively, one may use an infrared regularization, e.g. in infinitesimally small gluon mass: the inverse propagator in the Lagrangian in momentum space has the form

$$\begin{aligned} & -\left(g_{\mu\nu}\,(k^2 - m_g^2) - (1 - \frac{1}{\xi})\,k_\mu k_\nu \right) \\ = \ & -\left(g_{\mu\nu} - (1 - \frac{1}{\xi})\,\frac{k_\mu k_\nu}{k^2 - m_g^2} \right)\,(k^2 - m_g^2) \end{aligned}$$

such that the propagator takes the form

$$-\left(g_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2 - \xi\,m_g^2} \right)\,\frac{1}{k^2 - m_g^2 + \mathrm{i}\varepsilon}\ .$$

As a result

$$\int_\ell \frac{1}{\ell^4} \to \int_\ell \frac{1}{\ell^2 - m_g^2}\,\frac{1}{\ell^2 - \xi\,m_g^2} = B_0(m_g, \sqrt{\xi} m_g; 0)$$

and we find

$$T_2 = (1-\xi)\,B_0(\cdots) = (1-\xi)\,(\mathrm{Reg} + 1 - \ln m_g^2) + \xi\,\ln\xi\ ,$$

which is UV and IR singular! This phenomenon is known from QED. Because of the massless photon the interaction has infinite range (Coulomb interaction) and one cannot take the charged

leptons on–shell, because they never become truly free[25]. In perturbative QCD the interaction with the massless gluons prevents the on–shell limit to exist for all color charged "perturbative states", i.e., for quarks as well as for the gluons. Due to confinement of QCD, which only allows colorless hadrons as physical states, the non-existence of the on–shell limit in fact is an academic problem as free quarks and gluons are not physical. For off-shell quarks there are no IR singularities, however the UV singularity is always there. With

$$(T_2)_{\rm UV} = \gamma^\mu \; (-(1-\xi)\,{\rm Reg})$$

we find the complete UV term for diagram (1):

$$(T_1+T_2)_{\rm UV}\,(1) = \gamma^\mu\,\xi\,{\rm Reg}\ .$$

We may now summarize the result for the vertex form factor from diagram (1):

$$g\,\gamma^\mu\;(T_i)_{bc}\;(1+\delta g)\ \ \text{for}\ \ p^2=q^2=m^2\,(m_g\ \text{finite infinitesimal})$$

with

$$\begin{aligned}
\delta g^{(1)} &= \frac{g^2}{16\pi^2}\left(C_2(R)-\frac{1}{2}\,C_2(G)\right)\\
\times\Big\{&4\,B_0(0,m;m^2)-3\,B_0(m,m;k^2)-2+2\,(k^2-2m^2)\,C_0(m_g,m,m;m^2,k^2,m^2)\\
&-(1-\xi)\,B_0(m_g,\sqrt{\xi}m_g;0)\Big\}
\end{aligned}$$

Explicitely we have:

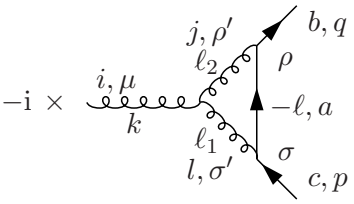
$$\begin{aligned}
B_0(0,m;m^2) &= {\rm Reg}+2-\ln m^2\\
B_0(m,m;k^2) &= {\rm Reg}+2-\ln m^2+2\left(\frac{4m^2}{k^2}-1\right)G\left(\frac{4m^2}{k^2}\right)\\
B_0(0,m;k^2) &\simeq {\rm Reg}+2-\ln\left(-(k^2+{\rm i}0)\right)\ ;\ \ m\to 0\\
B_0(m_g,\sqrt{\xi}m_g;0) &= {\rm Reg}+1-\ln m_g^2+\frac{\xi}{1-\xi}\,\ln\xi\\
C_0(m_g,m,m;m^2,k^2,m^2) &= \frac{2}{k^2\sqrt{1-y}}\left\{\ln x\,\ln\frac{m}{m_g}+{\rm Sp}(-x)+\frac{\pi^2}{12}-\frac{1}{4}\ln^2 x+\ln x\,\ln(1+x)\right\}\\
\text{where}\qquad & y=\frac{4m^2}{k^2}\ ;\ \ x=\frac{\sqrt{1-y}-1}{\sqrt{1-y}+1}\ ;\ \ {\rm Sp}(x)=-\int_0^1\frac{{\rm d}t}{t}\,\ln(1-xt)\ .
\end{aligned}$$

The new function ${\rm Sp}(x)$ is called dilogarithm or Spence function. The UV singular term reads

$$\delta g^{(1)}_{\rm UV} = \frac{g^2}{32\pi^2}\left(C_2(R)-\frac{1}{2}\,C_2(G)\right)2\xi\,{\rm Reg}\ .$$

[25] As we know from QED, it is a matter of looking for well defined realistic observables to avoid such singularities. One solution is the Bloch-Nordsieck prescription, which requires to include soft real photons in addition to the virtual ones, since they are indistinguishable experimentally.

We now calculate diagram (2): (we denote $\ell_2 = \ell + q$ and $\ell_1 = \ell + p$)



$$
\begin{aligned}
= \quad & -\mathrm{i}\,\mathrm{i}^6\,(\mathrm{i}g)\,g^2 \sum_{jl} f_{ijl}\,(T_j)_{ba}\,(T_l)_{ac} \times \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\,\gamma^\rho \frac{(-\not{\ell}+m)}{\ell^2-m^2}\gamma^\sigma \times \frac{1}{\ell_1^2}\frac{1}{\ell_2^2} \\
& \times\Big\{ g^{\rho'\sigma'}\,(\ell_1+\ell_2)^\mu - g^{\mu\rho'}(\ell_2+k)^{\sigma'} - g^{\mu\sigma'}(\ell_1-k)^{\rho'} \Big\} \\
& \times \left(g_{\rho\rho'} - (1-\xi)\,\frac{\ell_{2\rho}\ell_{2\rho'}}{\ell_2^2} \right) \left(g_{\sigma\sigma'} - (1-\xi)\,\frac{\ell_{1\sigma}\ell_{1\sigma'}}{\ell_1^2} \right) \ .
\end{aligned}
$$

Common factor here is:

$$N_2 = -\mathrm{i}\,\tfrac{g^3}{2}\,C_2(G)\;(T_i)_{bc}\;\tfrac{\mathrm{i}}{16\pi^2}\,,$$

and we already used

$$f_{ijl}\,T_jT_l = \frac{1}{2}\,f_{ijl}\;\{T_j,T_l\} + \frac{\mathrm{i}}{2}\,f_{ijl}f_{jlk}\,T_k = \frac{\mathrm{i}}{2}\,C_2(G)\,\delta_{ik}\,T_k \ .$$

We again sort contributions according to the terms of the gluon propagators. This time we have a product of two propagators.
1st term: $g_{\rho\rho'}g_{\sigma\sigma'}$
for the numerator yields

$$
\begin{aligned}
X_1 \quad &= \quad \gamma^\rho\,(-\not{\ell}+m)\,\gamma^\sigma \;\times\; \big\{ g_{\rho\sigma}\,(\ell_1+\ell_2)^\mu - g^\mu{}_\rho(\ell_2+k)_\sigma - g^\mu{}_\sigma(\ell_1-k)_\rho \big\} \\
&= \quad 2\,(d-2)\,\not{\ell}\ell^\mu + (d-2)\,\not{\ell}Q^\mu + 2\,m\,d\,\ell^\mu + m\,d\,Q^\mu + \gamma^\mu\not{\ell}\,(\not{\ell}+2\not{q}-\not{p}) + (\not{\ell}+2\not{p}-\not{q})\,\not{\ell}\gamma^\mu \\
&\quad\; -m\,(2\ell^\mu+4Q^\mu) + 6m^2\gamma^\mu \\
&= \quad 2\,(d-2)\,\not{\ell}\ell^\mu + (d-6)\,\not{\ell}Q^\mu + 2\,m\,d\,\ell^\mu + m\,(d-4)\,Q^\mu + 2\gamma^\mu\ell^2 + 4\,\ell Q\,\gamma^\mu + 6m^2\gamma^\mu \ .
\end{aligned}
$$

Here we again have used the on–shell condition and replaced $\not{p} = m$ and $\not{q} = m$ when sandwiched between the external quark spinors and collected different terms. If we use $2\ell Q = \ell_1^2+\ell_2^2-2\ell^2-2m^2$ and $p^2 = m^2$ and $q^2 = m^2$ we obtain

$$X_1 \quad = \quad -2\gamma^\mu\,(\ell^2-m^2) + 2\gamma^\mu\ell_1^2 + 2\gamma^\mu\ell_2^2 + (d-6)\,\not{\ell}Q^\mu + 2\,(d-2)\,\not{\ell}\ell^\mu + 2\,m\,d\,\ell^\mu + m\,(d-4)\,Q^\mu \ .$$

Denoting $(1) = \ell_1^2$, $(2) = \ell_2^2$ and $(3) = (\ell^2-m^2)$ we may write

$$
\begin{aligned}
T_1 \quad &= \quad \int_\ell \Big\{ \gamma^\mu \left[-2\frac{1}{(1)(2)} + 2\frac{1}{(1)(3)} + 2\frac{1}{(2)(3)} \right] \\
&\quad\; +m\,(d-4)\,Q^\mu \frac{1}{(1)(2)(3)} + \left((d-6)\,Q^\mu\gamma^\alpha + 2\,m\,d\,g^{\mu\alpha} \right) \frac{\ell_\alpha}{(1)(2)(3)} + 2\,(d-2)\,g^{\mu\alpha}\gamma^\beta\,\frac{\ell_\alpha\ell_\beta}{(1)(2)(3)} \Big\} \\
&= \quad \gamma^\mu\,\left[-2\,B_0(0,0;k^2) + 2\,B_0(m,0;p^2) + 2\,B_0(m,0;q^2) \right] \\
&\quad\; -m\,(d-4)\,Q^\mu\,C_0(m,0,0;p^2,k^2,q^2) \\
&\quad\; -\left((d-6)\,Q^\mu\gamma^\alpha + 2\,m\,d\,g^{\mu\alpha} \right)\,(p_\alpha C_{11} + k_\alpha\,C_{12}) \\
&\quad\; -2\,(d-2)\,g^{\mu\alpha}\gamma^\beta\;(p_\alpha p_\beta\,C_{21} + k_\alpha k_\beta\,C_{22} + (p_\alpha k_\beta + k_\alpha p_\beta)\,C_{23} - g_{\alpha\beta}\,C_{24}) \ .
\end{aligned}
$$

Setting $p = (Q-k)/2$, $q = (Q+k)/2$ and applying the on–shell condition (Dirac equation) we finally have

$$\begin{aligned} T_1 &= \gamma^\mu \left[2\,(d-2)\,C_{24} - 2\,B_0(0,0;k^2) + 2\,B_0(m,0;p^2) + 2\,B_0(m,0;q^2)\right] \\ &\quad + m\,Q^\mu \left[(4-d)\,C_0 + (6-2d)\,C_{11} - (d-2)\,C_{21}\right] \\ &\quad + m\,k^\mu \left[d\,C_{11} - 2\,d\,C_{12} + (d-2)\,C_{21} - 2\,(d-2)\,C_{23}\right] \ , \end{aligned}$$

where the last two terms are finite such that

$$(T_1)_{\rm UV}\,(2) = \gamma^\mu\, 3\,{\rm Reg} \ .$$

2nd term: $-(1-\xi)\,g_{\rho\rho'}\,\frac{\ell_{1\sigma}\ell_{1\sigma'}}{\ell_1^2}\,(+\ {\rm sym}\ \rho,\ell_1 \leftrightarrow \sigma,\ell_2)$ from the product of gluon propagators. Here we obtain a numerator

$$\begin{aligned} X_2 &= -(1-\xi)\,\gamma^\rho\,(-\not\ell + m)\,\not\ell_1 \frac{1}{\ell_1^2}\,(+\ {\rm sym}\ \ell_1 \leftrightarrow \ell_2) \\ &\quad \times \left\{ \ell_1^\rho\,(\ell_1+\ell_2)^\mu - g^{\mu\rho}\,\underbrace{(\ell_2^2 - k^2)}_{\ell_1(\ell_2+k)} - \ell_1^\mu\,(\ell_1-k)^\rho \right\} \\ &= -(1-\xi)\,\frac{1}{\ell_1^2} \times \left\{ \not\ell_1\,(-\not\ell+m)\,\not\ell_1 \ell_2^\mu - \gamma^\mu\,(-\not\ell+m)\,\not\ell_1\,(\ell_2^2-k^2) + \not k\,(-\not\ell+m)\,\not\ell_1 \ell_1^\mu \right\} \ , \end{aligned}$$

and using the on–shell relations $\not\ell_1 u(p) = (\not\ell + m)u(p)$, $\bar u(q)\not k u(p) = 0$ etc. we arrive at

$$X_2 = (1-\xi)\,\frac{1}{\ell_1^2}\,(\ell^2 - m^2) \times \left\{ \underbrace{\not\ell_1}_{(\not\ell_2 - \not k) = \not\ell_2}\ \ell_2^\mu - \gamma^\mu\,(\ell_2^2 - k^2) + \underbrace{\not k}_{=0}\ \ell_1^\mu \right\} \ .$$

We thus have to evaluate

$$T_2 = (1-\xi) \int_\ell \left\{ \frac{\not\ell_2 \ell_2^\mu}{\ell_1^4 \ell_2^2} - \gamma^\mu \frac{1}{\ell_1^4} + \gamma^\mu\,k^2\,\frac{1}{\ell_1^4\ell_2^2} \right\} \ .$$

First we note that here we have no IR problem if we consider the two IR singular terms together, indeed $(\ell_2^2 - k^2)/\ell_1^4/\ell_2^2 - \ell_1(\ell_2+k)/\ell_1^4/\ell_2^2$ is O.K. The integrals are

$$\begin{aligned} \int_{\ell_1} \frac{1}{\ell_1^4} &= 0 \\ k^2 \int_{\ell_1} \frac{1}{\ell_1^4 \ell_2^2}\bigg|_{\ell_2 = \ell_1 + k} &= \hat I_0(2,1) = -B_0 + 2 \\ \int_{\ell_2} \frac{\not\ell_2 \ell_2^\mu}{\ell_1^4 \ell_2^2}\bigg|_{\ell_1 = \ell_2 - k} &= \not k \frac{k^\mu}{k^2}\,\hat I_{21}(1,2) + \gamma^\mu\ \underbrace{\hat I_{22}(1,2)}_{\frac{1}{4}B_0} \ . \end{aligned}$$

The contribution from the first term of the last integral vanishes on–shell ($\not k = 0$). Again there is no IR problem in the γ^μ amplitude. Also note $\not\ell_2 \ell_2^\mu \to \not\ell_1 \ell_1^\mu$ + terms proportional to $\not k$, which vanish on–shell. We thus arrive at

$$T_2 = 2\,(1-\xi)\,\gamma^\mu \left\{ \frac{1}{4}\,B_0(0,0;k^2) - B_0(0,0;k^2) + 2 \right\}$$

with a factor 2 for including the symmetric term ($\ell_1 \leftrightarrow \ell_2$) and thus

$$(T_2)_{\rm UV}\,(2) = \gamma^\mu \left(-(1-\xi)\,\frac{3}{2}\,{\rm Reg} \right) \ .$$

3rd term: $(1-\xi)^2 \not{\ell}_2 (-\not{\ell}+m) \not{\ell}_1 \frac{1}{\ell_1^2} \frac{1}{\ell_2^2}$, from the product of the two gluon propagators, yields in the numerator

$$\begin{aligned} X_3 &= (1-\xi)^2 \not{\ell}_2 (-\not{\ell}+m) \not{\ell}_1 \frac{1}{\ell_1^2} \frac{1}{\ell_2^2} \\ &\quad \times \left\{ \underbrace{(\ell_1\ell_2)}_{\frac{1}{2}(\ell_1^2+\ell_2^2-k^2)} (\ell_1+\ell_2)^\mu - \ell_2^\mu \underbrace{(\ell_2(\ell_2+k))}_{\ell_2^2-k^2} - \ell_1^\mu \underbrace{((\ell_1-k)\ell_2)}_{\ell_1^2-k^2} \right\} \\ &= -(1-\xi)^2 \frac{1}{2} \frac{\ell^2-m^2}{\ell_1^2\ell_2^2} \not{\ell}_2 [= \not{\ell}_1 \text{ on } - \text{ shell}] \cdot \left\{ k^2 (\ell_1+\ell_2)^\mu + \ell_1^2 k^\mu - \ell_2^2 k^\mu \right\} \ , \end{aligned}$$

which yields

$$T_3 = -\frac{(1-\xi)^2}{2} \int_\ell \left\{ k^2 \frac{\not{\ell}_1 \ell_1^\mu}{\ell_1^4\ell_2^4} + k^2 \frac{\not{\ell}_2 \ell_2^\mu}{\ell_1^4\ell_2^4} + k^\mu \frac{\not{\ell}_1}{\ell_1^2\ell_2^4} - k^\mu \frac{\not{\ell}_2}{\ell_1^4\ell_2^2} \right\} \ .$$

Here we have no IR problem in the γ^μ amplitude. The problematic terms may be written as

$$\frac{\not{\ell}_1 \ell_2^\mu}{\ell_1^4\ell_2^4} \ \text{ and } \ \frac{\not{\ell}_2 \ell_1^\mu}{\ell_1^4\ell_2^4} \ .$$

Between spinors only γ^μ terms survive as $\bar{u}(q)\not{k}u(p) = 0$. Integration variables are ℓ_1 with $\ell_2 = \ell_2 + k$ or ℓ_2 with $\ell_1 = \ell_2 - k$. Only terms $g^{\mu\alpha}\gamma_\alpha = \gamma^\mu$ of the integrals:

$$k^2 \int_{\ell_1} \frac{\not{\ell}_1 \ell_1^\mu}{\ell_1^4\ell_2^4} = \not{k} \frac{k^\mu}{k^2} \hat{I}_{21}(2,2) + \gamma^\mu \underbrace{\hat{I}_{22}(2,2)}_{\frac{1}{2}}$$

contribute and with twice the same term we arrive at

$$T_3 = \gamma^\mu \left(-\frac{(1-\xi)^2}{2} \right) \quad \text{finite!}$$

and hence

$$(T_3)_{\rm UV}(2) = 0 \ .$$

Herewith we have the result for the vector form–factor from diagram (2) for $q^2 = p^2 = m^2$

$$\begin{aligned} \delta g^{(2)} &= \frac{g^2}{16\pi^2} \left(\frac{C_2(G)}{2} \right) \times \\ &\quad \left\{ 4\, C_{24}(m,0,0;m^2,k^2,m^2) - 1 - 2\, B_0(0,0;k^2) + 4\, B_0(m,0;m^2) \right. \\ &\quad \left. -2(1-\xi) \left(\frac{3}{4} B_0(0,0;k^2) - 2 \right) - (1-\xi)^2 \frac{1}{2} \right\} \\ &= \frac{g^2}{32\pi^2} \frac{C_2(G)}{2} (6 - 3(1-\xi)) \ \text{Reg} + \text{finite} \end{aligned}$$

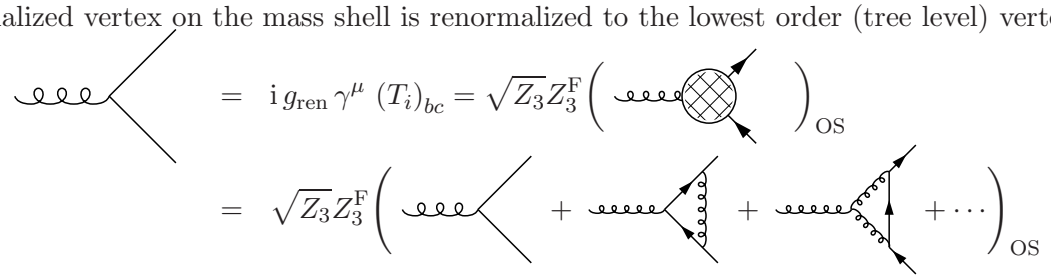
where $4C_{24} \to B_0(0,0;k^2) + 1$ for $m \to 0$. The ε–pole term reads

$$\delta g^{(2)}_{\rm UV} = \frac{g^2}{32\pi^2} \frac{C_2(G)}{2} (6 - 3(1-\xi)) \ \text{Reg} \ .$$

So far we only calculated the dimensionally regularized **bare** amplitudes. In order to obtain the physical amplitudes for $d = 4$ one has to perform the renormalization and take the limit $\varepsilon \to 0$. As we know, this limit only exists after renormalization.

Renormalization of the quark–gluon vertex

Again we first do what one would do in QED, namely, we try the on–shell renormalization. The renormalized vertex on the mass shell is renormalized to the lowest order (tree level) vertex



$$
\begin{aligned}
&= \mathrm{i}\, g_{\rm ren}\, \gamma^\mu\, (T_i)_{bc} = \sqrt{Z_3} Z_3^{\rm F} \Big(\quad \Big)_{\rm OS} \\
&= \sqrt{Z_3} Z_3^{\rm F} \Big(\quad + \quad + \quad + \cdots \Big)_{\rm OS} \\
&= \mathrm{i}\, \sqrt{Z_3} Z_3^{\rm F}\, g_0\, \gamma^\mu\, (T_i)_{bc} + \cdots = \mathrm{i}\, \sqrt{Z_3} Z_3^{\rm F}\, Z_g\, g_{\rm ren}\, \gamma^\mu\, (T_i)_{bc} + \cdots \\
&= \mathrm{i}\, g_{\rm ren}\, \Gamma^\mu_{i\,bc\ {\rm ren}}
\end{aligned}
$$

where OS means on–shell: $q^2 = p^2 \to m^2$ and $k^2 \to 0$, and we have used the parameter renormalization relations

$$
g_0 = g_{\rm ren} + \delta g = g_{\rm ren} \left(1 + \frac{\delta g}{g} \right) = Z_g\, g_{\rm ren} \ .
$$

We again denote $\delta Z_i = Z_i - 1$. The expression above is then expanded in $g_{\rm ren}$, which we usually simply write g, while in the bare calculation before the renormalization g meant actually the bare g_0. The expansion yields

$$
\Gamma^\mu_{i\,bc\ {\rm ren}} = (T_i)_{bc} \left(\gamma^\mu\, A_1 + Q^\mu\, A_2 + k^\mu\, A_3 \right)
$$

and we may write the only amplitude A_1 which picks external renormalization factors (the latter only enter proportional to the Born term)

$$
A_1 = 1 + \frac{1}{2}\, \delta Z_3 + \delta Z_3^{\rm F} + \frac{\delta g}{g} + \quad + \quad + \cdots
$$

In the OS renormalization scheme, after factorization of the renormalized vertex (defining the renormalized coupling constant), the renormalization condition for A_1 reads

$$
(A_1)_{\rm OS} \equiv 1 \ .
$$

This fixes the charge renormalization counterterm to be given by

$$
\frac{\delta g}{g} = \delta Z_g = -\left(\frac{1}{2}\, \delta Z_3 + \delta Z_3^{\rm F} + \quad + \quad + \cdots \right)_{\rm OS} \ .
$$

In QCD one mainly applies the (modified) minimal subtraction scheme ($\overline{\rm MS}$ –scheme) defined essentially by

$$
(\delta Z_g)_{\rm MS} = (\delta Z_g)_{\rm UV\ singular\ part}
$$

and collecting the different terms we have calculated above we obtain

$$
\begin{aligned}
(\delta Z_g)_{\rm MS} \quad = \quad & -\frac{g^2}{32\pi^2} \Bigg\{ \frac{C_2(G)}{2} \left(\frac{10}{3} + (1-\xi) \right) - \frac{4}{3}\, T(R)\, N_f + C_2(R)\, (-2\xi) + C_2(R)\, (2\xi) \\
& - \frac{C_2(G)}{2}\, (2\xi) + \frac{C_2(G)}{2}\, (6 - 3(1-\xi)) \Bigg\} {\rm Reg}
\end{aligned}
$$

and hence

$$\left\| \quad (\delta Z_g)_{\overline{\rm MS}} \;=\; 1 - \tfrac{g^2}{32\pi^2}\,\left\{\tfrac{11}{3}\,C_2(G) - \tfrac{4}{3}\,T(R)\,N_f\right\}\,{\rm Reg} \right. \tag{7.20}$$

a gauge invariant result! It was first published by Politzer, Gross and Wilczek in 1973. It may be not completely clear at this point why this result was honored with the Nobel prize in 2004!

Using the Slavnov-Taylor identities for the Z–factors we have a complete set of one-loop renormalization constants for QCD. They are the basic input or the renormalization group to be discussed next.

Summary of minimal subtraction renormalization constants:

$$\begin{aligned}
Z_q = Z_{\rm F} &= 1 + \frac{g^2}{16\pi^2}\,(-\xi\,C_2(R))\,{\rm Reg} \\
Z_{\rm G} = Z_3 &= 1 + \frac{g^2}{32\pi^2}\,\left(C_2(G)\,\left(\frac{13}{3} - \xi\right) - \frac{8}{3}\,T(R)\,N_f\right)\,{\rm Reg} \\
Z_\eta = \tilde{Z}_3 &= 1 + \frac{g^2}{32\pi^2}\,\left(\frac{1}{2}\,C_2(R)\,(3-\xi)\right)\,{\rm Reg} \\
Z_g &= 1 + \frac{g^2}{16\pi^2}\,\left(\frac{11}{3}\,C_2(R) - \frac{4}{3}\,T(R)\,N_f\right)\,{\rm Reg} \\
Z_m &= 1 + \frac{g^2}{16\pi^2}\,(-3\,C_2(R))\,{\rm Reg}
\end{aligned} \tag{7.21}$$

Remark on $\overline{\rm MS}$ renormalization:
In case we have omitted the factors $\mu^{\varepsilon/2}$ needed to define the dimensionless bare coupling $g_0 = \bar{g}_0\,\mu^{-\varepsilon/2}$ in $d \neq 4$, we have to identify g_0 with $\bar{g}_0$ and a bare amplitude at one loop [(1)] has the form

$$\begin{aligned}
A_0 &= 1 + \bar{g}_0^2\,\left(a_{0s}^{(1)}\,{\rm Reg} + a_{0r}^{(1)}\right) \\
&= 1 + g_0^2\,\mu^\varepsilon\,\left(a_{0s}^{(1)}\,{\rm Reg} + a_{0r}^{(1)}\right)\;,
\end{aligned}$$

with $a_{0s}^{(1)}$ the coefficient of the singular term (ε–poles) and $b_{0r}^{(1)}$ a regular one. The effect of the scale factors μ^{4-d} (which usually is included in the definitions of the standard integrals (see Fig. 6.2)) is a substitution:

$$\begin{aligned}
\mu^\varepsilon\,{\rm Reg} &= {\rm e}^{\frac{\varepsilon}{2}\,\ln\mu^2}\,{\rm Reg} \sim {\rm Reg} + \frac{\varepsilon}{2}\,\ln\mu^2\,{\rm Reg} \\
&\sim {\rm Reg} + \ln\mu^2 \;\text{ as }\; \varepsilon \to 0
\end{aligned}$$

where we used

$${\rm Reg} = \frac{2}{\varepsilon} - \gamma + \ln 4\pi\;.$$

Since $g_0 = g_{\rm ren} + \delta g$ (we simply write g for $g_{\rm ren}$ in the following) with $\delta g/g = O(g^2)$ the bare amplitude may be written in terms of the renormalized coupling as

$$A_0 = 1 + g^2\,\left(a_{0s}^{(1)}\,{\rm Reg} + a_{0s}^{(1)}\,\ln\mu^2 + b_{0r}^{(1)}\right)\;.$$

The renormalized amplitude for "minimal subtraction" then reads

$$A_{\rm ren}^{(\overline{\rm MS})} = 1 + g^2\,\left(a_{0s}^{(1)}\,\ln\mu^2 + b_{0r}^{(1)}\right)\;,$$

which means that renormalized quantities in the $\overline{\rm MS}$ scheme follow from the bare ones by the substitution

$$\mathrm{Reg} \to \ln \mu^2 \quad [\overline{\rm MS}] \qquad \text{or} \qquad \frac{2}{\varepsilon} \to \ln \mu^2 \quad [\rm MS] \ .$$

Appendix on group factors for $SU(3)$

The basic relations are

$$[T_i, T_k] = \mathrm{i}\, f_{ikl}\, T_l \ ; \ \ \{T_i, T_k\} = d_{ikl}\, T_l + \frac{1}{3}\, \delta_{ik}$$

Further useful relations are:

$$\mathrm{Tr}\ T_i T_k T_l = \frac{\mathrm{i}}{4}\, f_{ikl} + \frac{1}{4}\, d_{ikl} \ ,$$

$$\sum_{ikl} d_{ikl}^2 = \frac{40}{3} \ ; \ \ \sum_{ikl} f_{ikl}^2 = 24$$

and

$$\sum_{dei} \varepsilon_{ade}\ (T_i)_{bd}\ (T_i)_{ec} = -\frac{2}{3}\, \varepsilon_{abc} \ .$$

Casimir invariants are the following:
1. Casimir of the regular representation of $G = SU(n)$:

$$C_2(G)\, \delta_{ik} = \sum_{lm} f_{ilm}\, f_{klm} \ ; \ \ C_2(G) = n$$

2. Let r be the dimension of the group G (number of generators) and $d(r)$ the dimension of the representation R. With

$$T(R)\, \delta_{ik} = \mathrm{Tr}\ \ (T_i\, T_k) \ = \sum_{ab} (T_i)_{ab}\, (T_k)_{ba} \ \ ,$$

the Casimir of this representation then is

$$C_2(R) = \frac{r}{d(r)}\, T(R) \ ; \ \ C_2(R)\, \delta_{ab} = \left(\sum_i T_i\, T_i \right)_{ab}$$

For $R =$ fundamental representation in particular

$$T(R) = \frac{1}{2} \ ; \ \ C_2(R) = \frac{n^2 - 1}{2n} \ .$$

For QCD we have

$$G = SU(3)_c \ ; \ \ C_2(G) = 3 \ ; \ \ T(R) = \frac{1}{2}\, N_f \ ; \ \ C_2(R) = \frac{4}{3} \ .$$

Exercises: Section 7

① Calculate the group invariants $C_2(G)$ and $C_2(R)$ using the known properties of the $SU(n)$ groups.

② Derive from our one-loop QCD results the corresponding ones for QED. Discuss main differences.

③ Calculate the integral $C_0(m_g, m, m; m^2, k^2, m^2)$ in the regime $m_g^2 \ll m^2, |p^2|$ using the representation (6.38). Discuss typical properties of the result.

④ Elaborate the following statement [from 't Hooft and Veltman's Diagrammar].
Statement: *Order-by-order renormalization is* **not** *equivalent to "throwing away* ε*–poles and their residues" of the unrenormalized* S*–matrix.*

Illustrate this for the following example of a double lepton–loop contribution to the photon self–energy:

Up to a factor $p^2\, g^{\mu\nu} - p^\mu p^\nu$, the irreducible self–energy bubble may be written in the form

$$:= f(p^2) = \frac{1}{d-4}\, f_1(p^2) + f_2(p^2) + (d-4)\, f_3(p^3)$$

What is the correct two–loop contribution from the factorizable double–loop above?

8 The renormalization group, QCD and asymptotic freedom

The renormalization group is the best example which proves that the renormalization problem in QFT is not just a technical problem (some people refuse QFT or at least its perturbative definition and treatment as an ill defined or mathematically ambiguous field) but has a deep physics reason. The renormalization group on the one hand is intimately related to renormalization as o tool to remove singularities and on the other hand controls the scale dependence of physics. Before we give a systematic formal derivation of the renormalization group (RG), we present a simplified outline of the basic ideas and features. After our discussion of the basics of renormalization and radiative corrections by loop effects (quantum fluctuations), an obvious and simple question which we may naturally ask is: how does the theory depend on the scale parameter μ? and what role does it play in physics.

The *Renormalization Group* is a key instrument to control:

- the dependence on renormalization scale (parameter μ in the $\overline{\text{MS}}$ scheme),
- the resummation of the perturbation expansion (summing leading-, subleading-, $\cdots$ logarithms); RG–improved perturbation expansion,
- the physical interpretation of the renormalized parameters (important in case when no S–matrix exists [in perturbation theory]),
- the renormalization scheme dependence,
- the asymptotic behavior (for large momenta [asymptotically free theories like QCD] or small momenta [infrared free theories like QED]).
- The RG is to be considered as the Dilatation Ward identity; Dilatation symmetry implies conformal symmetry but these possible symmetries [strictly renormalizable massless QFT at classical level] are always broken by renormalization effects [renormalization anomaly similar to triangle anomaly].
- Further notions: renormalization fixed points, anomalous dimensions, asymptotic freedom, Bjorken scaling etc.

8.1 Renormalization Group: the general solution

The objects of our interest are the building blocks of a QFT, the bare dimensionally regularized $d = 4 - \varepsilon$ ($\varepsilon > 0$) dimensional 1PI vertex–functions (amputated Green functions):

$$\Gamma_0^{(n_G,2n_F)}(\{p\}; \bar{g}_0, m_0, \xi_0)_\varepsilon$$

with

$$\bar{g}_0 = g_0 \mu^{\varepsilon/2}$$

the bare coupling and

$$\dim\Gamma = d - n_G \frac{d-2}{2} - 2n_f \frac{d-1}{2}$$

the canonical dimension of the vertex–function. The bare vertex functions under a change of scale are **homogeneous** functions when we rescale all parameters according to their dimensions including the "reference scale parameter" μ:

$$\Gamma_0^{(n_G,2n_F)}(\{\kappa\, p\}; g_0\,(\kappa\,\mu)^{\varepsilon/2}, \kappa\, m_0, \xi_0)_\varepsilon \;=\; \kappa^{\dim\Gamma}\,\Gamma_0^{(n_G,2n_F)}(\{p\}; g_0\,(\mu)^{\varepsilon/2}, m_0, \xi_0)_\varepsilon$$

(engineers scaling). The renormalized vertex–functions needed for real physics in $d = 4$ dimensions we obtain by renormalization of the fields and parameters

$$G_0 = \sqrt{Z_3}\, G_{\rm ren}\;;\;\; \psi_0 = \sqrt{Z_3^{\rm F}}\,\psi_{\rm ren}\;;\;\; \eta_0 = \sqrt{\tilde{Z}_3}\,\eta_{\rm ren},$$
$$g_0 = Z_g\, g_r\;;\;\; m_0 = Z_m\, m_r\;;\;\; \xi_0 = Z_3\xi_r\;\;.$$

This translates into the relation

$$\Gamma_0^{(n_G,2n_F)}(\{p\}; g_0\,(\mu)^{\varepsilon/2}, m_0, \xi_0)_\varepsilon \;=\; (Z_3)_\varepsilon^{-n_G/2}\,\left(Z_3^{\rm F}\right)_\varepsilon^{-n_F}\,\Gamma_{\rm ren}^{(n_G,2n_F)}(\{p\}; g_r, m_r, \xi_r, \mu)_\varepsilon\;\;,$$

for the vertex–functions. The renormalization constants Z_i must be chosen such that

$$\lim_{\varepsilon\to 0}\Gamma_{\rm ren}^{(n_G,2n_F)}(\{p\}; g_r, m_r, \xi_r, \mu)_\varepsilon \qquad \text{exists}\;\;.$$

As a consequence of the renormalization $\Gamma_{\rm ren}$ depends in a very non–trivial manner on the scale parameter μ, also in the limit $\varepsilon \to 0$. While a complete rescaling of all dimensionful parameters

$$\Gamma_{\rm ren}^{(n_G,2n_F)}(\{\kappa\, p\}; g_r, \kappa\, m_r, \xi_r, \kappa\,\mu)_\varepsilon \;=\; \kappa^{\dim\Gamma}\,\Gamma_{\rm ren}^{(n_G,2n_F)}(\{p\}; g_r, m_r, \xi_r, \mu)_\varepsilon\;\;,$$

shows a perfect *homogeneous* scaling law, scaling of the physical parameters only, together with the momenta, spoils the previous relation

$$\Gamma_{\rm ren}^{(n_G,2n_F)}(\{\kappa\, p\}; g_r, \kappa\, m_r, \xi_r, \mu)_\varepsilon \;\neq\; \kappa^{\dim\Gamma}\,\Gamma_{\rm ren}^{(n_G,2n_F)}(\{p\}; g_r, m_r, \xi_r, \mu)_\varepsilon\;\;.$$

It means that the homogeneity is broken and this actually represents another renormalization anomaly: quantum fluctuations spoil naive, so called *canonical*, scaling laws expected at the classical level. The actual dependence of $\Gamma_{\rm ren}$ on μ, we obtain from the renormalization equation above by applying the chain rule of differentiation (the μ–dependence is not only continuous but also differentiable), and the fact that for fixed $\bar{g}_0$ Γ is completely independent of μ:

$$\mu\,\frac{\partial\Gamma_0}{\partial\mu}\bigg|_{\bar{g}_0} \equiv 0\;\;.$$

This translates into a highly non-trivial relation, the RG, for the dependence of the renormalized vertex–functions on the renormalized parameters[26]:

$$\left\{\mu\,\frac{\partial}{\partial\mu} + \beta\,\frac{\partial}{\partial g_r} + \omega\,\frac{\partial}{\partial\xi_r} + \gamma_m\, m_r\,\frac{\partial}{\partial m_r} - n_G\,\gamma_G - 2n_F\,\gamma_F\right\}\Gamma_{\rm ren}^{(n_G,2n_F)}(\{p\}; g_r, m_r, \xi_r, \mu)_\varepsilon = 0\;\;,$$

[26] Explicitly,

$$\mu\frac{\partial}{\partial\mu}\Gamma_0^{(n_G,2n_F)}(\{p\}; \bar{g}_0, m_0, \xi_0)\bigg|_{\bar{g}_0} \;=\; \left(\mu\frac{\partial}{\partial\mu}\,(Z_3)^{-n_G/2}\right)\bigg|_{\bar{g}_0}\,\left(Z_3^{\rm F}\right)^{-n_F}\,\Gamma_{\rm ren}^{(n_G,2n_F)}$$
$$+\;\; (Z_3)^{-n_G/2}\,\left(\mu\frac{\partial}{\partial\mu}\left(Z_3^{\rm F}\right)^{-n_F}\right)\bigg|_{\bar{g}_0}\,\Gamma_{\rm ren}^{(n_G,2n_F)} \;+\; (Z_3)^{-n_G/2}\,\left(Z_3^{\rm F}\right)^{-n_F}\,\mu\frac{\partial}{\partial\mu}\Gamma_{\rm ren}^{(n_G,2n_F)}\bigg|_{\bar{g}_0}\;\;.$$

Using

$$\mu\frac{\partial}{\partial\mu}\,(Z_3)^{-n_G/2}\bigg|_{\bar{g}_0} = -\frac{n_G}{2}\,(Z_3)^{-n_G/2}\,Z_3^{-1}\,\mu\frac{\partial}{\partial\mu}\,Z_3\bigg|_{\bar{g}_0} = -\frac{n_G}{2}\,(Z_3)^{-n_G/2}\,\mu\frac{\partial}{\partial\mu}\,\ln Z_3\bigg|_{\bar{g}_0} = -n_G\,\gamma_G\,(Z_3)^{-n_G/2}\;\;,$$

the famous RG–equation, which characterizes the renormalization properties of any QFT. We remind the reader that there is a Slavnov-Taylor identity which requires

$$\xi_0 = Z_3 \xi_r \ ,$$

with Z_3 the gauge field renormalization constant. This implies

$$\omega = -2\xi_r \, \gamma_G$$

such that in the **Landau–gauge**

$$\xi_r = 0$$

all RG–coefficient functions as dimensionless functions only depend on the renormalized coupling g_r:

$$\beta = \beta(g_r) \ ; \ \ \gamma_i = \gamma_i(g_r) \ ; \ \ \omega \equiv 0 \qquad \text{Landau gauge} \ .$$

Because the physical parameter renormalization factors Z_g and Z_m can be defined in a gauge independent way, also β and γ_m may always be defined in a gauge invariant manner, i.e. to be ξ independent. In order to determine the renormalization group coefficients, we have to make use of the fact that the bare quantities depend of μ only via

$$\bar{g}_0 = g_0 \, \mu^{\varepsilon/2} \ ,$$

such that for any bare function $F(g_0, \mu)$[27]

$$\mu \, \frac{\partial}{\partial \mu} \, F\left(\bar{g}_0 = g_0 \, \mu^{\varepsilon/2}\right)\Big|_{\bar{g}_0} = \left(\mu \, \frac{\partial}{\partial \mu} - \frac{\varepsilon}{2} \, g_0 \frac{\partial}{\partial g_0}\right) \, F\left(g_0, \mu\right) \doteq \mathrm{D}_{\mu,\varepsilon} \, F\left(g_0, \mu\right) \ .$$

Furthermore, we note that

$$F^{-1} \, \mathrm{D}_{\mu,\varepsilon} \, F = \mathrm{D}_{\mu,\varepsilon} \, \ln F \ ; \ \ \mathrm{D}_{\mu,\varepsilon} \, \mu = \mu \ .$$

From the chain rule we identify the coefficient functions as

$$\begin{aligned} \beta = \mathrm{D}_{\mu,\varepsilon} \, g_r &= g_r \left(-\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \, g_0 \, \frac{\partial}{\partial g_0} \, \ln Z_g\right) \\ \gamma_G = \mathrm{D}_{\mu,\varepsilon} \, \ln \sqrt{Z_3} &= -\frac{\varepsilon}{4} \, g_0 \, \frac{\partial}{\partial g_0} \, \ln Z_3(g_0, \xi_0) \end{aligned}$$

---

and the corresponding relation for Z_3^{F} together with

$$\begin{aligned} & \mu \frac{\partial}{\partial \mu} \Gamma_{\mathrm{ren}}(\{p\}; g_r, m_r, \xi_r, \mu)\Big|_{\bar{g}_0} = \mu \frac{\partial}{\partial \mu} \Gamma_{\mathrm{ren}}(\{p\}; g_r, m_r, \xi_r, \mu) + \mu \frac{\partial g_r}{\partial \mu}\Big|_{\bar{g}_0} \, \frac{\partial}{\partial g_r} \Gamma_{\mathrm{ren}}(\{p\}; g_r, m_r, \xi_r, \mu) \\ + \quad & \mu \frac{\partial \xi_r}{\partial \mu}\Big|_{\bar{g}_0} \, \frac{\partial}{\partial \xi_r} \Gamma_{\mathrm{ren}}(\{p\}; g_r, m_r, \xi_r, \mu) + \mu \frac{\partial m_r}{\partial \mu}\Big|_{\bar{g}_0} \, \frac{\partial}{\partial m_r} \Gamma_{\mathrm{ren}}(\{p\}; g_r, m_r, \xi_r, \mu) \end{aligned}$$

where the derivatives of the vertex–functions are taken at fixed renormalized parameters, the ones of the parameters and counterterms at fixed $\bar{g}_0$. With $\mu \frac{\partial g_r}{\partial \mu}\Big|_{\bar{g}_0} = \beta$ etc. the result follows.

[27] Using

$$\mu \frac{\partial}{\partial \mu} \, F(g_0, \mu)\Big|_{\bar{g}_0} = \mu \frac{\partial}{\partial \mu} \, F(g_0, \mu)\Big|_{g_0} + \mu \frac{\partial g_0}{\partial \mu}\Big|_{\bar{g}_0} \, \frac{\partial}{\partial g_0} F(g_0, \mu)\Big|_{\mu}$$

with

$$\mu \frac{\partial (g_0 = \bar{g}_0 \mu^{-\varepsilon/2})}{\partial \mu}\Big|_{\bar{g}_0} = -\frac{\varepsilon}{2} \bar{g}_0 \mu^{-\varepsilon/2} = -\frac{\varepsilon}{2} g_0 \ .$$

$$
\begin{aligned}
\gamma_F = \mathrm{D}_{\mu,\varepsilon}\,\ln\sqrt{Z_3^{\mathrm{F}}} &= -\frac{\varepsilon}{4}\,g_0\,\frac{\partial}{\partial g_0}\,\ln Z_3^{\mathrm{F}}(g_0,\xi_0) \\
\gamma_m\,m_r = \mathrm{D}_{\mu,\varepsilon}\,m_r &= \frac{\varepsilon}{2}\,m_r\,g_0\,\frac{\partial}{\partial g_0}\,\ln Z_m(g_0,\xi_0) \\
\omega = \mathrm{D}_{\mu,\varepsilon}\,\xi_r &= -\frac{\varepsilon}{2}\,g_0\,\frac{\partial}{\partial g_0}\,\xi_r = -2\,\xi_r\,\gamma_G\ .
\end{aligned}
$$

They all have finite limits for $\varepsilon \to 0$. The right hand sides may be recalculated in terms of renormalized parameters, where in MS type renormalization schemes we have

$$
Z_i = 1 + \sum_{n=1}^{\infty} \frac{z_{i,n}(g,\xi)}{\varepsilon^n}\ ,
$$

which together with the chain rule implies that the coefficient functions are determined by the $z_{i,1}(g,\xi)$ alone:

$$
\left\|
\begin{array}{lcl}
\beta(g) &=& \frac{g}{2}\,g\,\frac{\partial}{\partial g}\,z_{g,1}(g) \\
\gamma_m(g) &=& \frac{1}{2}\,g\,\frac{\partial}{\partial g}\,z_{m,1}(g) \\
\gamma_G(g) &=& -\frac{1}{4}\,g\,\frac{\partial}{\partial g}\,z_{3,1}(g,\xi) \\
\gamma_F(g) &=& -\frac{1}{4}\,g\,\frac{\partial}{\partial g}\,z_{3,1}^{\mathrm{F}}(g,\xi)\ .
\end{array}
\right. \tag{8.1}
$$

The coefficients of the ε–poles in higher orders are determined by recurrence relations like

$$
\begin{aligned}
\frac{g}{2}\,g\,\frac{\partial}{\partial g}\,z_{g,n+1}(g) &= \beta(g)\,\left\{z_{g,n} + g\,\frac{\partial}{\partial g}\,z_{g,n}\right\} \\
\frac{1}{2}\,g\,\frac{\partial}{\partial g}\,z_{m,n+1}(g) &= \left(\beta(g)\,\frac{\partial}{\partial g} + \gamma_m(g)\right)\,z_{m,n} \\
\frac{1}{2}\,g\,\frac{\partial}{\partial g}\,z_{3,n+1}(g,\xi) &= \left(\beta(g)\,\frac{\partial}{\partial g} - 2\gamma_G\,\xi\,\frac{\partial}{\partial \xi} - 2\gamma_G(g)\right)\,z_{3,n} \\
\frac{1}{2}\,g\,\frac{\partial}{\partial g}\,z_{3,n+1}^{\mathrm{F}}(g,\xi) &= \left(\beta(g)\,\frac{\partial}{\partial g} - 2\gamma_G\,\xi\,\frac{\partial}{\partial \xi} - 2\gamma_F(g)\right)\,z_{3,n}^{\mathrm{F}}\ .
\end{aligned}
$$

From the results for the one–loop renormalization counterterms (see previous section) we may read off the UV singular parts of the counter terms:

$$
\begin{aligned}
Z_g &= 1 - \tfrac{g^2}{16\pi^2}\,\left(11 - \tfrac{2}{3}\,N_f\right)\,\tfrac{1}{\varepsilon}\ , & Z_m &= 1 - \tfrac{g^2}{16\pi^2}\,8\,\tfrac{1}{\varepsilon}\ , \\
Z_3 &= 1 + \tfrac{g^2}{16\pi^2}\,\left(13 - 3\xi - \tfrac{4}{3}\,N_f\right)\,\tfrac{1}{\varepsilon}\ , & Z_3^{\mathrm{F}} &= 1 - \tfrac{g^2}{16\pi^2}\,\xi\tfrac{8}{3}\,\tfrac{1}{\varepsilon}\ ,
\end{aligned}
$$

from which the leading terms of the RG coefficient functions may be easily read off.

The RG equation is a partial differential equation which is homogeneous and therefore can be solved easily along so called characteristic curves. Let s parameterize such a curve, such that all quantities become functions of the single parameter s: $g = g(s),\ m = m(s),\ \mu = \mu(s)$ and

$$
\frac{\mathrm{d}\Gamma}{\mathrm{d}s}\,(\{p\}; g(s), m(s), \mu(s)) = \left\{\frac{\mathrm{d}\mu}{\mathrm{d}s}\,\frac{\partial}{\partial\mu} + \frac{\mathrm{d}g}{\mathrm{d}s}\,\frac{\partial}{\partial g} + \frac{\mathrm{d}m}{\mathrm{d}s}\,\frac{\partial}{\partial m}\right\}\,\Gamma = n\gamma\,\Gamma\ ,
$$

with

$$
\frac{\mathrm{d}\mu}{\mathrm{d}s} = \mu\ ,\ \frac{\mathrm{d}g}{\mathrm{d}s} = \beta(g)\ ,\ \frac{\mathrm{d}m}{\mathrm{d}s} = m\gamma_m(g)\ ,
$$

which is a set of ordinary differential equations the solution of which is solving the RG equation. For simplicity of notation and interpretation we have assumed the Landau gauge $\xi = 0$ and we abbreviated $n_G\gamma_G + 2n_F\gamma_F = n\gamma$. The successive integration then yields

1)

$$\frac{\mathrm{d}\mu}{\mathrm{d}s} = \mu \;\triangleright\quad \ln\mu = s + \text{constant} \;\triangleright\quad \mu = \mu_0\,\mathrm{e}^s = \mu_0\,\kappa \;,$$

where $\kappa = \mathrm{e}^s$ is a scale dilatation parameter

2)

$$\frac{\mathrm{d}g}{\mathrm{d}s} = \beta(g) \;\triangleright\quad \frac{\mathrm{d}g}{\beta(g)} = \mathrm{d}s = \frac{\mathrm{d}\mu}{\mu} \;\triangleright$$

$$\ln(\mu/\mu_0) = \ln\kappa = \int\limits_g^{g(\kappa)} \frac{\mathrm{d}g'}{\beta(g')} \;,$$

which is the implicit definition of the running coupling $g(\kappa)$ with $g = g(1)$ the coupling at reference scale μ_0 and $g(\kappa) = g(\mu/\mu_0)$ the coupling at scale μ.

3)

$$\frac{\mathrm{d}m}{\mathrm{d}s} = m\gamma_m \;\triangleright\quad \frac{\mathrm{d}m}{m} = \gamma_m(g)\,\mathrm{d}s = \gamma_m(g)\frac{\mathrm{d}g}{\beta(g)} \;\triangleright$$

$$\ln m + \text{constant} \;=\; \int\limits_g^{g(\kappa)} \frac{\gamma(g')\,\mathrm{d}g'}{\beta(g')} \;,$$

or

$$m(\kappa) = m\,\exp\int\limits_g^{g(\kappa)} \frac{\gamma(g')\,\mathrm{d}g'}{\beta(g')} \;,$$

4)

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}s} = n\gamma(g)\,\mathrm{d}s = n\gamma(g)\,\frac{\mathrm{d}\mu}{\mu} = n\gamma(g)\,\frac{\mathrm{d}g}{\beta(g)} \;\triangleright$$

$$\Gamma(\kappa) = \Gamma\,\exp\left\{ n \int\limits_g^{g(\kappa)} \frac{\gamma(g')\,\mathrm{d}g'}{\beta(g')} \right\} = \Gamma\; z_3(g,\kappa)^{n_G}\; z_3^{\mathrm{F}}(g,\kappa)^{2n_F} \;,$$

with $\Gamma = \Gamma(1)$, and

$$z_3(g,\kappa) = \exp\int\limits_g^{g(\kappa)} \frac{\gamma_G(g')\,\mathrm{d}g'}{\beta(g')} \;, z_3^{\mathrm{F}}(g,\kappa) = \exp\int\limits_g^{g(\kappa)} \frac{\gamma_F(g')\,\mathrm{d}g'}{\beta(g')} \;.$$

Altogether, we may write this as an equation which describes the response of the theory with respect to a change of the scale parameter μ:

$$\boxed{\Gamma\left(\{p\}; g, m, \mu/\kappa\right) = z_3(g,\kappa)^{-n_G}\; z_3^{\mathrm{F}}(g,\kappa)^{-2n_F}\;\Gamma\left(\{p\}; g(\kappa), m(\kappa), \mu\right).}$$

Thus **a change of the scale parameter μ is equivalent to a finite renormalization of the parameters and fields** and together with the homogeneity relation we have for the vertex functions with scaled momenta

$$\begin{aligned}\Gamma\left(\{\kappa p\}; g, m, \mu\right) &= \kappa^{\dim\Gamma}\,\Gamma\left(\{p\}; g(\kappa), \frac{m(\kappa)}{\kappa}, \frac{\mu}{\kappa}\right) \\ &= \kappa^{\dim\Gamma}\; z_3(g,\kappa)^{-n_G}\; z_3^{\mathrm{F}}(g,\kappa)^{-2n_F}\;\Gamma\left(\{p\}; g(\kappa), \frac{m(\kappa)}{\kappa}, \mu\right) \;,\end{aligned}$$

which is the basic relation for a discussion of the asymptotic behavior.

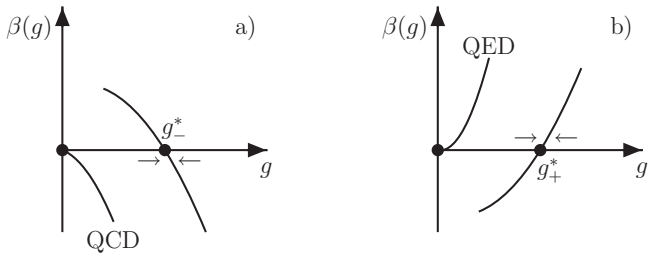


Figure 8.1: RG fixed points are zeros of the β–function: a) UV fixed points, b) IR fixed points

8.2 Asymptotic Behavior

Two regimes are of interest, the high energy (ultraviolet) behavior and the low energy (infrared) behavior.

8.2.1 Short distance or UV behavior

The ultraviolet behavior, which determines the short distance properties, is obtained by choosing $\kappa|p| \gg m, \mu$ thus

$$\ln \kappa = \int_g^{g(\kappa)} \frac{\mathrm{d}g'}{\beta(g')} \to +\infty \quad ; \quad \kappa \to \infty \ .$$

However, the integral can only become divergent for finite $g(\kappa)$ if $\beta(g)$ has a zero at $\lim_{\kappa\to\infty} g(\kappa) = g^*$: more precisely, in the limit $\kappa \to \infty$ the effective coupling has to move to a fixed point $g(\kappa) \to g^*_-$ if finite, and the fixed point coupling is characterized by $\beta(g^*_-) = 0$, $\beta'(g^*_-) < 0$. Thus g^*_- is an **ultraviolet fixed point** coupling. Note that by dilatation of the momenta at fixed m and μ, the effective coupling is automatically driven into a fixed point, a zero of the β–function with negative slope, if it exists. If $g^*_- = 0$ we have **asymptotic freedom**. This is how QCD behaves, which has a β–function

$$\beta_{\mathrm{QCD}}(g_s) = -g_s \left(\beta_0 \left(\frac{g^2}{16\pi^2} \right) + \beta_1 \left(\frac{g^2}{16\pi^2} \right)^2 + \cdots \right)$$

with $\beta_0 > 0$ (see Fig. 8.1 a). QCD will be considered in more detail later on.

A possible fixed point is accessible in perturbation theory provided g^* is sufficiently small, such that perturbation theory is sufficiency "convergent" as an asymptotic series. One may then expand about g^*:

$$\begin{aligned} \beta(g) &= (g - g^*_-)\, \beta'(g^*_-) + \cdots \\ \gamma(g) &= \gamma^* + (g - g^*_-)\, \gamma'(g^*_-) + \cdots \end{aligned}$$

and provided $\beta'(g^*_-) \neq 0$ we have

$$\begin{aligned} a(g,\kappa) &= \exp \int_g^{g(\kappa)} \frac{\gamma(g')}{\beta(g')}\, \mathrm{d}g' = \exp \int_g^{g(\kappa)} \frac{\gamma(g^*_-)}{\beta(g')}\, \mathrm{d}g' \ \cdot r(g,\kappa) \\ &= \kappa^{\gamma^*}\, r(g,\kappa) \end{aligned}$$

where the non–singular remainder

$$r(g,\kappa) \;\; = \;\; \exp \int\limits_{g}^{g(\kappa)} \frac{(\gamma(g') - \gamma^*)}{\beta(g')}\, \mathrm{d}g'$$

in the limit of large κ yields a finite scale independent wave function renormalization

$$\lim_{\kappa\to\infty} \; r(g,\kappa) = r(g,\infty) \;\; .$$

We thus find the asymptotic from

$$\Gamma(\{\kappa p\}; g, m, \mu) \stackrel{\rightarrow}{\sim} \kappa^d \, \Big(\kappa^{d_G}\, r_G(g,\infty)\Big)^{-n_G} \, \Big(\kappa^{d_F}\, r_F(g,\infty)\Big)^{-2n_F} \, \Gamma(\{p\}; g_-^*, 0, \mu)$$

which exhibits asymptotic *scaling*. In the first place it is given by the vertex functions of a *massless theory*. As expected, at high energies masses may be neglected, however on the expense that another mass scale remains in the game, the scale parameter μ. The first factor κ^d is trivial and is due to the d–momentum conservation which was factored out. Then each field exhibits a homogeneous (power–like) behavior in the dilatation factor κ, the exponent of which exhibits an *anomalous dimension* as a consequence of the dynamics of the theory:

$$d_G = \frac{d-2}{2} + \gamma_G^* \;\; , \quad d_F = \frac{d-1}{2} + \gamma_F^* \;\; .$$

The first term is the naive or engineers dimension the second part is the anomalous part which is a quantum effect, a relict of the breaking of scale invariance, when $g \neq g^*$. While naively we would expect that in $d = 4$ dimensions the massless theory has scaling: for example a scalar two–point function, the only dimensionful physical quantity being the momentum, one would expect $G(p;g) \sim 1/p^2$ as G has dimension 2. However, if there would be a non–trivial UV fixed point one would have $G(p,g,\mu) \sim (\mu^2)^{\gamma^*}/(p^2)^{1+\gamma^*}$ ($\gamma^* > 0$) which shows the role and unavoidability of the scale parameter μ, which has to eat up the extra dimension γ^* induced by the dynamics of the theory. Otherwise only truly free theories could have scaling, called *canonical scaling* in this case. The discovery of asymptotic freedom of QCD is the prime example of a dynamical theory, notabene of the theory of strong interactions, exhibiting asymptotic canonical scaling (Bjorken scaling) of liberated quarks (quark parton model). The latter was discovered before in the pioneering investigations concerning *Deep Inelastic Scattering* (DIS) of electrons on protons and bound neutrons by Friedman, Kendall and Taylor (Nobel prize 1990). These experiments have been of essential importance for the development of the quark model and to the discovery of QCD as the theory of the strong interactions.

8.2.2 Long distance or IR behavior

The infrared behavior corresponds to the long distance properties of a system. Here the regime of interest is $\kappa|p| \ll m, \mu$ and the discussion proceeds essentially as before: now

$$\ln \kappa = \int\limits_{g}^{g(\kappa)} \frac{\mathrm{d}g'}{\beta(g')} \to -\infty \;\; ; \quad \kappa \to 0 \;\; ,$$

is required. Now as $\kappa \to 0$ the effective $g(\kappa) \to g_+^*$ where g_+^* is a zero of the β–function with positive slope, see Fig. 8.1 b), $\beta(g_+^*) = 0$ and $\beta'(g_+^*) > 0$. This is the typical situation in the

construction of low energy effective theories, particularly in the discussion of critical phenomena of statistical systems (keywords: critical behavior, critical exponents, scaling laws, universality). If $g_+^* = 0$ the effective theory is infrared free (the opposite of *asymptotic freedom*), also called Gaussian (Gaussian fixed point). Here the well known examples are QED

$$\beta_{\rm QED}(e) = \frac{e^3}{12\pi^2}\sum_f N_{cf} Q_f^2 + \cdots$$

or the self–interacting scalar field ϕ^4–theory

$$\beta(\lambda) = -\varepsilon\lambda + \frac{3\lambda^2}{16\pi^2} + \cdots$$

in $d = 4$ dimensions. For QED the running coupling to leading order thus follows from

$$\ln\kappa \;=\; \int_e^{e(\kappa)} \frac{1}{\beta(e')}\,{\rm d}e' = \frac{12\pi^2}{\sum_f N_{cf}Q_f^2}\int_e^{e(\kappa)} \frac{1}{(e')^3}\,{\rm d}e' = \frac{24\pi^2}{\sum_f N_{cf}Q_f^2}\left(\frac{1}{e^2} - \frac{1}{e(\kappa)^2}\right)$$

where the sum extends over all light flavors $f:\ m_f < \mu$[28]. The running fine structure constant thus at leading order is given by

$$\alpha(\mu) = \frac{\alpha}{1 - \frac{2\alpha}{3\pi}\,\sum_f N_{cf}Q_f^2\,\ln\mu/\mu_0}$$

where μ_0 is the scale where the lightest particle starts to contribute, which is the electron $\mu_0 = m_e$. We then may identify $\alpha(\mu_0) = \alpha$ the classical low energy value of the fine structure constant, with the proviso that only logarithmic accuracy is taken into account (see below). The running α is equivalent to the Dyson summation of the transversal part of the photon self–energy to the extent that only the logs are kept. The RG running takes into account the leading radiative corrections in case the logs are dominating over constant terms, i.e., provided large scale changes are involved.

As $\alpha(\mu)$ is increasing with μ in the resummed perturbation theory running coupling exhibits a pole, the so called *Landau pole* at which the coupling becomes infinite: $\lim_{\mu\stackrel{<}{\to}\mu_L}\alpha(\mu) = \infty$ The "fixed point" very likely is an artifact of perturbation theory, which of course cease to be valid when the one–loop correction approaches 1. What this tells us is that we actually do not know what the high energy asymptotic behavior of QED is.

We must stress here that the discovery of the RG (Stueckelberg, Petermann 1953, Gell-Mann, Low 1954, Bogoliubov, Shirkov 1957, Ken Wilson 1971) had and still has far reaching consequences for understanding the scaling structure of QFTs. Equally important the RG provides an indispensable tool for the resummation of leading quantum effects, without running into conflict with basic QFT properties like unitarity and locality. In contrast, Dyson resummation of propagator effects, for example, leads to such conflicts, as it treats propagators and vertices on different footing. Anomalous dimensions (anomalous power laws) would never have been seen in an order by order perturbative treatment. In condensed matter physics it took 70 years to understand long distance scaling because of lack in understanding the emergence of anomalous dimensions. Wilson's version of the RG finally provided the solution. Due to poor convergence (numerically

[28] This latter restriction takes into account the decoupling of heavy flavors, valid in QED and QCD. Since in the $\overline{\rm MS}$ scheme, i.e., renormalization by the substitution ${\rm Reg} \to \ln\mu^2$, which we are considering here, decoupling is not automatic, one has to impose it by hand. At a given scale one is thus considering an effective theory, which includes only those particles with masses below the scale μ.

relatively large coupling) perturbative QCD would be almost useless, if renormalization group improvement (running coupling, running masses, anomalous dimensions) would not be available. In Grand Unified Theories the RG is a key tool for understanding possible coupling unification.

Last but not least, also in QED, for example in high precision calculations of $g-2$, where leading 5–loop contributions are required to match the experimental precision, the RG is an indispensable tool which allows us to calculate in a simple way leading higher order effects, just by replacing the fixed order-by-order coupling by the running one. In QED vertex corrections cancel against electron wave function renormalization contributions, a famous QED Ward-Takahashi identity, and only photon vacuum polarization (VP) effects contribute to the renormalization of the charge. Therefore, in QED the running α is equivalent to the Dyson summation of the transversal part of the photon self–energy to the extent that only the logs are kept. The RG running takes into account the leading radiative corrections in case the logs are dominating over constant terms, i.e., provided large scale changes are involved.

An example: in the calculation of the contributions from electron loops in photon propagators to the muon anomaly a_μ, such large scale changes from m_e to m_μ are involved and indeed one may calculate such two–loop contributions starting from the lowest order result

$$a_\mu^{(2)} = \frac{\alpha}{2\pi} \quad \text{via the substitution} \quad \alpha \to \alpha(m_\mu)$$

where

$$\alpha(m_\mu) = \frac{\alpha}{1 - \frac{2}{3}\frac{\alpha}{\pi}\,\ln\frac{m_\mu}{m_e}} = \alpha\,\left(1 + \frac{2}{3}\frac{\alpha}{\pi}\,\ln\frac{m_\mu}{m_e} + \cdots\right)$$

such that we find

$$a_\mu^{(4)\,\mathrm{LL}}(\mathrm{vap}, e) = \frac{1}{3}\,\ln\frac{m_\mu}{m_e}\,\left(\frac{\alpha}{\pi}\right)^2$$

which indeed agrees with the leading log result obtained by the direct calculation. In the calculation of a_μ only the electron VP insertions are governed by the RG and the corresponding one–flavor QED β–function has been calculated to three loops

$$\beta(\alpha) = \frac{2}{3}\left(\frac{\alpha}{\pi}\right) + \frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2 - \frac{121}{144}\left(\frac{\alpha}{\pi}\right)^3 + \cdots$$

which thus allows to calculate leading $\alpha^n\,(\ln m_\mu/m_e)^n$, next–to–leading $\alpha^n\,(\ln m_\mu/m_e)^{n-1}$ and next–to–next–to–leading $\alpha^n\,(\ln m_\mu/m_e)^{n-2}$ log corrections.

In $d = 4$ QFTs we are usually limited to perturbative methods. This also applies to RG applications. Since the $g = 0$ is always a RG fixed point the RG allows us to study asymptotically free theories at large enough energies (QCD) and infrared free theories (QED) at long distances. In the electroweak SM the initial couplings are weak enough such that calculations up to rather high energies usually are possible without problems.

Summary: the RG of QCD in Short

The renormalization group, introduced above, for QCD plays a particularly important role for a quantitative understanding of AF as well as a tool for improving the convergence of the perturbative expansion. For QCD the RG is given by

$$\begin{aligned} \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g_s(\mu) &= \beta\left(g_s(\mu)\right) \\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}m_i(\mu) &= -\gamma\left(g_s(\mu)\right)\, m_i(\mu) \end{aligned}$$

with

$$\begin{aligned}\beta(g) &= -\beta_0\frac{g^3}{16\pi^2} - \beta_1\frac{g^5}{(16\pi^2)^2} + O(g^7)\\ \gamma(g) &= \gamma_0\frac{g^2}{4\pi^2} + \gamma_1\frac{g^4}{(4\pi^2)^2} + O(g^6)\end{aligned}$$

where, in the $\overline{\rm MS}$ scheme,

$$\begin{aligned}\beta_0 &= 11 - \tfrac{2}{3}N_f \quad ; \quad \gamma_0 = 2\\ \beta_1 &= 102 - \tfrac{38}{3}N_f \quad ; \quad \gamma_1 = \tfrac{101}{12} - \tfrac{5}{18}N_f\end{aligned}$$

and N_f is the number of quark flavors. The RG for QCD is known to 4 loops. It allows to define effective parameters in QCD, which incorporate the summation of leading logarithmic (1–loop), next–to–leading logarithmic (2–loop), $\cdots$ corrections (RG improved perturbation theory). The solution of (8.5) for the running coupling constant $\alpha_s(\mu) = g_s^2(\mu)/(4\pi)$ yields

$$\begin{aligned}&\frac{4\pi}{\beta_0\alpha_s(\mu)} - \frac{\beta_1}{\beta_0^2}\,\ln\left(\frac{4\pi}{\beta_0\alpha_s(\mu)} + \frac{\beta_1}{\beta_0^2}\right) =\\ &\ln\mu^2/\mu_0^2 + \frac{4\pi}{\beta_0\alpha_s(\mu_0)} - \frac{\beta_1}{\beta_0^2}\,\ln\left(\frac{4\pi}{\beta_0\alpha_s(\mu_0)} + \frac{\beta_1}{\beta_0^2}\right) \equiv \ln\mu^2/\Lambda^2\end{aligned}$$

with reference scale (integration constant)

$$\Lambda_{\rm QCD} = \Lambda_{\overline{\rm MS}}^{(N_f)} = \mu\,\exp\left\{-\frac{4\pi}{2\beta_0\alpha_s(\mu)}\left(1 + \frac{\alpha_s(\mu)}{4\pi}\,\frac{\beta_1}{\beta_0}\,\ln\frac{\beta_0\alpha_s(\mu)}{4\pi + \frac{\beta_1}{\beta_0}\alpha_s(\mu)}\right)\right\}$$

which can be shown easily to be independent of the reference scale μ. It is RG invariant

$$\mu\frac{\rm d}{{\rm d}\mu}\,\Lambda_{\rm QCD} = 0\ , \tag{8.2}$$

and thus QCD has its own intrinsic scale $\Lambda_{\rm QCD}$ which is related directly to the coupling strength (dimensional transmutation). This is most obvious at the one–loop level where we have the simple relation

$$\alpha_s(\mu) = \frac{1}{\frac{\beta_0}{4\pi}\ln\frac{\mu^2}{\Lambda^2}}\ .$$

Thus $\Lambda_{\rm QCD}$ incorporates the reference coupling $\alpha_s(\mu_0)$ measured at scale μ_0 in a scale invariant manner, i.e., each experiment measures the same $\Lambda_{\rm QCD}$ irrespective of the reference energy μ_0 at which the measurement of $\alpha_s(\mu_0)$ is performed.

The solution of (8.5) for the effective masses $m_i(\mu)$ reads

$$m_i(\mu) = m_i(\mu_0)\frac{r(\mu)}{r(\mu_0)} \equiv \overline{m}_i r(\mu)$$

with

$$r(\mu) = \exp -2\left\{\frac{\gamma_0}{\beta_0}\ln\frac{4\pi}{\beta_0\alpha_s(\mu)} + \left(\frac{\gamma_0}{\beta_0} - \frac{4\gamma_1}{\beta_1}\right)\,\ln(1 + \frac{\beta_1}{\beta_0}\frac{\alpha_s(\mu)}{4\pi})\right\}\ .$$

Note that also the $\overline{m}_i$ are RG invariant masses (integration constants) and for the masses play a role similar to $\Lambda_{\rm QCD}$ for the coupling. The solution of the RG equation may be expanded in the large log $L \equiv \ln\frac{\mu^2}{\Lambda^2}$, which of course only makes sense if L is large ($\mu \gg \Lambda$),

$$\begin{aligned}\alpha_s(\mu) &= \frac{4\pi}{\beta_0\,L}\,\left(1 - \frac{\beta_1}{\beta_0^2}\,\frac{\ln(L + \frac{\beta_1}{\beta_0^2})}{L} + \cdots\right)\\ m_i(\mu) &= \overline{m}_i\,\left(\frac{L}{2}\right)^{-\frac{\gamma_0}{\beta_0}}\,\left(1 - \frac{2\beta_1\gamma_0}{\beta_0^3}\,\frac{\ln L + 1}{L} + \frac{8\gamma_1}{\beta_0^2 L} + \cdots\right)\ .\end{aligned}$$

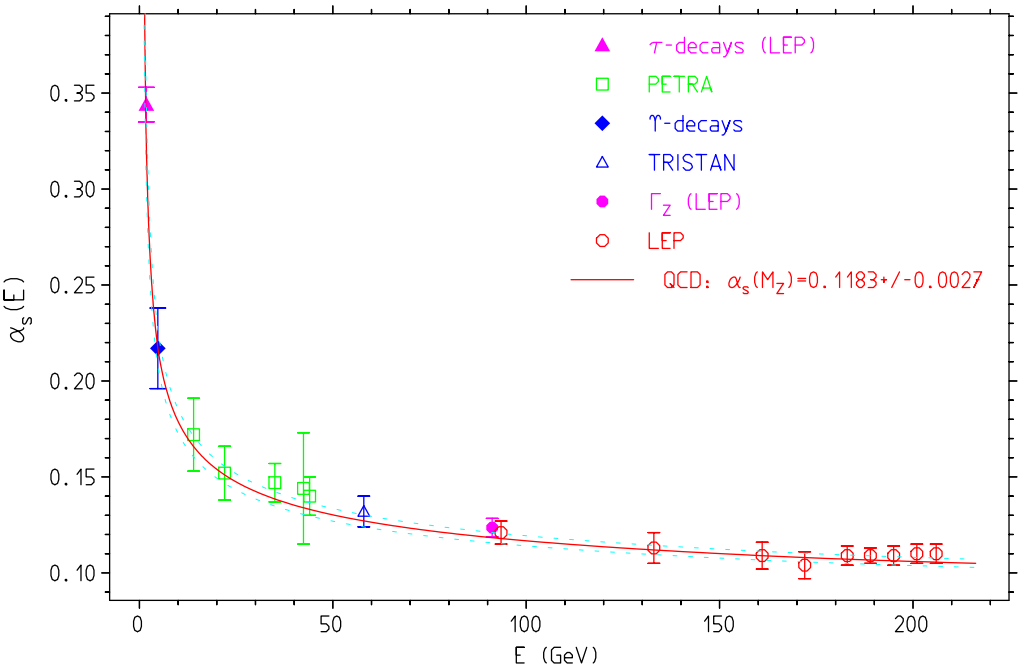


Figure 8.2: The QCD running coupling "constant" as a function of the energy scale.

If L is not large one should solve (8.2) or its higher order version numerically by iteration for $\alpha_s(\mu)$. For the experimental prove of the running of the strong coupling constant see Fig. 8.2. We note that in agreement with RG improved perturbative QCD the experimentally determined values of the strong coupling constant at different energy scales show huge effects as α_s varies from $\alpha_s(M_\tau) = 0.322 \pm 0.03$ at the τ mass near 2 GeV to $\alpha_s(M_Z) = 0.1183 \pm 0.0027$ at the Z mass $M_Z = 91.1876 \pm 0.0021$ GeV.

The non-perturbative calculations in lattice QCD are able to demonstrate a surprisingly good agreement with perturbative results. The corresponding result from the Alpha Collaboration is shown in Fig. 8.5 below.

Exercise: Consider the 2–loop QCD β–function as a function of α_s and discuss its dependence on the number of quark flavors N_f.

8.3 Universality of the asymptotic behavior

What happens to the RG if we use a different renormalization procedure, which leads to a reparametrization of the Green's functions. What happens is that the Z_i renormalization constants change together with the redefinition of the parameters:

$$Z_i \to \hat{Z}_i = F_i\, Z_i\ ; \qquad F_i \quad \text{a finite function}\ . \tag{8.3}$$

Of course, the reparametrization functions F_i are finite since the UV singularities of the bare theories are identical. For simplicity we assume here a mass independent reparametrization of the type

$$(g, m, \mu) \ \to (\hat{g}, \hat{m}, \mu)\ ; \qquad \hat{Z}_i(g_0, \xi_0) = F_i(g_0, \xi_0)\, Z_i(g_0, \xi_0)\ ,$$

which, by applying the chain rule of differentiation yields,

$$\begin{aligned}
\hat{\beta}(\hat{g},\hat{\xi}) &= \beta(g)\,\frac{\partial \hat{g}}{\partial g} + \omega(g,\xi)\,\frac{\partial \hat{g}}{\partial \xi} \\
\hat{\gamma}(\hat{g},\hat{\xi}) &= \gamma(g) - \left(\beta(g)\,\frac{\partial \ln F_m}{\partial g} + \omega(g,\xi)\,\frac{\partial \ln F_m}{\partial \xi}\right) \\
\hat{\omega}(\hat{g},\hat{\xi}) &= \omega(g,\xi)\,F_3^{-1} - \hat{\xi}\,\left(\beta(g)\,\frac{\partial \ln F_3}{\partial g} + \omega(g,\xi)\,\frac{\partial \ln F_3}{\partial \xi}\right) \\
\hat{\gamma}_i(\hat{g},\hat{\xi}) &= \gamma(g)_i - \frac{1}{2}\,\left(\beta(g)\,\frac{\partial \ln F_3^i}{\partial g} + \omega(g,\xi)\,\frac{\partial \ln F_3^i}{\partial \xi}\right) \;, i = G, F \;.
\end{aligned}$$

Note that in case we chose gauge independent physical parameter renormalizations the last terms of the first two equations are absent, because then $\partial\hat{g}/\partial\xi = 0$ and $\partial F_m/\partial\xi = 0$. Remember that in contrast, the field renormalization factors in general are gauge dependent. If we again chose for simplicity the Landau–gauge, we have

$$\xi = 0 \Rightarrow \omega = 0 \quad \text{such that} \quad \hat{\xi} = 0 \Rightarrow \hat{\omega} = 0 \;. \tag{8.4}$$

Beyond the special assumptions made, it turns out that RG fixed points and anomalous dimensions are **universal** in the following sense: If a fixed point $g = g^*$ exists with $\beta(g^*) = 0$ then in the new scheme there exists fixed point coupling $\hat{g}^* = \hat{g}(g^*)$ with $\beta(\hat{g}^*) = 0$ and the above chain rule relations imply

$$\hat{\beta}'(g^*) = \beta'(g^*) \;, \hat{\gamma}_i(\hat{g}^*) = \gamma_i(g^*) \;;\;\; i = m, G, F \;.$$

This proves the universality of asymptotic behavior. Universality plays a very important role in the theory of critical phenomena describing the long range behavior of of statistical mechanics systems near the critical pint of a second or higher order phase transition. In statistical physics universality hold under much more general conditions (Wilson's renormalization semi group).

8.4 RG-invariance of physical observables

Let $R(s)$ be an observable (a measurable quantity) where $\sqrt{s}$ stands for a set of scales like the invariant energy, momentum transfers etc. Assume we have calculated $R(s)$ in a specific renormalization scheme (i.e. for a given parametrization) where

$$R(s) = R(s; g, m, \xi, \mu) \;.$$

As an observable $R(s)$ must be independent of μ and of ξ: which means

$$\begin{aligned}
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\,R &= \left\{\mu\frac{\partial}{\partial\mu} + \tilde{\beta}\,\frac{\partial}{\partial g} + \tilde{\omega}\,\frac{\partial}{\partial\xi} + \tilde{\gamma}_m\,m\,\frac{\partial}{\partial m}\right\} R(s; g, m, \xi, \mu) = 0 \\
\frac{\mathrm{d}}{\mathrm{d}\xi}\,R &= \left\{\frac{\partial}{\partial\xi} + \tilde{\rho}\,\frac{\partial}{\partial g} + \tilde{\tau}_m\,m\,\frac{\partial}{\partial m}\right\} R(s; g, m, \xi, \mu) = 0 \;.
\end{aligned}$$

This property is called GR–invariance and is special because there is no amplitude renormalization analogous to the wave function renormalization factors. The parameters are, as far as they explicitly show up (e.g. in a reasonable renormalization scheme no explicit gauge parameter dependence would show up), all dependent of each other, with

$$\tilde{\beta} = \mu\,\frac{\partial}{\partial\mu}\,g \;;\;\; \tilde{\omega} = \mu\,\frac{\partial}{\partial\mu}\,\xi \;;\;\; \tilde{\gamma}\,m = \mu\,\frac{\partial}{\partial\mu}\,m \;;$$
$$\tilde{\rho} = \frac{\partial}{\partial\xi}\,g \;;\;\; \tilde{\tau}\,m = \frac{\partial}{\partial\xi}\,m \;.$$

It means that the two coupled equations (8.5) eliminate the two redundant parameters μ and ξ. As it should be, the theory only has g and m as relevant physical parameters.

8.5 The "running" parameters of QCD

The relevant QCD parameters are the strong interaction coupling constant $g_s(\mu)$ or $\alpha_s(\mu) \doteq \frac{g_s^2(\mu)}{4\pi}$, respectively, and the quark masses $m_i(\mu)$ $(i = u, d, s, c, b, t)$. For QCD the RG is given by

$$g(\kappa) \quad : \qquad \ln\kappa = \int_{g(1)}^{g(\kappa)} \tfrac{1}{\beta(g')}\,\mathrm{d}g' \qquad \Leftrightarrow \qquad \mu\tfrac{\mathrm{d}}{\mathrm{d}\mu} g_s(\mu) = \beta\left(g_s(\mu)\right)$$

$$m_i(\kappa) \quad : \quad \tfrac{m(\kappa)}{m(1)} = \exp\int_{g(1)}^{g(\kappa)} \tfrac{\gamma_m(g')}{\beta(g')}\,\mathrm{d}g' \quad \Leftrightarrow \quad \mu\tfrac{\mathrm{d}}{\mathrm{d}\mu} m_i(\mu) = -\gamma_m\left(g_s(\mu)\right)\, m_i(\mu)$$

with

$$\begin{aligned} \beta(g) &= -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} + O(g^7) \\ \gamma_m(g) &= \gamma_0 \frac{g^2}{4\pi^2} + \gamma_1 \frac{g^4}{(4\pi^2)^2} + O(g^6) \end{aligned}$$

where, in the $\overline{\mathrm{MS}}$ scheme,

$$\begin{aligned} \beta_0 &= 11 - \tfrac{2}{3}N_f \quad ; \quad \gamma_0 = 2 \\ \beta_1 &= 102 - \tfrac{38}{3}N_f \quad ; \quad \gamma_1 = \tfrac{101}{12} - \tfrac{5}{18}N_f \end{aligned}$$

and N_f is the number of quark flavors.

Minimal subtraction type renormalization schemes are mass m and gauge parameter ξ independent. The following theorem holds.
Theorem: For any m and ξ independent renormalization scheme the leading RG coefficients b_0, b_1 and c_0 in the perturbative expansions

$$\beta(g) = b_0\, g^3 + b_1\, g^5 + \cdots \;;\; \gamma_m(g) = c_0\, g^2 + \cdots$$

are renormalization scheme independent and thus *universal*.
Proof: the perturbative expansion is an expansion in $\alpha = \frac{g^2}{4\pi}$, which means that corrections are $O(g^2)$ for succeeding terms in the expansion. The couplings of different schemes are thus related by

$$\hat{g} = g + a_1\, g^3 + \cdots \quad \Rightarrow \quad g = \hat{g} - a_1\, \hat{g}^3 + \cdots \;.$$

The change of scheme $(g, m) \to (\hat{g}, \hat{m})$ yields

$$\hat{\beta}(\hat{g}) = \beta(g)\,\frac{\partial\hat{g}}{\partial g} \quad ; \quad \hat{\gamma}_m(\hat{g}) = \gamma_m(g) - \beta(g)\,\frac{\partial \ln F_m}{\partial g} \;,$$

where $F_m = 1 + f_1\, g^2 + \cdots$. The explicit calculation then yields

$$\hat{b}_0 = b_0 \;, \quad \hat{b}_1 = b_1 \quad \text{and } \hat{c}_0 = c_0$$

as inferred by the theorem. This theorem in particular applies for going from the minimal to the modified minimal subtraction scheme MS $\to \overline{\mathrm{MS}}$! It thus covers only a very limited set of renormalization schemes, which, however, includes the formally simple (but not very physical) $\overline{\mathrm{MS}}$ scheme which is utilized in most QCD applications.

8.6 α_s in perturbation theory

Here we work out the effective strong interaction coupling constant in pQCD, in successive approximations obtained by taking perturbative approximations for the coefficient functions β, γ etc. in solving the RG equations.

a) Leading order $O(\alpha_s)$

With the lowest order beta function we find

$$\ln\kappa = \int_{g(1)}^{g(\kappa)} \frac{\mathrm{d}g'}{\left(-\frac{\beta_0}{(4\pi)^2}g'3\right)} = \left.\frac{1}{\frac{2\beta_0}{(4\pi)^2}g'2}\right|_{g(1)}^{g(\kappa)} ,$$

i.e.,

$$\frac{4\pi}{(g(\kappa))^2} - \frac{4\pi}{(g(1))^2} = \frac{\beta_0}{4\pi}\,\ln\kappa^2 \ .$$

This we may write in the form

$$\left\| \ \alpha_s^{-1}(\mu) = \alpha_s^{-1}(\mu_0) + \tfrac{\beta_0}{4\pi}\,\ln\mu^2/\mu_0^2 \right. \tag{8.5}$$

or

$$\begin{aligned} \alpha_s(\mu) &= \frac{\alpha_s(\mu_0)}{1+\alpha_s(\mu_0)\,\frac{\beta_0}{4\pi}\,\ln\mu^2/\mu_0^2} \\ &= \alpha_s(\mu_0)\cdot\sum_{k=0}^{\infty}\left(-\alpha_s(\mu_0)\,\frac{\beta_0}{4\pi}\,\ln\frac{\mu^2}{\mu_0^2}\right)^k \ . \end{aligned} \tag{8.6}$$

The last reexpanded result of the RG–solution demonstrates that the latter corresponds to a *resummation* of the *leading log's* in the perturbative expansion.

Exercise: Study the convergence properties of the RG-resummation.

Example: we may determine $\alpha_s(M_Z)$ at the Z boson mass scale $M_Z = 91.1876$ GeV, using the observable $R(s) = \sum_{f\le\sqrt{s}} Q_f^2 N_{cf}\cdot\left\{1 + c_1\,\frac{\alpha_s(s)}{\pi}\right\}$ (see below), where for $s \gg 4m_f^2$ we have $c_1 = 1$,and given $R(M_Z) \simeq 3.811 \pm 0.003$ as an (pseudo-) experimental value. We may then calculate $\alpha_s(M_B)$ at the B meson mass scale $M_B \simeq 5.3$ GeV using the RG. Note that in this energy region $N_f = 5$ is the number of active flavors. To lowest order $R_{N_f}^{(0)} = \sum Q_f^2 N_{cf}$ is given by $R_3^{(0)} = 2$, $R_4^{(0)} = 3\,\frac{1}{3}$ and $R_5^{(0)} = 3\,\frac{2}{3}$. Using $R^{(1)}(M_Z) = R(M_Z)$, we obtain $\alpha_s^{(1)}(M_Z) = 0.124 \pm 0.002$ and using Eq. (8.5) we find $\alpha_s^{(1)}(M_B) = 0.218 \pm 0.006$. Not only the coupling gets stronger as we go to lower energies, also the uncertainty gets magnified correspondingly.

The QCD–scale

We note that in the 1–loop approximation $\alpha_s^{-1}(\mu)$ plotted against $\ln\mu^2/\mu_0^2$ is a straight line, as illustrated in Fig. 8.3:

$$\alpha_s^{-1}(\mu) = \alpha_s^{-1}(\mu_0) \ + \ \frac{\beta_0}{4\pi}\,\ln\mu^2/\mu_0^2 \ .$$

A very interesting point to be stressed here is the fact that the effective coupling and the scale

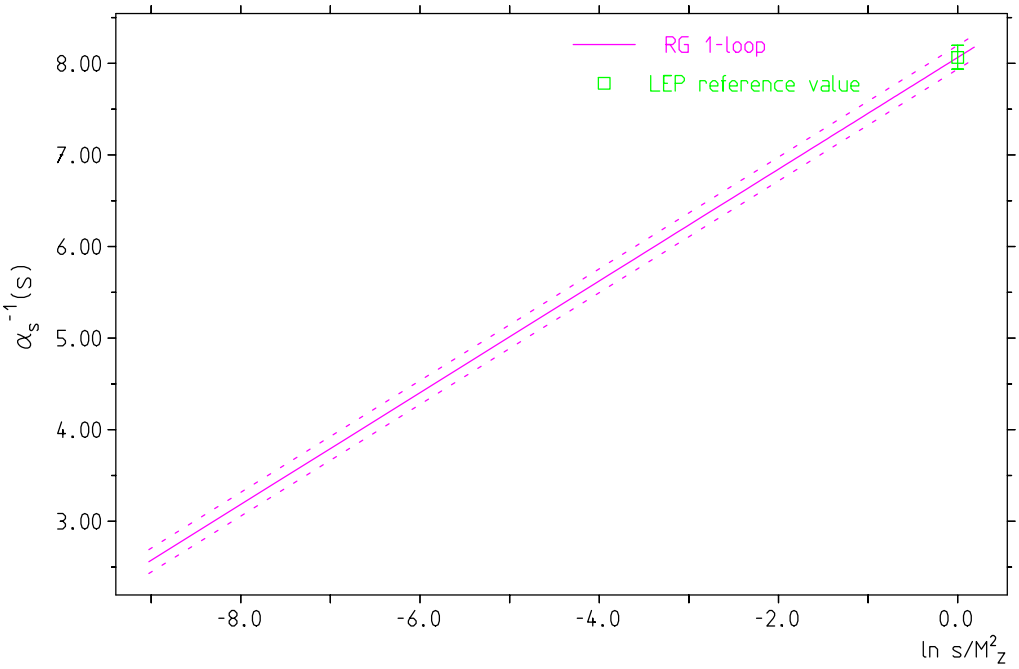


Figure 8.3: $\alpha_s^{-1}(\mu)$ vs. $\ln \mu^2/\mu_0^2$ in the one–loop approximation a straight line.

determine each other (up to neglected higher order effects). It is therefore natural to define an intrinsic QCD scale $\Lambda = \Lambda_{\rm QCD}$ by

$$\alpha_s^{-1}(\mu_0) \doteq \frac{\beta_0}{4\pi} \ln \mu_0^2/\Lambda^2 \ ,$$

where $\Lambda < \mu_0$. Consequently, we have

$$\alpha_s^{-1}(\mu) = \frac{\beta_0}{4\pi} \ln \mu^2/\Lambda^2$$

for arbitrary values of μ and we generally obtain

$$\left\| \ \alpha_s^{-1}(\mu) = \frac{1}{\frac{\beta_0}{4\pi} \ln \mu^2/\Lambda^2} \ . \right. \tag{8.7}$$

The relation between the QCD coupling α_s and the QCD scale Λ is called dimensional transmutation. If $\mu_0 = \sqrt{s_0}$ is the invariant energy at which the coupling $\alpha_s(\mu_0)$ is measured, then (in the considered approximation) the QCD scale is given by

$$\Lambda_{\rm QCD} = \sqrt{s_0} \exp\left(-\frac{4\pi}{2\beta_0 \alpha_s(\sqrt{s_0})}\right)$$

as an intrinsic physical scale of QCD. Dimensional transmutation is a quantum phenomenon related to the breaking of scale (and hence of conformal) invariance:

coupling		scale
$\alpha_s(\mu)$	$\Leftrightarrow$	$\Lambda_{\rm QCD}$
(dimensionless)		(dimension of a mass)

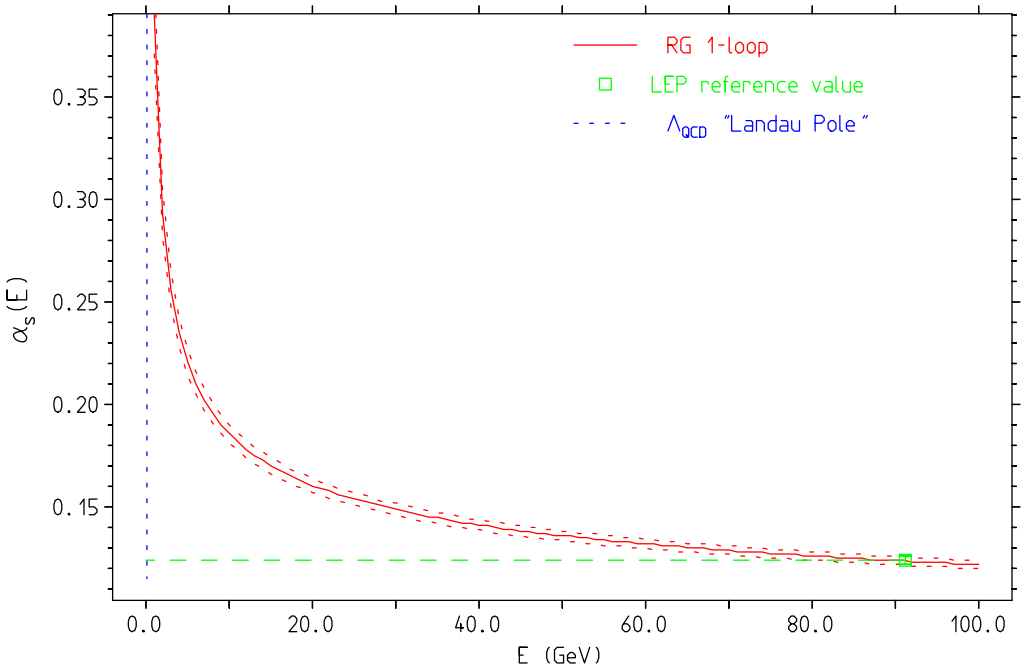


Figure 8.4: $\alpha_s(E)$ as a function of the energy scale E. As E approaches the QCD scale $\Lambda \simeq 123$ MeV, the effective coupling diverges (so called *Landau pole*), signaling the breakdown of perturbation theory as we go to too low energies.

For the example given above we find for $N_f = 5$ flavors at one–loop

$$\Lambda_{\rm QCD} \simeq (0.123 \pm 0.013)\ {\rm GeV}\ .$$

Exercise: Elaborate on the question where is the best scale to determine the reference strong interaction coupling constant. Compare the situation with the fine structure constant in QED.

• The crucial point is the following: $\Lambda_{\rm QCD}$ is independent of μ at which it was determined,in other words, different physicists determine the same $\Lambda_{\rm QCD}$ also if they measure the reference coupling at different energies!

Verification:

$$\begin{aligned}
\mu \frac{\rm d}{{\rm d}\mu} \Lambda_{\rm QCD} &= \mu \exp\left(-\frac{4\pi}{2\beta_0 \alpha_s(\mu)}\right) \\
&+ \mu \exp\left(-\frac{4\pi}{2\beta_0 \alpha_s(\mu)^2}\right) \times \frac{4\pi}{2\beta_0 \alpha_s(\mu)} \mu \frac{\rm d}{{\rm d}\mu} \alpha_s(\mu) \\
&= 0 \qquad (8.8)
\end{aligned}$$

since

$$\mu \frac{\rm d}{{\rm d}\mu} \alpha_s(\mu) = \frac{2g}{4\pi} \mu \frac{{\rm d}g}{{\rm d}\mu} = \frac{2g}{4\pi} \beta(g) = -\frac{2\beta_0}{4\pi} \alpha_s(\mu)^2 \qquad (8.9)$$

in the given approximation.

Note: all "derived" QCD quantities, like $\alpha_s(\mu)$ or $\Lambda_{\rm QCD}$, extracted by confronting a pQCD calculation (theory) with a measured quantity (experiment) depend on the approximation used. This is of course generally true when confronting theory and experiment. However, because of

the relatively poor convergence of pQCD (much larger coupling strength in QCD vs. QED) in strong interaction physics numerical values for derived quantities usually depend substantially on the renormalization scheme and the order of perturbation theory used.

b) Next to leading order $O(\alpha_s^2)$

The solution of the RG equation with the next to leading order beta function Eq. (8.5) we may easily find (see **c)** below)

$$\begin{aligned} & \frac{4\pi}{\beta_0\,\alpha_s(\mu)} - \frac{\beta_1}{\beta_0^2}\,\ln\left(\frac{4\pi}{\beta_0\,\alpha_s(\mu)} + \frac{\beta_1}{\beta_0^2}\right) \\ = \;& \ln\mu^2/\mu_0^2 + \frac{4\pi}{\beta_0\,\alpha_s(\mu_0)} - \frac{\beta_1}{\beta_0^2}\,\ln\left(\frac{4\pi}{\beta_0\,\alpha_s(\mu_0)} + \frac{\beta_1}{\beta_0^2}\right) \\ = \;& \ln\mu^2/\Lambda^2 \end{aligned}$$

with

$$\left\| \Lambda = \Lambda_{\rm QCD} = \Lambda_{\overline{\rm MS}}^{(N_f)} = \mu\,\exp\left(-\tfrac{1}{2}\,\left(\tfrac{4\pi}{\beta_0\,\alpha_s(\mu)} - \tfrac{\beta_1}{\beta_0^2}\,\ln\left(\tfrac{4\pi}{\beta_0\,\alpha_s(\mu)} + \tfrac{\beta_1}{\beta_0^2}\right)\right)\right)\;. \right. \tag{8.10}$$

Again, Λ is RG–invariant in the given approximation and $\Lambda \leq \mu$ as $(\beta_0 > 0\;, \alpha_s(\mu) > 0)$.

Remember: $\Lambda_{\rm QCD}$ depends on

1) the renormalization scheme ($\overline{\rm MS}$, MOM, $\cdots$)
2) the number of flavors $N_f = 3, 4, 5, 6$
3) further approximations made when solving the RG–equation (see below).

An approximation made frequently is an expansion in powers of the large logarithm

$$L \doteq \ln\mu^2/\Lambda^2$$

when $\mu \gg \Lambda$. Note that this is a further approximation, beyond truncating the perturbative series. One easily works out

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 L}\,\left(1 + \frac{\beta_1}{\beta_0^2}\,\frac{\ln\left(L + \frac{\beta_1}{\beta_0^2}\right)}{L} + \cdots\right)^{-1}\quad .$$

For the effective mass at next to leading order the solution of the RG–equation takes the form

$$m_i(\mu) = m_i(\mu_0)\,\frac{r(\mu)}{r(\mu_0)}$$

with

$$\left\| r(\mu) = \exp -2\,\left(\tfrac{\gamma_0}{\beta_0}\,\ln\tfrac{4\pi}{\beta_0\,\alpha_s(\mu)} + \left(\tfrac{\gamma_0}{\beta_0} - \tfrac{4\gamma_1}{\beta_1}\right)\,\ln(1 + \tfrac{\beta_1}{\beta_0}\,\tfrac{\alpha_s(\mu)}{4\pi}\right) \right. \tag{8.11}$$

and an expansion in L^{-1} yields

$$m_i(\mu) = \overline{m}_i\,\left(\frac{L}{2}\right)^{-\frac{2\gamma_0}{\beta_0}}\,\left(1 - \frac{2\beta_1\gamma_0}{\beta_0^3}\,\frac{L+1}{L} + \frac{8\gamma_1}{\beta_0^2 L} + \cdots\right)\;.$$

Like $\Lambda_{\rm QCD}$ the $\overline{m}_I$ are RG–invariant (serving as integration constants). As a corollary we may check that

$$\frac{m_i^{(N_f)}(\mu)}{m_k^{(N_f)}(\mu)} = \frac{\overline{m}_i}{\overline{m}_k}\;.$$

c) Higher orders We first rewrite the RG–solution for g in terms of α_s:

$$\ln \kappa = \int\limits_{g(1)}^{g(\kappa)} \frac{\mathrm{d}g'}{\beta(g')} \Rightarrow \ln \kappa^2 = \int\limits_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\mathrm{d}\alpha}{\beta(\alpha)} \tag{8.12}$$

where

$$\beta(\alpha) = \frac{g\beta(g)}{4\pi} = -b_0\,\alpha^2 - b_1\,\alpha^3 - \cdots$$

with

$$b_0 = \frac{\beta_0}{4\pi}\ , b_1 = \frac{\beta_1}{(4\pi)^2}$$

We may write

$$\frac{1}{\beta} = -\frac{1}{b_0\,\alpha^2}\,\frac{1}{1+\frac{b_1}{b_0}\,\alpha} + \left(\frac{1}{\beta(\alpha)} + \frac{1}{b_0\,\alpha^2 + b_1\,\alpha^3}\right)\ ,$$

where the first two (leading plus next–to–leading) terms may be written as

$$\begin{aligned}\frac{1}{\beta(\alpha)^{(2)}} &= \frac{1}{-b_0\,\alpha^2 - b_1\,\alpha^3} = \frac{1}{-b_0\,\alpha^2}\,\frac{1}{1+b_1/b_0\,\alpha} \\ &= -\frac{1}{b_0\,\alpha^2} + \frac{b_1}{b_0^2}\left(\frac{1}{\alpha} - \frac{1}{\alpha+\frac{b_0}{b_1}}\right)\ .\end{aligned}$$

Integration then yields

$$\begin{aligned}\ln \kappa^2 &= \left(\frac{1}{b_0\,\alpha} - \frac{b_1}{b_0^2}\,\ln\left(\frac{1}{b_0\,\alpha} + \frac{b_1}{b_0^2}\right) - \frac{b_1}{b_0^2}\,\ln\frac{b_0^2}{b_1}\right)\Bigg|_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \\ &\quad + F(\alpha_s(\mu)) - F(\alpha_s(\mu_0))\end{aligned}$$

with

$$F(\alpha_s) = \int\limits_0^{\alpha_s} \mathrm{d}\alpha\,\left(\frac{1}{\beta(\alpha) + \frac{1}{b_0\,\alpha^2 + b_1\,\alpha^3}}\right)\ .$$

Note that the last integrand is

$$\frac{\beta(\alpha) + b_0\,\alpha^2 + b_1\,\alpha^3}{(b_0\,\alpha^2 + b_1\,\alpha^3)\,\beta(\alpha)} = O(1) \quad \text{for small} \quad \alpha$$

i.e, it is regular at $\alpha = 0$. Thus the third term of Eq. (8.13) contributes

$$\int\limits_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \mathrm{d}\alpha\ (\cdots) = \left(\int\limits_{\alpha_s(\mu_0)}^{0} \mathrm{d}\alpha\ - \int\limits_{\alpha_s(\mu_0)}^{0} \mathrm{d}\alpha\right)\ (\cdots) = F(\alpha_s(\mu)) - F(\alpha_s(\mu_0))\ .$$

In contrast to the singular leading term $1/\alpha_s$ and next–to–leading term $\ln \alpha_s$, the function $F(\alpha_s)$ is regular and has a convergent power series expansion with $F(\alpha_s) = O(\alpha_s)$ as $\alpha_s \to 0$.

The result thus may be written in as

$$\begin{aligned} f(\alpha_s(\mu)) &\doteq \frac{4\pi}{\beta_0\,\alpha_s(\mu)} - \frac{\beta_1}{\beta_0^2}\,\ln\left(\frac{4\pi}{\beta_0\,\alpha_s(\mu)} + \frac{\beta_1}{\beta_0^2}\right) + F(\alpha_s(\mu)) \\ &= \ln\mu^2/\mu_0^2 + f(\alpha_s(\mu_0)) \doteq \ln\mu^2/\Lambda^2 \ . \end{aligned} \tag{8.13}$$

Note that $f(\alpha_s(\mu_0)) = \ln\mu_0^2/\Lambda^2$, and that the extra higher order term (contributions to β beyond 2–loop) just amount to a redefinition of $\Lambda_{\rm QCD}$:

$$\underbrace{\frac{4\pi}{\beta_0\,\alpha_s(\mu)} - \frac{\beta_1}{\beta_0^2}\,\ln\left(\frac{4\pi}{\beta_0\,\alpha_s(\mu)} + \frac{\beta_1}{\beta_0^2}\right)}_{\ln\mu^2/\Lambda^2_{(2)}(\mu)} + \underbrace{F(\alpha_s(\mu))}_{\ln\Lambda^2_{(2)}(\mu)/\Lambda^2}$$

The consequence of the analysis is the following:

1) $\Lambda_{\rm exact}$ is invariant under the exact RG, i.e., strictly renormalization independent

2) since $F(\alpha_s(\mu))$ depends on the renormalization scale μ the finite order pQCD Λ's like $\Lambda_{(2)}(\mu)$ must depend on μ.

Let us stress once more: the "measured" reference coupling $\alpha_s(\mu_0)$ is obtained by comparing an observable, an experimentally determined number, with a scheme dependent finite order pQCD prediction (usually RG improved), and thus itself depends on the precise theory input. For given $\alpha_s(\mu_0)$, furthermore $\Lambda_{\rm QCD}$ and $\alpha_s(\mu)$ depend in a essential manner on the approximation used.

Note: for self-consistency one should strictly use the same scheme and the same order of RG–improved pQCD for both extracting $\alpha_s(\mu_0)$ from the data and for the prediction of $\alpha_s(\mu)$.

The following example may illustrate the sensitivity to higher order effects: we consider QCD with $N_f = 3$ flavors a scale $\mu_0 = 2$ GeV in the $\overline{\rm MS}$ scheme and going from two to three loops. We then have

$$\begin{aligned} \beta_0 &= 11 - \frac{2}{3}\,N_f = 9 \\ \beta_1 &= 102 - \frac{38}{3}\,N_f = 64 \\ \beta_2 &= \frac{2857}{2} - \frac{5033}{18}\,N_f + \frac{325}{54}\,N_f^2 = \frac{3863}{6} \simeq 643.833\ldots \end{aligned}$$

and thus

$$\beta(\alpha) = -b_0\,\alpha^2 - b_1\,\alpha^3 - b_2\,\alpha^4 - \cdots$$

with

$$b_0 = \frac{\beta_0}{4\pi} \simeq 0.716\ ; \qquad b_1 = \frac{\beta_1}{(4\pi)^2} \simeq 0.405\ ; \qquad b_2 = \frac{\beta_2}{(4\pi)^3} \simeq 0.324\ .$$

The regular higher order (here 3–loop) correction term stems from integrating

$$\frac{1}{\beta(\alpha)} + \frac{1}{b_0\,\alpha^2 + b_1\,\alpha^3} = \frac{\beta(\alpha) + b_0\,\alpha^2 + b_1\,\alpha^3}{\beta(\alpha)\,(b_0\,\alpha^2 + b_1\,\alpha^3)} \simeq \frac{b_2}{b_0^2}$$

and thus

$$F(\alpha_s(\mu_0)) \sim \int_0^{\alpha_s(\mu_0)} {\rm d}\alpha\,\frac{b_2}{b_0^2} = \frac{b_2}{b_0^2}\,\alpha_s(\mu_0) = \ln\frac{\Lambda^2_{(2)}}{\Lambda^2_{(3)}} \simeq 0.203\ .$$

The reference scale chosen in our example $\mu_0 = 2$ GeV is close to the τ lepton mass $M_\tau = (1.77684 \pm 0.00017)$ GeV at the boarder of applicability of pQCD. Here we may use $\alpha_s(M_\tau)$ extracted from hadronic τ decays: $\alpha_s(\mu_0) \simeq \alpha_s(M_\tau) \simeq 0.322 \pm 0.030$. We then obtain

$$\frac{\Lambda_{(2)}}{\Lambda_{(3)}} = \exp\left(\frac{b_2}{2b_0^2}\,\alpha_s(\mu_0)\right) \simeq 1.11 \ ,$$

which is a 11 % correction. More precisely, using the exact value $F(\alpha_s(\mu_0)) \sim 0.170 \pm 0.013$, the correction is 9 %. As a typical result we find for $N_f = 3$

$$\Lambda^{(3)}_{\overline{\rm MS}} = (355 \pm 55)\ \text{MeV} \ , \ \text{while}\ \Lambda^{(2)}_{\overline{\rm MS}} = (387 \pm 66)\ \text{MeV}$$

Present day pQCD analyses generally use the known 4–loop RG results (van Ritbergen, Vermaseren, Larin 1997, Chetyrkin 2004, Czakon 2004). Numerical routines including these results are available (e.g. `RHAD` by Harlander, Steinhauser 2002)

d) the non-perturbative $\alpha_s(\mu)$ from lattice QCD:
Discretized QCD on a $d = 4$ Euclidean "space–time" lattice allows to calculate α_s in a non–perturbative manner by numerical simulation. The QCD path integral is directly evaluated by means of Metropolis Monte Carlo integration (with appropriate importance sampling of the gauge configurations). Results from the Alpha Collaboration translated into the $\overline{MS}$ scheme are sown in Fig. 8.5.

Exercise: Work out the 2–loop result for the mass renormalization factor Eq. (8.11) using the techniques outlined above.

8.7 How does the RG work for asymptotically free theories?

Because of its importance for principal as well as practical purposes it may be helpful to illustrate once more in a slightly different way what is the essence of RG improved perturbation theory. Let us consider as a simple example the two–point function $G^{(2)}(p; g, 0, \mu)$ of the massless theory. Perturbation theory at k-th order yields a *leading* contribution

$$C_k \left(\frac{\alpha_s}{\pi}\,\ln\frac{p^2}{\mu^2}\right)^k \ ; \ \ \alpha_s \doteq \frac{g^2}{4\pi}$$

and hence

$$G^{(2)}(p; g, 0, \mu) = \sum_{k=0}^{\infty} C_k \left(\frac{\alpha_s}{\pi}\,\ln\frac{p^2}{\mu^2}\right)^k + \cdots$$

Now let $p = \kappa\, p_0$ where $p_0 = O(\mu)$. For large $\kappa \gg 1$ we obtain

$$\alpha_s\,\ln\frac{p^2}{\mu^2} \simeq \alpha_s\,\ln\kappa^2 + \alpha_s\,\ln\frac{p_0^2}{\mu^2}$$

with a negligible second term, and $\alpha_s\,\ln\kappa^2$ acts as an effective coupling:

$$G^{(2)}(\kappa p_0; g, 0, \mu) = \sum_{k=0}^{\infty} C_k \left(\frac{\alpha_s}{\pi}\,\ln\kappa^2\right)^k + \cdots = \sum_{k=0}^{\infty} C_k \left(\frac{\alpha_s(\kappa)}{\pi}\right)^k + \cdots$$

Obviously the naive perturbation series only makes sense if

$$\alpha_s\,\ln\kappa^2 < 1 \quad (\text{better} \quad \ll 1) \ ,$$

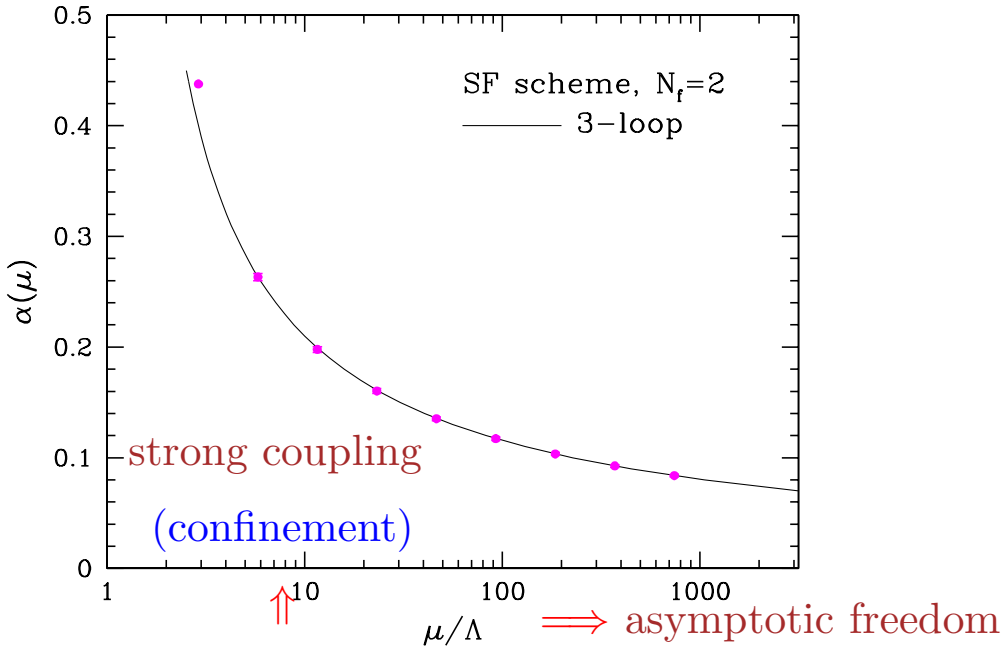


Figure 8.5: The strong interaction coupling $\alpha_s(\mu)$ as a function of the renormalization scale obtained non–perturbatively from lattice QCD vs. the 3–loop pQCD prediction. The non-perturbative lattice result for $\Lambda_{\rm QCD}$ for $N_f = 2$ is $\Lambda^{(2)}_{\overline{\rm MS}} = 245(16)(16)$ MeV. From Ref. [1] (ALPHA Collaboration).

i.e., for large κ the naive perturbation expansion breaks down, in spite of asymptotic freedom which tells us that the effective coupling should become small. The latter fact can only be obtained by applying the RG. As a solution of the latter we have

$$G^{(2)}(\kappa p_0; g, 0, \mu) = \kappa^{2d_G^0}\, a(\kappa, g)^2\, G^{(2)}(p_0; g(\kappa), 0, \mu)$$

with

$$\alpha_s(\kappa) = \frac{\alpha_s}{1 + \alpha_s \frac{\beta_0}{4\pi} \ln \kappa^2} \ll 1$$

which means that up to calculable singular factors $\kappa^{2d_G^0}\, a(\kappa, g)^2$ we obtain

$$G^{(2)}(p_0; g(\kappa), 0, \mu) = \sum_{k=0}^{\infty} C_k \left(\frac{\alpha_s(\kappa)}{\pi} \ln \frac{p_0^2}{\mu^2} \right)^k + \cdots \ ,$$

which actually converges the better the larger κ is! The important point here is that $\ln p_0^2/\mu^2 = O(1)$ is kept fixed. Note that the order by order (i.e. not RG resummed) pQCD result does not suggest the correct structure of the theory in the asymptotic region.

Exercises I: Section 8

① Consider the 2–loop QCD β–function as a function of α_s and discuss its dependence on the number of quark flavors N_f.

② Study the convergence properties of the RG-resummation Eq. (8.6) (see also Subsect. 8.7).

③ Elaborate on the question "Where is the best scale to determine the reference strong interaction coupling constant?". Compare the situation with the fine structure constant in QED.

④ Work out the 2–loop result for the mass renormalization factor Eq. (8.11) using the techniques outlined in the text.

8.8 Decoupling of heavy states and the decoupling theorem

Mass independent renormalization schemes are not very physical, because they do not take into account the physical mass dependence of full QCD with $N_f = 6$. In a more physical renormalization scheme like on-shell renormalization in QED one observes automatic decoupling of heavier states and fields. For example one can do very precise Lamb–shift calculations in atomic spectra without any knowledge about the existence of the muon or even of the τ–lepton. Similarly, low energy hadron physics like the existence of the pions and their properties certainly do not depend on the existence of the charm, bottom or top quark. Even the much lighter strange quark which is much closer to the up and down quarks, but still clearly heavier, has no substantial effect on the pion system. Now, applying massless renormalization, means all quarks are taken massless in first place, which of course would change real physics at the low scales. Unfortunately, in QCD we cannot apply in a physical sound way on–shell renormalization, because in pQCD a physical S–matrix does not exist, the physical states are hadrons not quarks and gluons. So what to do? Formally, one can try to implement to some extent an on-shell renormalization procedure, however, as in QED already, infrared singularities spoil also this possibility because of the massless gluons. An infinitesimally small gluon mass allows to get IR regulated answers for intermediate quantities which are not infrared save by themselves. One advantage of such a scheme would be that one would actually obtain automatic decoupling, because each state or field is taken into account only at its intrinsic scale.

For a long time is has been taken for granted that the physics at low energies is not influenced in any sense by the possible existence of particles at much higher energy scales. In a quantum field theory though this is not trivial because heavy particles in general contribute to quantum corrections as they appear in loops as virtual particles. Let us consider QCD, specifically. In the QCD Lagrangian only quarks have mass. However, the scale of quark masses covers a very large range from a few MeV of the up and down quarks to almost 200 GeV for the top quark. The strong coupling constant is a parameter entering the QCD Lagrangian independent of the quark masses. In contrast to electroweak theory where couplings and masses, generated by the Higgs mechanism, are strongly correlated. Unlike in electroweak theory, in QCD we can choose the particle masses arbitrary heavy at a fixed value of the coupling constant. As seen from the world of the light particles a heavy quark can contribute to the light quark and gluon amplitudes at the 2–loop level only, by diagrams of the type shown in Fig. 8.6. In QCD like theories[29], at the level of the bare theory, masses of a heavy quark for example only enter in the denominator of the quark propagator. The latter, in principle, may be expanded as $1/(\not{q} - m_t) = (\not{q} + m_t)/(q^2 - m^2) \sim -1/m_t\,(1+\not{q}/m_t)(1+q^2/m_t^2+\cdots)$. However, such an expansion makes little sense in a loop integral

[29] the other well known example is QED, where the leptons play the role of the quarks

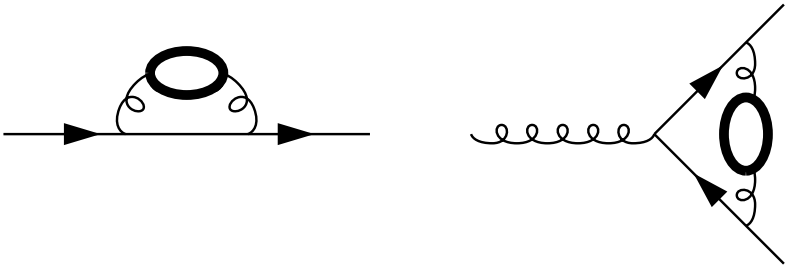


Figure 8.6: Heavy virtual particles (fat lines) in light field (thin lines) amplitudes. In the "light sector' the heavy fields only appear "integrated out".

where q, or a linear combination of q with other momenta, is a loop momentum to be integrated over. Indeed, the expansion produces fake UV singularities. This shows that a slightly more sophisticated consideration is necessary. Such an analysis was performed by Symanzik 1973 [13] and Appelquist and Carazzone 1975 [14]. The result is known as the Appelquist–Carazzone theorem, which we present in the following.

The question is how heavy particles of mass $M_i \gg \mu$ manifest themselves in the energy range $\sqrt{s} \leq \mu$. An example are the RG–coefficients in the $\overline{\text{MS}}$ scheme:

$$\begin{aligned} \beta_0 &= 11 - \tfrac{2}{3} N_f \quad ; \quad \gamma_0 = 2 \\ \beta_1 &= 102 - \tfrac{38}{3} N_f \quad ; \quad \gamma_1 = \tfrac{101}{12} - \tfrac{5}{18} N_f \end{aligned}$$

What is N_f to be taken here? In a mass independent renormalization scheme each flavor yields a mass independent contribution, thus for full QCD $N_f = 6$. One could ask the question: is it possible to determine N_f from the experimentally determined evolution of $\alpha_s(\mu)$ as a function of μ? In massless renormalization the theory at low energies seems to look like full QCD far beyond the top mass scale, beyond 0.5 TeV say. Is this reasonable? The decoupling theorem helps answer such questions.

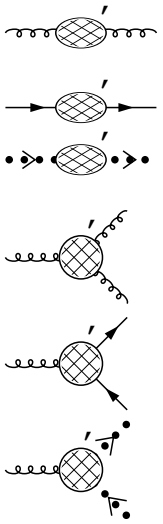
The Decoupling Theorem: if all external momenta of a process or a corresponding amplitude are small relative to the mass M of a heavy state, then the Green functions of the full theory which however exhibit light fields only as external lines differ from the theory which has no heavy fields at all only by finite renormalizations of couplings, masses and fields of the light theory, up to terms which are suppressed by inverse powers of the heavy mass. Thus further corrections are of the form $(\mu/M)^x$ with $x \geq 1$.

It means that only the renormalization subtraction constants are dependent on M (logarithms) and this M–dependence gets renormalized away by physical subtraction conditions. In other words, effects from degrees of freedom which are not "excited" at a given scale μ can be dropped. Essentially, we may always take N_f to be the number of flavors with masses $m_i < \mu$.

In view of the decoupling theorem what can we say about the $\overline{\text{MS}}$ renormalization scheme applied to full QCD? In addition to typical large logarithms $\ln s/\mu^2$ (μ the renormalization scale , $\sqrt{s}$ a typical physical process energy) also even larger logarithms $\ln M^2/\mu^2$ or $\ln M^2/s$ are showing up which are fake (an artifact of massless renormalization) and potentially affect the convergence of the perturbation expansion. Such a scheme thus is not recommendable to do physics at scales $\mu, \sqrt{s} \ll M$. It is thus certainly more adequate to drop the heavy field from the consideration from the very beginning and consider an effective theory exhibiting the light fields only. The drawback will be that different effective theories are needed, the parametrizations of which have to be made consistent by hand at the "matching points" (see below).

8.9 MOM scheme, automatic decoupling

Momentum (MOM) subtraction schemes are a generalized version of the on–shell scheme conventionally used in QED (more generally in electroweak theory), where quasi physical S-matrix elements are subtracted to determine the physical meaning of the parameters. This was the route we followed in Sect. 7, when discussing one–loop renormalization besides the $\overline{\rm MS}$ renormalization also considered there. MOM schemes are mass–dependent, like the on–shell scheme but avoid IR singularities by not subtracting directly on the mass–shell (which leads necessarily to IR singularities in the wave–function renormalization factors) but at some Euclidean reference scale. More precisely the conditions applied read:



$$\Pi_{\rm ren}(k^2)\big|_{k^2=-\mu^2} = 0$$

$$\Sigma_{1\,\rm ren}(p^2)\big|_{p^2=-\mu^2} = \Sigma_{2\,\rm ren}(p^2)\big|_{p^2=-\mu^2} = 0$$

$$b_{\rm ren}(q^2)\big|_{q^2=-\mu^2} = 0$$

$$\Gamma'^{\rho\sigma\mu}_{kli\,\rm ren}(k,p,q)\Big|_{\rm sp:-\mu^2} = 0 \qquad \text{MOM}_I$$

$$\Gamma'^{\mu}_{i,bc\,\rm ren}(k,p,q)\Big|_{k^2=p^2=q^2=-\mu^2} = 0 \qquad \text{MOM}_{II}$$

$$\Gamma'^{\mu}_{i,kl\,\rm ren}(k,p,q)\Big|_{k^2=p^2=q^2=-\mu^2} = 0 \qquad \text{MOM}_{III}\ .$$

The prime (') indicates that the non–trivial higher order part only is to be taken. By "sp" we denote the symmetry point, for a symmetric 3–point function (like ggg) sp: $p_ip_j = \frac{1}{2}\,(3\,\delta_{ij}-1)\,\mu^2$, for a symmetric 4–point function (e.g $gggg$) sp: $p_ip_j = \frac{1}{3}\,(4\,\delta_{ij}-1)\,\mu^2$. The expressions are obviously symmetric and take into account 4–momentum conservation of the three and four momenta,respectively.

In a MOM scheme we obtain mass dependent renormalization constants

$$Z_{i\,\rm MOM} = Z_i(g,\xi,\frac{m_i}{\mu})$$

and the RG takes the form

$$\begin{aligned} \mu\tfrac{\partial}{\partial\mu}\overline{g} &= \beta(\overline{g},\overline{\xi},\tfrac{\overline{m}_i}{\mu})\,; \qquad \beta = -\beta_0\,\tfrac{\overline{g}^3}{16\pi^2}+\cdots \\ \mu\tfrac{\partial}{\partial\mu}\overline{m}_i &= -\gamma(\overline{g},\overline{\xi},\tfrac{\overline{m}_i}{\mu})\,\overline{m}_i\,; \qquad \gamma = \gamma_0\,\tfrac{\overline{g}^2}{4\pi^2}+\cdots \end{aligned}$$

of a coupled system, which can be solved by numerical integration. We should stress that MOM schemes using off–shell subtraction (to avoid IR problems) per se are leading to gauge dependent effective parameters and gauge dependent RG coefficients. Once more we encounter here a situation where unphysical redundancy of the theoretical formalism cannot be avoided. There is no real problem though, as only true observable quantities must be free of all these diseases.

It is not too difficult to investigate explicitly the mass effects in the renormalization constants. Physical mass effects in the coupling constant renormalization defined via the $gq\bar{q}$–vertex we actually have calculated in Sect. 7.4. In QCD like in any non-Abelian gauge theory the gauge coupling constant g_s may be defined in different ways. At the tree level we may choose the ggg

(MOM$_I$), the $gq\bar{q}$ (MOM$_{II}$) or even the unphysical $g\eta\bar{\eta}$ (MOM$_{III}$) vertex. Let us compare the first two here. At one loop in the Landau gauge $\overline{\xi} = 0$ one obtains

$$\beta_0 = 11 - \frac{2}{3}\sum_f K\left(\frac{\overline{m}_f^2}{\mu^2}\right) \;\; ; \;\; \gamma_{i0} = 2 \cdot G\left(\frac{\overline{m}_i^2}{\mu^2}\right)$$

where in the MOM$_I$ we have

$$\begin{aligned} K = K_{ggg}(x^2) &= 1 - 18\,x^2 + \frac{36x^4}{\sqrt{1+4\,x^2}}\,\ln\frac{\sqrt{1+4\,x^2}+1}{\sqrt{1+4\,x^2}-1} \\ &\quad -2\int\limits_0^{1/3} \mathrm{d}y\,\rho(y)\,\left[\frac{x^2}{y+x^2} - \frac{3\,x^4}{(y+x^2)^2}\right] \end{aligned}$$

with

$$\rho(y) = \begin{cases} 2\sqrt{3}\,\arctan\left(\sqrt{3}\,\frac{1-\sqrt{1-4\,y}}{1+3\,\sqrt{1-4\,y}}\right) & ; \quad 0 \le y \le 1/4 \\ \frac{2\pi}{\sqrt{3}} & ; \quad 1/4 \le y \le 1/3 \;. \end{cases}$$

For MOM$_{II}$ we have the simpler expression

$$K = K_{gq\bar{q}}(x^2) = 1 - 6\,x^2 + \frac{12x^4}{\sqrt{1+4\,x^2}}\,\ln\frac{\sqrt{1+4\,x^2}+1}{\sqrt{1+4\,x^2}-1} \;.$$

The mass renormalization we may get from the quark self–energy function

$$G(x^2) = 1 - x^2\,\ln\left(1+\frac{1}{x^2}\right) \;.$$

We easily evaluate

$$\begin{aligned} K(x^2)\,,\; G(x^2) \to 1 \quad &\text{for } m_i^2 \ll \mu^2 \;: \quad \beta_0 \to \beta_{0\,\overline{\rm MS}}(N_f) \\ K(x^2)\,,\; G(x^2) \to 0 \quad &\text{for } m_i^2 \gg \mu^2 \;: \quad \beta_0 \to \beta_{0\,\overline{\rm MS}}(N_f - 1) \;. \end{aligned}$$

For the mass renormalization in the heavy limit we obtain

$$\gamma_{i0}\,m_i\frac{\partial}{\partial m_i} \to \frac{1}{2}\,\frac{\mu^2}{m_i^2}\,m_i\frac{\partial}{\partial m_i} \to 0 \;.$$

The behavior of the physical threshold functions which switch on or off the contribution from the heavy field is depicted in Fig. 8.7. The $\overline{\rm MS}$ scheme in contrast requires decoupling by hand, which means

$$K(x^2)\,,\; G(x^2) \to \Theta(\mu^2 - m_i^2) \;.$$

with the Heaviside Θ–function. At a given scale $\sqrt{s} = \mu$, e.g. an energy $E = \sqrt{s}$ involved in a process, as an approximation one may consider the theory $\mathcal{L}_{\rm QCD}^{(N_f)}$, with N_f the number of "excited" flavors: $2\mu_i \lesssim \mu$ $(i = 1, 2, \cdots, N_f)$ which yields a particular parametrization (convention!) valid in the range $2m_{N_f} \lesssim \mu \lesssim 2m_{N_f+1}$. This is a parametrization which optimizes the convergence of pQCD in that particular range. Crossing a new threshold at $\mu_{N_f+1} \simeq 2m_{N_f+1}$ we go to a new effective theory

$$\mathcal{L}_{\rm QCD}^{(N_f)} \to \mathcal{L}_{\rm QCD}^{(N_f+1)} \tag{8.14}$$

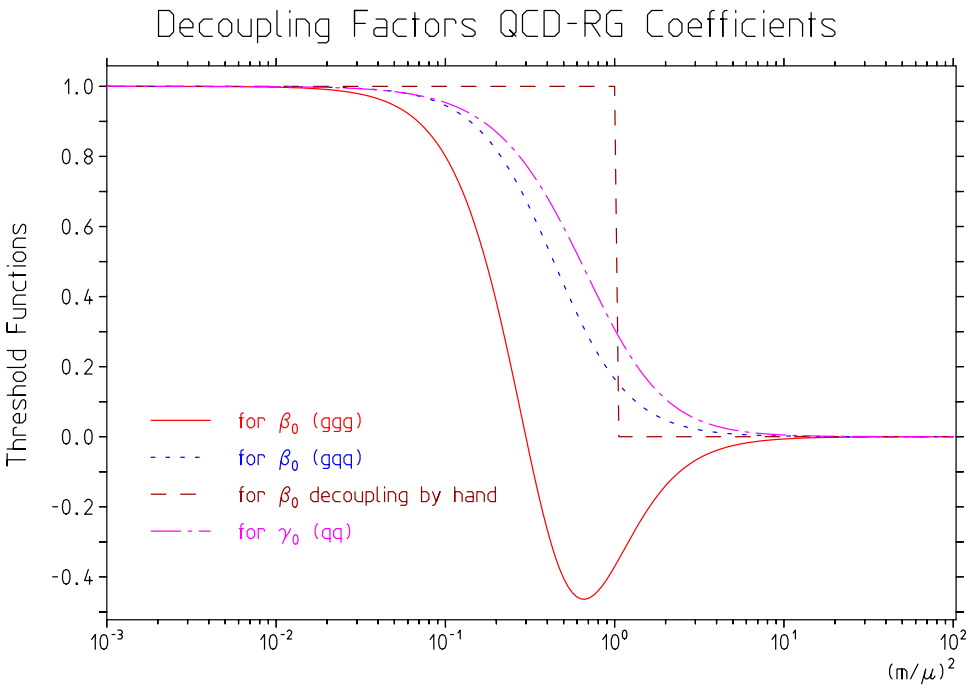


Figure 8.7: Threshold behavior of MOM RG coefficient functions.

with matching conditions [15] (to be discussed below) at the new threshold. The thresholds typically are (not very well defined)

$q\bar{q}-$threshold		typical mass	$N_f \to N_f+1$
$(s_0)_{ud}$	$\simeq 4\,m_\pi^2$	$(\simeq 0.1\ \mathrm{GeV}^2)$	2
$(s_0)_{ss}$	$\simeq 4\,m_K^2$	$(\simeq 1.0\ \mathrm{GeV}^2)$	$2 \to 3$
$(s_0)_{cc}$	$\simeq M_{J/\psi}^2$	$(\simeq 10.\ \mathrm{GeV}^2)$	$3 \to 4$
$(s_0)_{bb}$	$\simeq M_\Upsilon^2$	$(\simeq 100\ \mathrm{GeV}^2)$	$4 \to 5$
$(s_0)_{tt}$	$\simeq 4\,m_t^2$	$\simeq (350\ \mathrm{GeV})^2$	$5 \to 6$.

Often in the past as a condition the transition (8.14) has been supplemented by a requirement of continuity of the parameters when crossing the threshold: e.g. continuity of $\alpha_s(\mu)$ at $\mu_0 = \sqrt{s}_{N_f+1}$, which yields $\Lambda_{\overline{\mathrm{MS}}}^{(N_f+1)}$ when $\Lambda_{\overline{\mathrm{MS}}}^{(N_f)}$ is known: (this prescription is ad hoc and know accepted to be obsolete!)

$$\ln\frac{\mu_0}{\Lambda_{\overline{\mathrm{MS}}}^{(N_f)}} = \frac{4\pi}{2\beta_0\alpha_s(\mu_0)}\left(1-\frac{\alpha_s(\mu_0)}{4\pi}\frac{\beta_1}{\beta_0}\ln\left(\frac{4\pi}{\beta_0\alpha_s(\mu_0)}+\frac{\beta_1}{\beta_0^2}\right)\right)$$

$$\ln\frac{\mu_0}{\Lambda_{\overline{\mathrm{MS}}}^{(N_f+1)}} = \frac{4\pi}{2\beta_0^+\alpha_s(\mu_0)}\left(1-\frac{\alpha_s(\mu_0)}{4\pi}\frac{\beta_1^+}{\beta_0^+}\ln\left(\frac{4\pi}{\beta_0^+\alpha_s(\mu_0)}+\frac{\beta_1^+}{\beta_0^{+2}}\right)\right)$$

where

$$(\beta_0^+,\beta_1^+) = (\beta_0,\beta_1)|_{N_f\to N_f+1} \quad .$$

Results are displayed in Fig. 8.8.

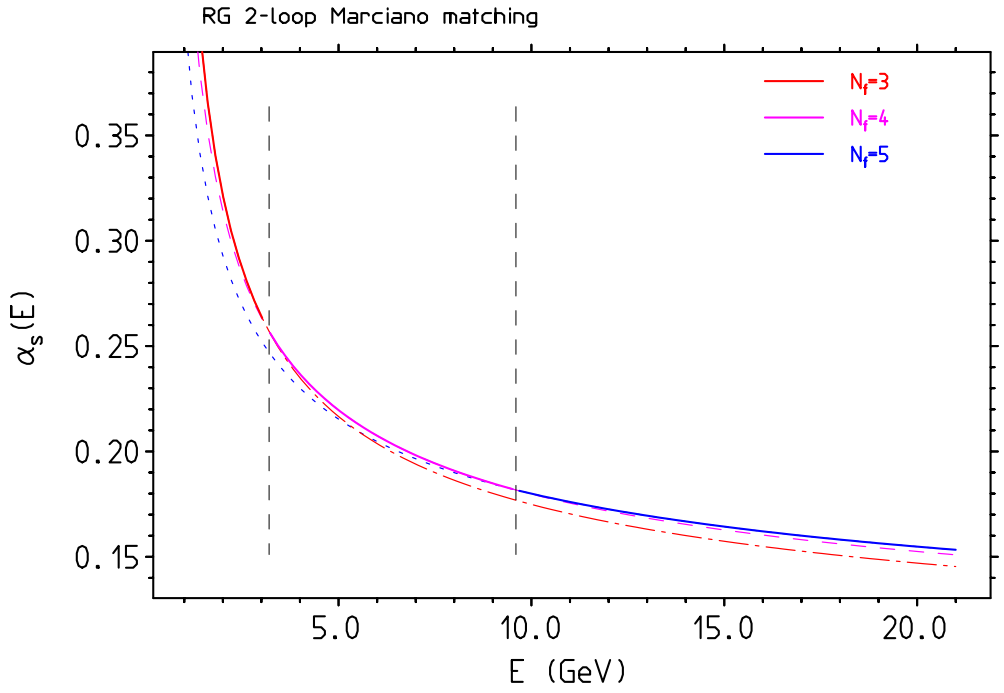


Figure 8.8: Matching by continuity (obsolete!): input $\alpha_s(M_Z) = 0.1189$; $\Lambda^{(5)}_{\overline{\rm MS}} = 223$ MeV, $\Lambda^{(4)}_{\overline{\rm MS}} = 342$ MeV, $\Lambda^{(3)}_{\overline{\rm MS}} = 435$ MeV. Furthermore: $\alpha_s(2m_b) = 0.1817$ and $\alpha_s(2m_c) = 0.2569$. Thresholds: $3 \to 4$ at 3.2 GeV and $4 \to 5$ at 9.6 GeV.

8.10 N_f - flavor effective QCD and matching conditions

Given the problems with MOM scheme renormalization, the alternative way, which is commonly used by QCD practitioners, is the $\overline{\rm MS}$ scheme together with decoupling by hand. As QCD by itself is a vector–like theory (no axial currents participate in QCD dynamics, although weak axial currents of course play a role by interference between weak and strong interaction processes) QCD is well defined for whatever the number of flavors we take. As the physical thresholds $\mu_{\rm thres}$ (again in QCD the notion of a threshold is not very well defined) we may either consider the quark masses $m_c \simeq 1.3$ GeV, $m_b \simeq 4.2$ GeV and $m_t \simeq 171$ GeV or the thresholds for charm, bottom or top pair–production in e^+e^-–annihilation, which lie at about twice the corresponding quark mass, or more physically vector meson masses $M_{J/\psi} \simeq 3.1$ GeV and $M_\Upsilon \simeq 9.5$ GeV, which are the lowest flavor neutral $\bar{c}c$ and $\bar{b}b$ states, respectively. The top is so unstable that it decays weakly before it is able to form a hadronic bound state and one uses naturally $2\,m_t$ as a pair production threshold.

What one then does is the following: at the energy region between two successive thresholds M_{N_f} and M_{N_f+1} one takes N_f–flavor QCD as an effective theory, i.e., the first N_f quarks are the light ones (theory of the light fields) while the other heavier fields are completely ignored. So, for scales $E = \sqrt{s}$ with $M_{N_f} < E \leq M_{N_f+1}$ one solves the RG-equations for the N_f flavors, for $M_{N_f+1} < E \leq M_{N_f+2}$ with $N_f + 1$ and so forth. This is what we have done before when we discussed the running of QCD parameters. The crucial question is what is the proper procedure to continue the effective theory when we pass a threshold and go form N_f to $N_f + 1$ flavors. In other words, we are looking for the appropriate *matching conditions*[30].

[30]One should keep in mind that in many of the earlier papers the matching has not been implemented in a consistent way. For example, as mentioned earlier, just requiring continuity of the parameters at $\mu_{\rm thres}$ is not a proper matching prescription.

What it means is that we are comparing two different theories which exhibit their own parameters and renormalization constants. The condition is that at the threshold the two theories have to agree. In pQCD one set of parameters is a perturbative series in the other set of parameters: (see [17])

$$\begin{aligned} a_{N_f+1}(\mu_{\rm thres}) &= a_{N_f}(\mu_{\rm thres}) \left[1+\sum_{k=1}^{\infty} C_k(x)\, a_{N_f}^k(\mu_{\rm thres})\right] \\ m_{q,N_f+1}(\mu_{\rm thres}) &= m_{q,N_f}(\mu_{\rm thres}) \left[1+\sum_{k=1}^{\infty} H_k(x)\, a_{N_f}^k(\mu_{\rm thres})\right] \end{aligned}$$

where the coefficients depend on $x = \ln \mu^2_{\rm thres}/m^2$ where m is an RG-invariant mass of the heavy quark which is switched on at the threshold.

All logarithmic terms are actually fixed by the fact that a_{N_f} and m_{q,N_F} satisfy the RG equations of the effective theory with N_f flavors:

$$\begin{aligned} \frac{{\rm d}a}{{\rm d}\ln\mu^2} &= \beta(a) = -a^2\,\left(\beta_0+\beta_1\,a+\beta_2\,a^2+\beta_3\,a^3\right)+O(a^6) \\ \frac{{\rm d}\ln m_q}{{\rm d}\ln\mu^2} &= \gamma_m(a) = -a\,\left(\gamma_0+\gamma_1\,a+\gamma_2\,a^2+\gamma_3\,a^3\right)+O(a^5)\ , \end{aligned}$$

where μ is to be taken in units of a RG invariant reference scale, like $\Lambda_{\rm QCD}$ or a RG invariant mass $\overline{m}$. The coefficients of the QCD beta function have been calculated recently in the $\overline{\rm MS}$ scheme up to four loops[31] (see [2]– [12])

$$\begin{aligned} \beta_0 &= \frac{1}{4}\left[11-\frac{2}{3}N_f\right]\ , \qquad \beta_1 = \frac{1}{16}\left[102-\frac{38}{3}N_f\right]\ , \\ \beta_2 &= \frac{1}{64}\left[\frac{2857}{2}-\frac{5033}{18}N_f+\frac{325}{54}N_f^2\right]\ , \\ \beta_3 &= \frac{1}{256}\left[\left(\frac{149753}{6}+3564\zeta_3\right)-\left(\frac{1078361}{162}+\frac{6508}{27}\zeta_3\right)N_f \right. \\ &\quad \left. +\left(\frac{50065}{162}+\frac{6472}{81}\zeta_3\right)N_f^2+\frac{1093}{729}N_f^3\right]\ , \end{aligned} \tag{8.15}$$

and also the coefficients of the quark-mass anomalous dimension have been calculated at the same order,

$$\begin{aligned} \gamma_0 &= 1\ , \qquad \gamma_1 = \frac{1}{16}\left[\frac{202}{3}-\frac{20}{9}N_f\right]\ , \\ \gamma_2 &= \frac{1}{64}\left[1249+\left(-\frac{2216}{27}-\frac{160}{3}\zeta_3\right)N_f-\frac{140}{81}N_f^2\right]\ , \\ \gamma_3 &= \frac{1}{256}\left[\frac{4603055}{162}+\frac{135680}{27}\zeta_3-8800\zeta_5\right. \\ &+ \left(-\frac{91723}{27}-\frac{34192}{9}\zeta_3+880\zeta_4+\frac{18400}{9}\zeta_5\right)N_f \\ &+ \left.\left(\frac{5242}{243}+\frac{800}{9}\zeta_3-\frac{160}{3}\zeta_4\right)N_f^2+\left(-\frac{332}{243}+\frac{64}{27}\zeta_3\right)N_f^3\right]\ . \end{aligned} \tag{8.16}$$

[31] In this section we adopt a slightly different normalization for the RG coefficients, the one used in the relevant papers on the subject, which are quoted in the references to this chapter. All factors $1/\pi$ are absorbed into $a = \alpha_s/\pi$, all remaining factors into the coefficients.

Here ζ_n is the Riemann zeta-function ($\zeta_2 = \pi^2/6$, $\zeta_3 = 1.202056903\ldots$, $\zeta_4 = \pi^4/90$ and $\zeta_5 = 1.036927755\ldots$) and N_f is the number of quark flavors with mass lower than the renormalization scale μ.

Taking the derivatives of the matching equations Eqs. (8.15), denoting derivatives by dots ($\dot{}$) and $N_f + 1$ quantities by hats ($\hat{}$), we obtain

$$\begin{aligned}
\dot{\hat{a}} &= \dot{a}\,[1 + \sum_k C_k\, a^k] + a\,[\sum_k \dot{C}_k\, a^k] + a\,[\sum_k k\, C_k\, \dot{a}\, a^{k-1}] \\
\dot{\hat{m}} &= \dot{m}\,[1 + \sum_k H_k\, a^k] + m\,[\sum_k \dot{H}_k\, a^k] + m\,[\sum_k k\, H_k\, \dot{a}\, a^{k-1}]
\end{aligned} \tag{8.17}$$

A comparison of the coefficients of the powers a^k then yields

$$\begin{aligned}
a^2: \quad \dot{C}_1 &= \beta_0 - \hat{\beta}_0 \\
a^3: \quad \dot{C}_2 &= \beta_1 - \hat{\beta}_1 + 2\,(\beta_0 - \hat{\beta}_0)\, C_1 \\
a^4: \quad \dot{C}_3 &= \beta_2 - \hat{\beta}_2 + (2\beta_1 - 3\hat{\beta}_1)\, C_1 - \hat{\beta}_0\, C_1^2 + (3\beta_0 - 2\hat{\beta}_0)\, C_2 \\
&\vdots \\
m\,a: \quad \dot{H}_1 &= \gamma_0 - \hat{\gamma}_0 \\
m\,a^2: \quad \dot{H}_2 &= \gamma_1 - \hat{\gamma}_1 - \hat{\gamma}_0\, C_1 + (\beta_0 + \gamma_0 - \hat{\gamma}_0)\; H_1 \\
m\,a^3: \quad \dot{H}_3 &= \gamma_2 - \hat{\gamma}_2 - 2\,\hat{\gamma}_1\, C_1 - \hat{\gamma}_0\, C_2 + (\beta_1 + \gamma_1 - \hat{\gamma}_1 - \hat{\gamma}_0\, C_1)\; H_1 + (2\,\beta_0 + \gamma_0 - \hat{\gamma}_0)\; H_2 \\
&\vdots
\end{aligned}$$

or, inserting the coefficient functions,

$$\begin{aligned}
\dot{C}_1 &= \frac{1}{6} \\
\dot{C}_2 &= \frac{19}{24} + \frac{1}{3}\, C_1 \\
\dot{C}_3 &= \frac{7387}{1728} - \frac{325}{1728}\, N_f - \left(4 - \frac{19}{24}\, N_f\right)\, C_1 - \left(\frac{31}{12} - \frac{1}{6}\, N_f\right)\, C_1^2 + \left(\frac{37}{12} - \frac{1}{6}\, N_f\right)\, C_2 \\
&\vdots \\
\dot{H}_1 &= 0 \\
\dot{H}_2 &= \frac{5}{36} - C_1 + \left(\frac{11}{4} - \frac{1}{6}\, N_f\right)\, H_1 \\
\dot{H}_3 &= \frac{1697}{1296} + \frac{5\,\zeta_3}{6} + \frac{35}{648}\, N_f - \left(\frac{293}{36} - \frac{5}{18}\, N_f\right)\, C_1 - C_2 \\
&\quad + \left(\frac{469}{72} - \frac{19}{24}\, N_f\right)\, H_1 - C_1 H_1 + \left(\frac{11}{2} - \frac{1}{3}\, N_f\right)\, H_2 \\
&\vdots
\end{aligned} \tag{8.18}$$

The system can be solved and has a unique solution up to the integration constants. The latter have to be determined by a direct calculation of the self-energies and vertex functions which define Z_g and Z_m. In fact for the leading terms C_1 and H_1 the integration constants vanish, i.e. $c_{1,0} = 0$ and $h_{1,0} = 0$. This we will use in the following. Integration of the above system of equations for C_k yields, ($x = \ln \mu^2/\Lambda^2$)

$$C_1 = \frac{1}{6}\, x$$

$$
\begin{aligned}
C_2 &= c_{2,0} + \frac{19}{24}\,x + \frac{1}{36}\,x^2 \\
C_3 &= c_{3,0} + c_{2,0}\left(\frac{37}{12} - \frac{1}{6}\,N_f\right) x + \frac{7387}{1728}\,x - \frac{325}{1728}\,N_f\,x + \frac{511}{576}\,x^2 + \frac{1}{216}\,x^3 \\
&\vdots \\
H_1 &= 0 \\
H_2 &= h_{2,0} + \frac{5}{36}\,x - \frac{1}{12}\,x^2 \\
H_3 &= h_{3,0} + h_{2,0}\left(\frac{11}{2} - \frac{1}{3}\,N_f\right) x - c_{2,0}\,x \\
&\quad + \frac{1697}{1296}\,x + \frac{5\,\zeta_3}{6}\,x + \frac{35}{648}\,N_f\,x - \frac{299}{432}\,x^2 - \frac{35}{216}\,x^3 + \frac{1}{108}\,N_f\,x^3 \\
&\vdots
\end{aligned}
\tag{8.19}
$$

As a result we learn that most of the matching terms are just determined by the RG. Not so easy is the evaluation of the integration constants. Their determination requires to calculate the renormalization constants $g_{s0} = Z_g\,g_s$, $m_{q0} = Z_m\,m_q$, $\psi_{q0} = \sqrt{Z_F}\,\psi_q$, $G^\mu_{a0} = \sqrt{Z_3}\,G^\mu_a$ and $\eta_0 = \sqrt{\tilde{Z}_3}\,\eta_a$ once in the full and once on the effective theory, corresponding quantities we denote by a prime, such that the effective parameters read

$$
\begin{aligned}
\alpha'_s(\mu) &= \left(\frac{Z_g}{Z'_g}\,\zeta_g^0\right)^2 \alpha_s(\mu) = \zeta_g^2\,\alpha_s(\mu) \\
m'_q(\mu) &= \frac{Z_m}{Z'_m}\,\zeta_m^0\,m_q(\mu) = \zeta_m\,m_q(\mu)
\end{aligned}
$$

in terms of the full QCD parameters. The relative renormalization factors ζ_i are finite and mediate the corresponding change of the renormalization scheme. What we need to know is what happens when we take the heavy field masses in full QCD to go to infinity and compare the result with what one obtains for the same Green functions in the N_f - effective theory. This provides a matching up to terms of order $O(1/M)$. As usual, for simplicity, we adopt dimensional regularization together with the $\overline{\rm MS}$ scheme and the convention $\mathrm{Tr}\,\mathbf{1} = f(d) = 4$ for the trace of the unit matrix in Dirac spinor space. This implies $c_{1,0} = d_{1,0} = 0$, which we have used already above. The other non-trivial constant depend on the choice of the RG invariant reference scale.

If the RG-invariant $\overline{\rm MS}$ mass is used as a reference scale, that is $m = \bar{m}(\bar{m})$ the coefficients one obtains are [10, 16, 18]

$$
c_{2,0} = -\frac{11}{72}\;, \quad c_{3,0} = \frac{82043}{27648}\zeta_3 - \frac{575263}{124416} + \frac{2633}{31104}n_f\;, \quad d_{2,0} = -\frac{89}{432}\;. \tag{8.20}
$$

If the pole mass is used as a reference scale, that is $m = M$ the coefficients one obtains are [16,18]

$$
\begin{aligned}
c_{2,0} &= \frac{7}{24} \\
c_{3,0} &= \frac{80507}{27648}\zeta_3 + \frac{1}{9}\zeta_2\,(2\log(2) + 7) + \frac{68849}{124416} - \frac{n_f}{9}\left(\zeta_2 + \frac{2479}{3456}\right)\;.
\end{aligned}
\tag{8.21}
$$

while $d_{2,0}$ does not change, and $d_{3,0}$, when known, has to be shifted by $+10/27$.

In [18] the relations Eq. 8.20 are directly calculated and given in compact numerical form. The number of light flavors is denoted by n_l (corresponding to our N_f above). If the RG-invariant

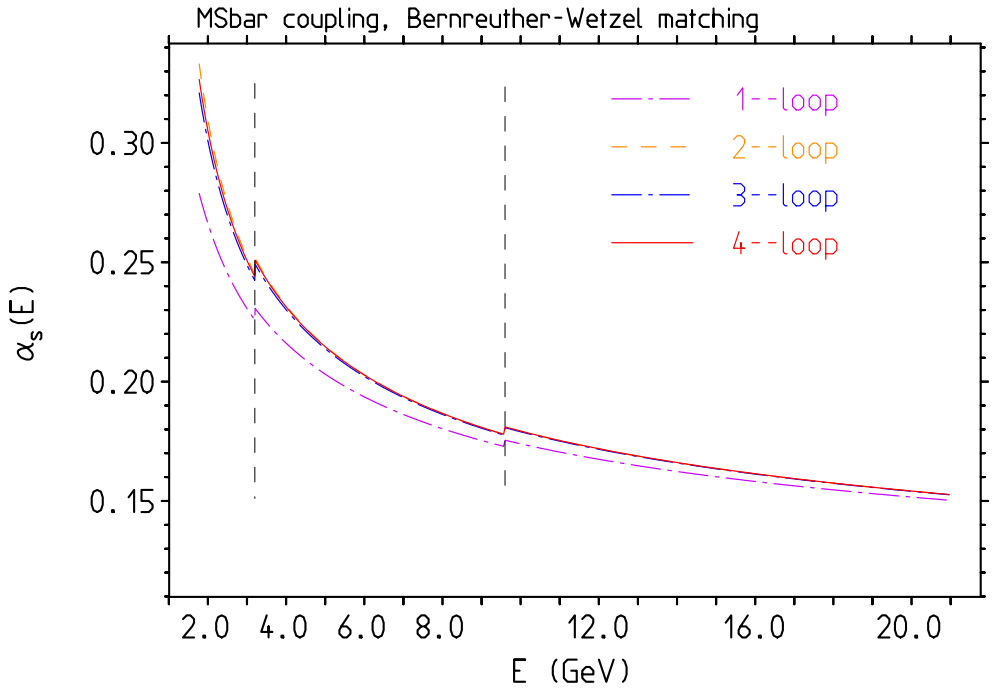


Figure 8.9: Matching via observables in different orders of the perturbation expansion. Input: $\alpha_s(M_Z) = 0.1189$, $m_c = 1.3$ GeV, $m_b = 4.2$ GeV. The dashed vertical lines mark the threshold positions. The discontinuities are quite substantial and clearly visible. Now: $\Lambda^{(5)}_{\overline{\rm MS}} = 219$ MeV, $\Lambda^{(4)}_{\overline{\rm MS}} = 330$ MeV, $\Lambda^{(3)}_{\overline{\rm MS}} = 411$ MeV. Furthermore: $\alpha_s(2m_b) = 0.1891$ and $\alpha_s(2m_c) = 0.2887$, while the OS masses read $M_c = 1.15$ GeV and $M_b = 4.48$ GeV.

$\overline{\rm MS}$ mass $\mu_h = \bar{m}(\bar{m})$ is used as a reference scale one obtains

$$(\zeta_g^{\rm MS})^2 \stackrel{(\mu=\mu_h)}{=} 1 + 0.152778\, a_s^2(\mu_h) + a_s^3(\mu_h)\,(0.972057 - 0.0846515\ n_l) + a_s^4(\mu_h)\left(5.17035 - 1.00993\ n_l - 0.0219784\ {n_l}^2\right), \tag{8.22}$$

$$\frac{1}{(\zeta_g^{\rm MS})^2} \stackrel{(\mu=\mu_h)}{=} 1 - 0.152778\, a_s'^2(\mu_h) + a_s'^3(\mu_h)\,(-0.972057 + 0.0846515\ n_l) + a_s'^4(\mu_h)\left(-5.10032 + 1.00993\ n_l + 0.0219784\ {n_l}^2\right). \tag{8.23}$$

If the perturbative pole mass $m = M$ is used as a reference scale, the result is given by

$$(\zeta_g^{\rm OS})^2 \stackrel{(\mu=M)}{=} 1 - 0.291667\, a_s^2(M) + a_s^3(M)\,(-5.32389 + 0.262471\ n_l) + a_s^4(M)\left(-85.875 + 9.69229\ n_l - 0.239542\ {n_l}^2\right), \tag{8.24}$$

$$\frac{1}{(\zeta_g^{\rm OS})^2} \stackrel{(\mu=M)}{=} 1 + 0.291667\, a_s'^2(M) + a_s'^3(M)\,(5.32389 - 0.262471\ n_l) + a_s'^4(M)\left(86.1302 - 9.69229\ n_l + 0.239542\ {n_l}^2\right). \tag{8.25}$$

After proper matching of N_f- to $N_f + 1$-flavor effective QCD, the result for the running $\overline{\rm MS}$ coupling is displayed in Fig. 8.9, which compares with Fig. 8.8. The discontinuities across the matching points at a given order of the perturbation expansion are a result of the change of the effective Lagrangian (discontinuous change of N_f). In fact the discontinuities are small relative to

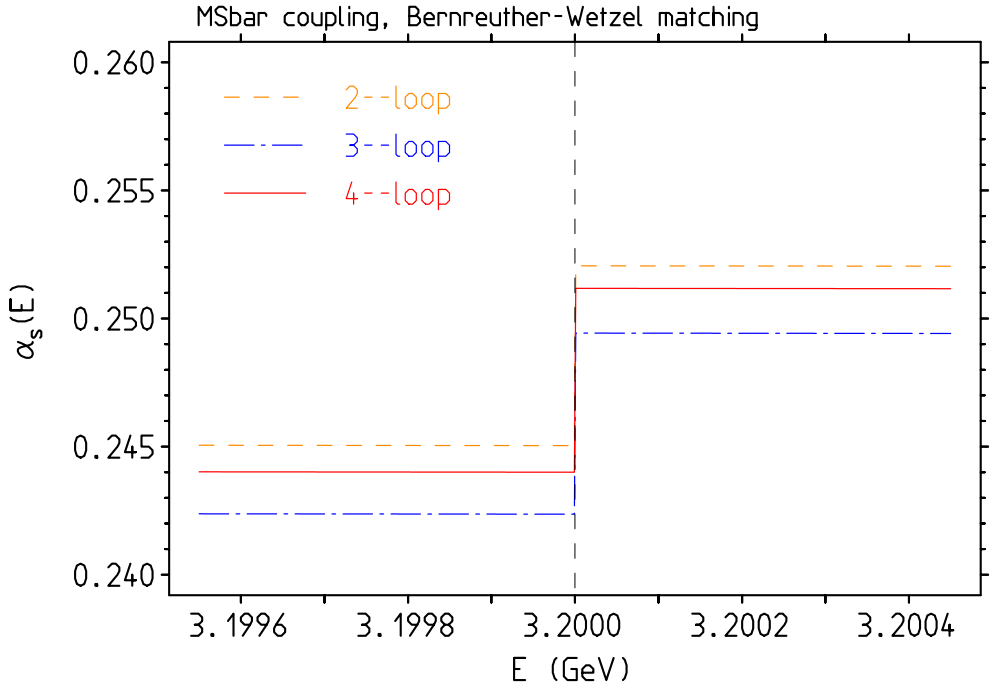


Figure 8.10: The discontinuous behavior of $\alpha_s^{\overline{\rm MS}}$ across the charm threshold chosen at 3.2 GeV.

the running effects as is shown here for passing the charm threshold in Fig. 8.10. In this approach the matching observable is a given number at each selected matching point, such that the "re-predicted" observable is continuous, while $\alpha_s^{N_f}(E)$ is a scale dependent effective parameter of the N_f-flavor effective theory, which, depending on the renormalization scheme, in general is not an observable itself.

8.11 Renormalization Scheme Dependence

We already have encountered a variety of renormalization schemes (RS), the choice each was motivated by formal or physical arguments. Different RS's represent different parametrization of a given theory and at the end of the same physics. In finite order perturbation theory, a change of a RS formally correspond to a reshuffling of higher order terms in the prediction of various quantities. Predictions of physical quantities of course should not depend on the specific choice of the input parameters and they in fact do not if we include all orders of the perturbation expansion. Actually, the reparametrization invariance is inferred by renormalization group invariance. However, practical perturbative calculations are *approximations* obtained by truncation of the perturbation series. The accuracy of the finite order approximations depends on the choice of the input parameters i.e. finite order results are scheme dependent. Let us illustrate this point by an example: Suppose we compute a matrix element M in the α_s-scheme (I) to one-loop order yielding a result

$$M^{(1)} = \alpha_s^n \, C \, [1 + b \, \alpha_s].$$

Now, suppose we calculate the same quantity in the α_s'-scheme (II) which amounts to a replacement of α_s by $\alpha_s' = \alpha_s \, (1 + \Delta\alpha_s)$ i.e. to one-loop order $\alpha_s' = \alpha_s[1 + a\alpha_s]$ and

$$M'^{(1)} = \alpha_s'^n \, C \, [1 + b' \, \alpha_s'].$$

Inserting α_s' we get

$$M'^{(1)} = M^{(1)} + \delta M$$

with $b' = b - na$ and

$$\delta M = \alpha_s^n \, C \, \left[(\frac{n(n-1)}{2} a^2 + (n+1)ab')\alpha_s^2 + \cdots + a^{n+1} b' \alpha_s^{n+2} \right].$$

Thus the result differs by δM. If we do not actually calculate the higher orders

$$\delta M = M'^{(1)} - M^{(1)}$$

must be considered as an uncertainty due to unknown higher order effects. In QCD in particular, RG resummation is mandatory and reduces the SD substantially. Nevertheless, we are always dealing with approximations which also in RG improved finite order pQCD are never RG invariant. The residual μ–dependence of a prediction is an important measure for the quality of an approximation. The study of the scheme dependence of resummation improved results is a way to estimate missing higher order contributions (*educated guess*). Of course only an actual n-loop calculation can tell us what the full n-loop answer is.

Let us summarize the content of this subsection by the following conclusions:

- If a physical quantity is calculated with different input parameters the answer is the same if we calculate it to arbitrary high orders. Thus RS independence only applies for the exact result.

However:

- Calculating a quantity to a given order the omitted higher order terms differ for different parametrizations. This leads to a *scheme dependence* of the result (approximate) due to different truncation errors. Thus RS dependence is is the result of omitted higher order terms which depend on the RS.

- Differences can also be due to different resummation prescriptions (see below).

- Note that the "convergence" of the asymptotic perturbation expansion may be substantially affected by the choice of the RS (remember that the perturbation series as an asymptotic series diverges beyond some unknown order).

Here we look at a few very simple lowest order examples where effects are particularly large (low order=large truncation errors):

1) Various definitions of $\alpha_s(\mu)$:

MS, $\overline{\text{MS}}$	β_0, β_1 identical, β_2 different
MOM_I	
MOM_{II}	mass dependent, but also gauge dependent
MOM_{III}	
$\vdots$	???

2) Identification of μ with a physical scale $\sqrt{s}$:
If $\mu_{\overline{\text{MS}}} = \sqrt{s}$ what is μ_{MS}, μ_{MOM}, $\cdots$???

For example: we have

$$\ln \mu^2_{\text{MS}} = \ln \mu^2_{\overline{\text{MS}}} + \gamma - \ln 4\pi$$

thus

$$\mu_{\overline{\text{MS}}}/\mu_{\text{MS}} = \exp \frac{\ln 4\pi - \gamma}{2} \simeq 2.66$$

which tells us that we should be aware of the fact that the identification of the renormalization scale with a physical scale is fairly arbitrary (largely a commonly agreed convention!).
Values for $\mu_{\text{MOM}}/\mu_{\overline{\text{MS}}}$ at one-loop (Celmaster, Gonsalves 1979):

Gauge	N_f	MOM_I	MOM_{II}	MOM_{III}
Landau	3	2.46	2.10	2.33
($\xi = 0$)	4	2.16	2.09	2.33
	5	1.85	2.07	2.33
Feynman	3	2.07	1.83	2.69
($\xi = 1$)	4	1.79	1.80	2.73
	5	1.51	1.76	2.76

- For different processes or observables, in general, different parametrizations are optimal with respect to "convergence" properties.
- However, is makes not sense to consider isolated processes described in a particular scheme. To see whether the theory works requires a convention for a *universal* parametrization, for example the $\overline{\rm MS}$ one, for its simplicity and relatively good convergence properties, together with the *decoupling by hand* prescription.

In any case: specification of the RS is indispensable, as otherwise numbers make little sense. What we always need to know:

- renormalization scheme
- identification of the scale with the scale of a process
- order of perturbation theory used
- number of flavors used (with $\overline{\rm MS}$)

More interesting and realistic examples of renormalization scheme dependence will be given in the later course of these lectures.

As a last remark, let me mention that the well established "running" of the strong interaction constant, as established by experimental results collected in Fig. 8.11, made tremendous progress thanks to the LEP experiments at CERN, especially by high precision hadron physics at the Z peak and in hadronic decays of τ leptons produced with high statistics at LEP. The energy range spans scales form $M_\tau \sim 1.8$ GeV to the Z mass $M_Z \sim 91.19$ GeV and at LEP-II up to 200 GeV. Pre LEP status of α_s measurements is shown in Fig. 8.12. Besides the e^+e^- physics at LEP, deep inelastic eN–scattering experiments (H1 and ZEUS) with HERA at DESY (see below) contributed significantly to the progress in determining α_s.

Epilogue on the RG and running couplings:
As we have learned the RG and the running of couplings and other parameters is intimately related to the UV singularities of the bare QFT. Very often, the occurrence of UV and the need for regularization and renormalization is considered as something obsolete, a lack of having had the right tricks to avoid such mathematical diseases and in fact there have been a number of proposals which technically avoid the need to deal with such singularities like the BPHZ approach or methods based on the use of Cutkosky cutting rules. From the point of view of physics we have learned that the UV singularities are not an artifact of missing the appropriate mathematics, they must be there for deep physics reasons, namely, that quantum effects break scaling properties. For any massless renormalizable theory dilatation invariance and hence conformal invariance is broken by the dynamics, and there is a one–to–one correspondence of between the would–be singularities and the renormalization group coefficients. In fact it also "proves" that nature has a built in regularization, presumably a sort of Planck cut-off. What we observe as a renormalizable QFT is a low energy effective theory, the low energy tail of a structure which at the Planck scale has a built in fundamental cut-off. It is well known that non–Abelian local gauge field theories are a natural consequence of such a scenario (see [20] and references therein).

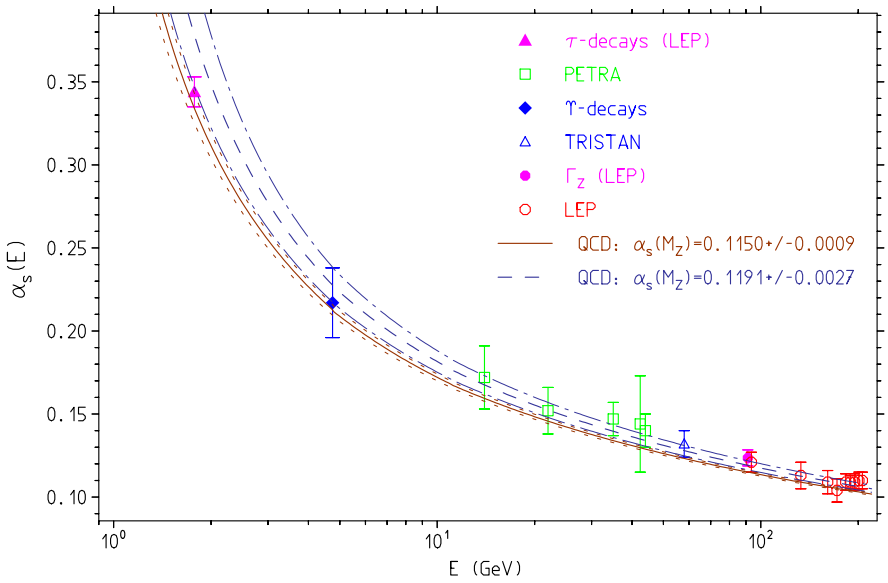


Figure 8.11: Present experimental status of the QCD running coupling. Shown are RG solutions with $\alpha_s(M_\tau)$ extracted from τ–decays and the one with $\alpha_s(M_Z)$ measured at LEP as start values. The corresponding central values agree at the 1 σ level.

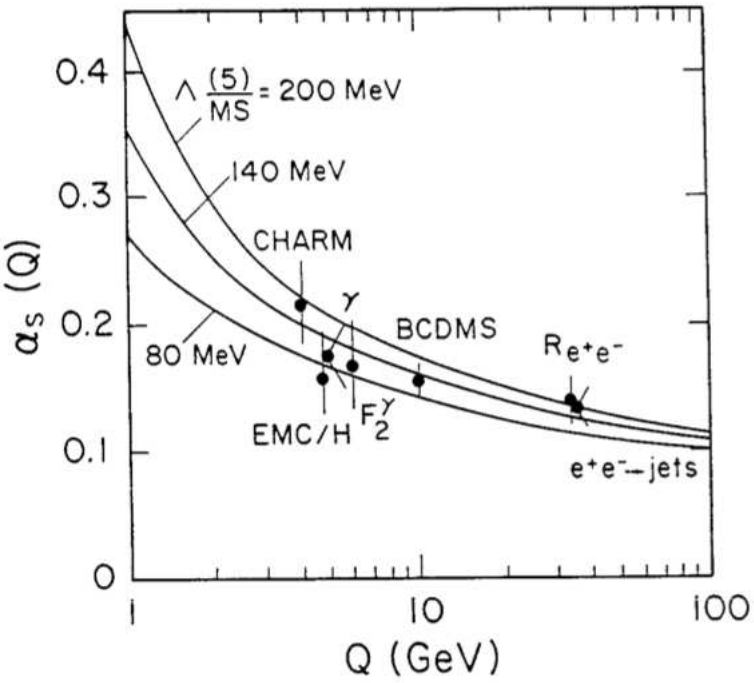


Figure 8.12: Status of α_s measurements 1989, before LEP went into operation. The curves show the best estimate for the strong coupling $\alpha_s^{(5)}(M_Z) = 0.11 \pm 0.01$ (corresponding to $\Lambda^{(5)}_{\overline{\rm MS}} = 140 \pm 60$ MeV).[Taken from [19]].

Exercises II: Section 8

⑤ Solve the matching relations Eq. (8.15) by power expansion in a on both sides of the equation for the first three non-trivial terms. While the first two terms can be worked out by hand, the third conveniently may be be calculated using computer algebra (maple, mathematica or maxima).

⑥ Verify the numerical formulae (8.22) and (8.24) by evaluating the corresponding analytical expressions.

⑦ Given $\alpha_s(M_Z) = 0.1215 \pm 0.0012$ calculate $\alpha_s(M_\tau)$ using the 4–loop RG equations together with 3–loop matching. This is the consistent combination of the two aspects: explain this statement (i.e. why don't we need 4–loop matching relations as well?).

References

[1] M. Della Morte, R. Frezzotti, J. Heitger, J. Rolf, R. Sommer, U. Wolff [ALPHA Collaboration], Nucl. Phys. B **713** (2005) 378 [arXiv:hep-lat/0411025].

[2] D. J. Gross, F. Wilczek, Phys. Rev. Lett. **30** (1973) 1343.

[3] H. D. Politzer, Phys. Rev. Lett. **30** (1973) 1346.

[4] W. E. Caswell, Phys. Rev. Lett. **33** (1974) 244.

[5] D. R. T. Jones, Nucl. Phys. B **75** (1974) 531.

[6] E. Egorian, O. V. Tarasov, Teor. Mat. Fiz. **41** (1979) 26 [Theor. Math. Phys. **41** (1979) 863].

[7] O. V. Tarasov, A. A. Vladimirov, A. Y. Zharkov, Phys. Lett. B **93** (1980) 429.

[8] S. A. Larin, J. A. M. Vermaseren, Phys. Lett. B **303** (1993) 334 [arXiv:hep-ph/9302208].

[9] K. G. Chetyrkin, M. Misiak, M. Münz, Nucl. Phys. B **518** (1998) 473 [arXiv:hep-ph/9711266].

[10] T. van Ritbergen, J. A. M. Vermaseren, S. A. Larin, Phys. Lett. B **400** (1997) 379 [arXiv:hep-ph/9701390].

[11] M. Czakon, Nucl. Phys. B **710** (2005) 485 [arXiv:hep-ph/0411261].

[12] K. G. Chetyrkin, Nucl. Phys. B **710** (2005) 499 [arXiv:hep-ph/0405193].

[13] K. Symanzik, Commun. Math. Phys. **34** (1973) 7.

[14] T. Appelquist, J. Carazzone, Phys. Rev. D **11** (1975) 2856.

[15] W. Bernreuther, W. Wetzel, Nucl. Phys. B **197** (1982) 228 [Erratum-ibid. B **513** (1998) 758].

[16] K. G. Chetyrkin, B. A. Kniehl, M. Steinhauser, Nucl. Phys. B **510** (1998) 61 [arXiv:hep-ph/9708255].

[17] G. Rodrigo, A. Pich, A. Santamaria, Phys. Lett. B **424** (1998) 367 [arXiv:hep-ph/9707474].

[18] K. G. Chetyrkin, J. H. Kühn, C. Sturm, Nucl. Phys. B **744** (2006) 121 [arXiv:hep-ph/0512060].

[19] G. Altarelli, Ann. Rev. Nucl. Part. Sci. **39** (1989) 357.

[20] F. Jegerlehner, The "Ether-World" and Elementary Particles, in *Theory of Elementary Particles*, Eds. H. Dorn, D. Lüst, G. Weigt, (WILEY-VCH, Berlin, 1998) p. 386 arXiv:hep-th/9803021.

[21] B. V. Geshkenbein, JETP Lett. **77** (2003) 603 [Pisma Zh. Eksp. Teor. Fiz. **77** (2003) 728]

[22] B. V. Geshkenbein, V. L. Morgunov, Phys. Lett. B **352** (1995) 456

[23] B. V. Geshkenbein, Phys. Rev. D **61** (2000) 033009

[24] F. M. Renard, Nucl. Phys. B **82** (1974) 1; N. N. Achasov, N. M. Budnev, A. A. Kozhevnikov, G. N. Shestakov, Sov. J. Nucl. Phys. **23** (1976) 320; N. N. Achasov, A. A. Kozhevnikov, M. S. Dubrovin, V. N. Ivanchenko, E. V. Pakhtusova, Int. J. Mod. Phys. A **7** (1992) 3187

[25] J. Bijnens, G. Colangelo, J. Gasser, Nucl. Phys. B427 (1994) 427

[26] H.W. Fearing, S. Scherer, Phys. Rev. D53 (1996) 315

[27] B.R. Holstein, Phys. Lett. B244 (1990) 83;
W.J. Marciano, Annu. Rev. Nucl. Part. Sci. 41 (1991) 469;
W.J. Marciano, A. Sirlin, Phys. Rev. Lett. 71 (1993) 3629;
M. Finkemeier, hep-ph/9501286, hep-ph/9505434

[28] J. J. Sakurai, Annals of Phys. (NY) **11** (1960) 1

[29] H. Joos, Acta Phys. Austriaca Suppl. **4** (1967) ???

[30] N. M. Kroll, T. D. Lee, B. Zumino, Phys. Rev. **157** (1967) 1376

[31] G. 't Hooft, Nucl. Phys. B **75** (1974) 461;
E. Witten, Nucl. Phys. B **160** (1979) 57

[32] B. V. Geshkenbein, V. L. Morgunov, arXiv:hep-ph/9410252

[33] G. J. Gounaris, J. J. Sakurai, Phys. Rev. Lett. **21** (1968) 244

[34] R. R. Akhmetshin et al [CMD-2 Collaboration], Phys. Lett. B **527**, 161 (2002)

[35] M. Gell-Mann, R. J. Oakes, B. Renner, Phys. Rev. **175** (1968) 2195

[36] S. Peris, M. Perrottet, E. de Rafael, JHEP **9805** (1998) 011

[37] F. Jegerlehner, O.V. Tarasov, Nucl. Phys. B **549** (1999) 481

[38] V. A. Khoze, M. I. Konchatnij, N. P. Merenkov, G. Pancheri, L. Trentadue, O. N. Shekhovtzova, Eur. Phys. J. C **25** (2002) 199

[39] K. Maltman, Phys. Lett. B **633** (2006) 512; K. Maltman, C. E. Wolfe, Phys. Rev. D **73** (2006) 013004; K. Maltman, Int. J. Mod. Phys. A **21** (2006) 813; hep-ph/0512331

[40] D. Stöckinger, hep-ph/0609168.

[41] J. Erler, G. T. Sanchez, hep-ph/0605052.

8.12 General $O(\alpha^2)$ framework for predicting $R(s)$

Quarks are charged particles and couple to the photon via the quark contribution to the electromagnetic current. The latter is given by

$$J_\mu^\gamma = \sum_i \left(\frac{2}{3}\,\bar{u}_i \gamma_\mu u_i - \frac{1}{3}\,\bar{d}_i \gamma_\mu d_i \right) \tag{8.1}$$

where $i = 1, 2, 3$ labels the three quark doublets:

$$\left(\begin{array}{c} u_i \\ d_i \end{array} \right) = \left(\begin{array}{c} u \\ d \end{array} \right) \;, \left(\begin{array}{c} c \\ s \end{array} \right) \;, \left(\begin{array}{c} t \\ b \end{array} \right) \;.$$

The Lagrangian density for strong and electromagnetic interactions is given by

$$\mathcal{L}_{\text{strong\&electromagnetic}} = \mathcal{L}_{\text{QCD}} + e J_\mu^\gamma A^\mu - \frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} \;.$$

Since $\alpha = \frac{e^2}{4\pi} = (137.036)^{-1}$ is small, it will be sufficient to consider the one–photon exchange approximation in many cases. In the following we denote by X a hadronic state (mesons, baryons) of appropriate quantum numbers. We then are interested in the S–matrix element

$$\langle X|S|e^+e^-\rangle = \langle X|T\left\{ \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\mathcal{L}_{\text{QCD,int}} + \mathcal{L}_{\text{QED,int}}\right)} \right\} |e^+e^-\rangle$$

of the process

$$e^+e^- \to X \;. \tag{8.2}$$

It should be stressed that we consider here pure hadronic states X only, which means a state which is obtained (e.g. in a lattice QCD calculation) in QCD alone. The much weaker electromagnetic and weak interactions are switched off, for what concerns X. Also experimentally we assume hat a hadronic state can be unambiguously identified[32]. Additional photons radiated by hadrons are corrections which count as $O(\alpha^3)$ effect, which is discarded in this leading $O(\alpha^2)$ approximation. Since α is small a systematic expansion in α is always assumed in the following. As perturbative QCD fails at low energies, we treat strong interaction effects non-perturbatively in a first step. Later we will argue where pQCD can be applied.

Since we are interested in the process (8.2), where at this order no external photon occurs, the process must be second order in QED. We thus have in lowest non–trivial order $O(\alpha^2)$

$$\langle X|S|e^+e^-\rangle = -\frac{1}{2!}\int \mathrm{d}^4x_1 \mathrm{d}^4x_2\, \langle X|T\left\{ \mathcal{L}_{\text{QED,int}}(x_1), \mathcal{L}_{\text{QED,int}}(x_2)\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\text{QCD,int}}} \right\} |e^+e^-\rangle$$

where for our purpose (we do not consider muons and taus here)

$$\mathcal{L}_{\text{QED,int}} = \left(-e\bar{\psi}_e \gamma_\mu \psi_e + e J_\mu^\gamma\right)\, A^\mu = e J_\mu A^\mu \;.$$

In addition the photons must appear contracted such that

$$\langle X|S|e^+e^-\rangle = -\frac{e^2}{2!}\int \mathrm{d}^4x_1 \mathrm{d}^4x_2\, \mathrm{i}\, D_{\mu\nu} \langle X|T\left\{ J^\mu(x_1), J^\nu(x_2)\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\text{QCD,int}}} \right\} |e^+e^-\rangle \;, \tag{8.3}$$

where $D_{\mu\nu}$ denotes the photon propagator (see Fig. 8.1).

[32] Especially at low energies this is not completely trivial for states which have masses in a similar range like $\pi^+\pi^-$– and $\mu^+\mu^-$–pairs produced at the same time in e^+e^-–annihilation. At high energies hadronic states typically have high multiplicities (hadronic jets) and can be neatly distinguished from lepton–pair production. Higher multiplicity lepton states are of higher order in the electroweak couplings and hence are suppressed due to the small coupling.

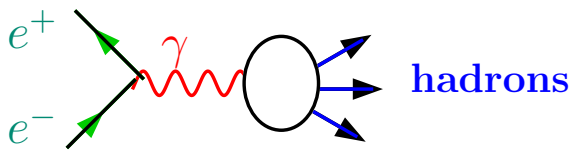


Figure 8.1: Hadron production in electron–positron annihilation.

The product of the two currents yields four pieces

$$J^\mu(x_1)\,J^\nu(x_2) = \Big\{ J^\mu_{\rm lep}(x_1)\,J^\nu_{\rm lep}(x_2) + J^\mu_{\rm lep}(x_1)\,J^\nu_{\rm had}(x_2) + J^\mu_{\rm had}(x_1)\,J^\nu_{\rm lep}(x_2) + J^\mu_{\rm had}(x_1)\,J^\nu_{\rm had}(x_2) \Big\} \quad .$$

The electron and positron states are created by external LSZ fields $\psi_{e,\rm ext}$ and $\bar\psi_{e,\rm ext}$ which contract with the fields of the currents, where the LSZ states are represented by

$$\begin{aligned} &\mathrm{i}\int \mathrm{d}^4y\,\bar\psi(y)(\mathrm{i}\gamma^\mu\,\overleftarrow{\partial}_\mu + m_e)\,u(p_-,s_-)\,\mathrm{e}^{-\mathrm{i}p_-y} && \text{incoming electron} \\ &\mathrm{i}\int \mathrm{d}^4y\,\mathrm{e}^{-\mathrm{i}p_+y}\,\bar v(p_+,s_+)\,(\mathrm{i}\gamma^\mu\,\overrightarrow{\partial}_\mu - m_e)\,\psi(y) && \text{incoming positron} \quad . \end{aligned}$$

The states $|e^+e^-\rangle$ and $|X\rangle$ are to be considered as products in the Hilbert space

$$\mathcal{H}_{\rm lep,\ had} = \mathcal{H}_{\rm lep} \otimes \mathcal{H}_{\rm had} \quad ,$$

i.e.,

$$\begin{aligned} |e^+e^-\rangle &= |e^+e^-\rangle_{\rm lep} \otimes |0\rangle_{\rm had} \\ |X\rangle &= |0\rangle_{\rm lep} \otimes |X\rangle_{\rm had} \quad . \end{aligned}$$

In detail:

$$\begin{aligned} & \langle X|J^\mu(x_1)J^\nu(x_2)|e^+e^-\rangle \\ = \;& \langle X|0\rangle_{\rm had}\langle 0|J^\mu_{\rm lep}(x_1)J^\nu_{\rm lep}(x_2)|e^+e^-\rangle_{\rm lep} + \langle X|J^\mu_{\rm had}(x_1)|0\rangle_{\rm had}\langle 0|J^\nu_{\rm lep}(x_2)|e^+e^-\rangle_{\rm lep} \\ + \;& \langle X|J^\nu_{\rm had}(x_2)|0\rangle_{\rm had}\langle 0|J^\mu_{\rm lep}(x_1)|e^+e^-\rangle_{\rm lep} + \langle X|J^\mu_{\rm had}(x_1)J^\nu_{\rm had}(x_2)|0\rangle_{\rm had}\langle 0|e^+e^-\rangle_{\rm lep} \\ = \;& 2\,\langle X|J^\mu_{\rm had}(x_1)0\rangle_{\rm had}\langle 0|J^\nu_{\rm lep}(x_2)|e^+e^-\rangle_{\rm lep} \quad , \end{aligned}$$

because, $\langle X|0\rangle_{\rm had} = 0$ as well as $\langle 0|e^+e^-\rangle_{\rm lep} = 0$ and by the symmetry $x_1,\mu \leftrightarrow x_2,\nu$. Furthermore, we use the Fourier representation of the photon propagator

$$D_{\mu\nu}(x_1 - x_2) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4q\,\mathrm{e}^{-\mathrm{i}\,q(x_1-x_2)}\,\tilde D_{\mu\nu}(q)$$

and translational invariance

$$J^\mu(x) = U(1,x)\,J^\mu(0)\,U^{-1}(1,x) = \mathrm{e}^{\mathrm{i}\,Px}\,J^\mu(0)\,\mathrm{e}^{-\mathrm{i}\,Px}$$

with P the total four momentum operator. Assuming that X is an eigenstate of momentum p_X we may use

$$\begin{aligned} \mathrm{e}^{\pm\mathrm{i}\,Px}|0\rangle &= |0\rangle \\ \mathrm{e}^{-\mathrm{i}\,Px}|X\rangle &= \mathrm{e}^{-\mathrm{i}\,p_Xx}|X\rangle \\ \mathrm{e}^{-\mathrm{i}\,Px}|e^+e^-\rangle &= \mathrm{e}^{-\mathrm{i}\,p_{e^+e^-}x}|e^+e^-\rangle \quad , \end{aligned}$$

such that altogether we have

$$\begin{aligned}\langle X|S|e^+e^-\rangle &= -e^2\frac{\mathrm{i}}{(2\pi)^4}\int \mathrm{d}^4q\,\tilde{D}_{\mu\nu}(q)\cdot\langle X|J^\mu_{\rm had}(0)|0\rangle\langle 0|J^\nu_{\rm lep}(0)|e^+e^-\rangle \\ &\qquad\cdot\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{-\mathrm{i}qx_1}\,\mathrm{e}^{\mathrm{i}qx_2}\,\mathrm{e}^{\mathrm{i}p_Xx_1}\,\mathrm{e}^{-\mathrm{i}p_{e^+e^-}x_2} \\ &= -\mathrm{i}\,e^2\,(2\pi)^4\,\delta^{(4)}(p_X-p_+-p_-)\,\tilde{D}_{\mu\nu}(q)\times\langle X|J^\mu_{\rm had}(0)|0\rangle\langle 0|J^\nu_{\rm lep}(0)|e^+e^-\rangle\ .\end{aligned}$$

We have used here that the x_i integrals over the exponentials yield a product of two delta functions

$$(2\pi)^4\,\delta^{(4)}(p_X-q)\ \times\ (2\pi)^4\,\delta^{(4)}(q-p_{e^+e^-})$$

which allows to perform the q integration trivially, leaving the overall four–momentum conservation and q is fixed to $q=p_X=p_{e^+e^-}$. Current conservation of the leptonic current $q_\nu J^\nu_{\rm lep}=0$ and the hadronic one $q_\mu J^\mu_{\rm had}=0$ implies that only the $g_{\mu\nu}$–part of the photon propagators contributes to the matrix element:

$$\tilde{D}_{\mu\nu}(q)\to -g_{\mu\nu}\,\frac{1}{q^2+\mathrm{i}\varepsilon}\ .$$

For the relevant T–matrix element, defined by $S=\mathbf{1}+\mathrm{i}\,(2\pi)^4\,\delta^{(4)}(\sum p_i)$ we find

$$T_{e^+e^-\to X}=\frac{e^2}{q^2}\,\langle X|J^\mu_{\rm had}(0)|0\rangle\langle 0|J_{\mu\,\rm lep}(0)|e^+e^-\rangle$$

where the leptonic matrix element is given at leading order by

$$\langle 0|J_{\mu\,\rm lep}(0)|e^+e^-\rangle=-\bar{v}(p_+,s_+)\,\gamma_\mu\,u(p_-,s_-)$$

and therefore

$$T(e^+e^-\to\gamma^*\to X)=-\frac{e^2}{q^2}\,\bar{v}(p_+,s_+)\,\gamma_\mu\,u(p_-,s_-)\ \cdot\ \langle X|J^\mu_{\rm had}(0)|0\rangle\ . \tag{8.4}$$

The total cross–section is

$$\sigma_h(s)=\sum_\Gamma\!\!\!\!\!\!\int \mathrm{d}\sigma(e^+e^-\to X,\ s=(p_++p_-)^2)$$

with the differential one given by (denoting $\mathrm{d}\mu(p)=\frac{\mathrm{d}^3p}{(2\pi)^3\,2\,\omega(p)}$, $\omega(p)=\sqrt{m^2+\vec{p}^{\,2}}$)

$$\mathrm{d}\sigma=\frac{(2\pi)^4\,\delta^{(4)}(p_++p_--p_X)}{2\sqrt{\lambda(s,m_e^2,m_e^2)}}\,|T|^2\,\mathrm{d}\mu(p_1')\cdots$$

For unpolarized e^+e^--beams, the square matrix element is given by the average over the initial spins of e^- and e^+ and can easily be calculated by standard techniques:

$$\overline{|T|^2}=\frac{1}{4}\sum_{s_\pm}|T|^2=\frac{e^4}{q^4}\,\ell_{\mu\nu}\,h^{\mu\nu} \tag{8.5}$$

where

$$\begin{aligned}\ell_{\mu\nu} &= \frac{1}{4}\sum_{s_\pm}\bar{v}(p_+,s_+)\,\gamma_\mu\,u(p_-,s_-)\,(\bar{v}(p_+,s_+)\,\gamma_\nu\,u(p_-,s_-))^* \\ &= \frac{1}{4}\,\mathrm{Tr}\,(\not{p}_+-m_e)\,\gamma_\mu\,(\not{p}_-+m_e)\,\gamma_\nu \\ &= \frac{1}{2}\left\{q_\mu q_\nu-q^2\,g_{\mu\nu}-(p_+-p_-)_\mu(p_+-p_-)_\nu\right\}\end{aligned}$$

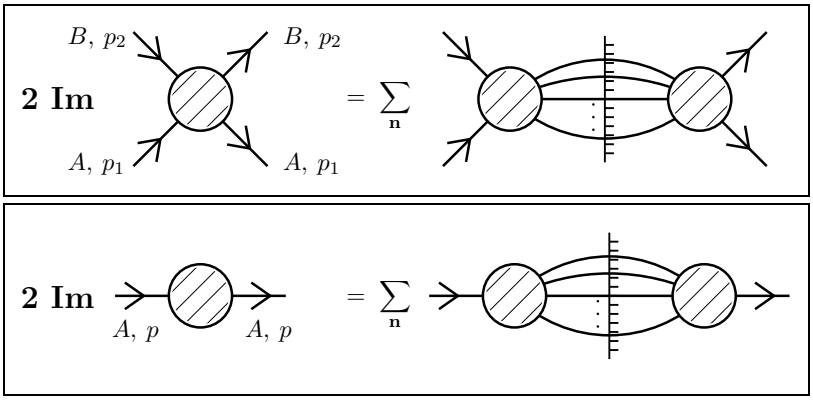


Figure 8.2: Optical theorem for scattering and propagation.

and

$$h^{\mu\nu} = \langle 0|J^\mu(0)|X\rangle\langle X|J^\nu(0)|0\rangle \ .$$

Writing $\sqrt{\lambda(s,m_e^2,m_e^2)} = s\,\sqrt{1-4m_e^2/s}$, we obtain

$$\sigma_h(s) = \frac{\alpha^2}{2s^3}\,\frac{16\pi^2}{\sqrt{1-4m_e^2/s}}\,\ell_{\mu\nu}\sum\!\!\!\!\!\!\int_X (2\pi)^4\,\delta^{(4)}(p_+ + p_- - p_X)\,h^{\mu\nu}$$

where summation/integration is over the hadronic final states. The leptonic tensor $\ell_{\mu\nu}$ describes the initial state and is fixed. For what concerns the hadronic part we try to be general before we will resort to pQCD which only applies at sufficiently high energy. At this point the optical theorem helps to proceed at the non-perturbative level.

Exploiting unitarity:

The optical theorem derives directly from the unitarity of the S–matrix. For T–matrix elements the unitarity condition reads

$$\mathrm{i}\left\{T_{if}^* - T_{fi}\right\} = \sum\!\!\!\!\!\!\int_n (2\pi)^4\,\delta^{(4)}(P_n - P_i)\,T_{nf}^* T_{ni}\ , \tag{8.6}$$

and the *optical theorem*, is obtained from this relation in the limit of elastic forward scattering $|f\rangle \to |i\rangle$ where

$$2\,\mathrm{Im}\,T_{ii} = \sum\!\!\!\!\!\!\int_n (2\pi)^4\,\delta^{(4)}(P_n - P_i)\,|T_{ni}|^2 \ . \tag{8.7}$$

Graphically, this relation may be represented as shown in Fig. 8.2. For the photon propagator it tells us that the imaginary part of the photon propagator is proportional to the total cross section $\sigma_{\rm tot}(e^+e^- \to \gamma^* \to \text{anything})$ ("anything" means any possible state):

$$\mathrm{Im}\Pi'_\gamma(s) = \frac{s}{4\pi\alpha}\,\sigma_{\rm tot}(e^+e^- \to \text{anything}) \ . \tag{8.8}$$

In first place, applying the optical theorem to e^+e^-–annihilation we have

$$2\,\mathrm{Im}\,T(e^+e^- \to e^+e^-)_{\rm forward} = (2\pi)^4 \sum_{QN's} \int \Pi_{i=1}^n\,\mathrm{d}\mu(p_i')\,\delta^{(4)}(P_n' - P_i)\ \cdot\ |T(p_1',\cdots,p_n'|p_+,p_-)|^2$$

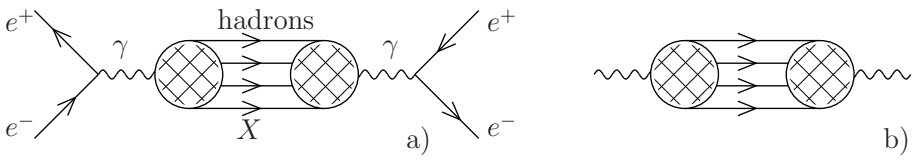


Figure 8.3: The optical theorem in e^+e^-–annihilation: a) including the QED part, b) the QCD part isolated.

which in the given one–photon exchange approximation has the structure shown in Fig. 8.3. The sum extends over spin and other possible quantum numbers (QN). The isolated QCD part, representing $(2\pi)^4\,\delta^{(4)}(p_+ + p_- - p_X)\,h^{\mu\nu}$, thus corresponds to the hadronic excitations (hadronic vacuum polarization) in the photon propagator. The hadronic part corresponds to the photon propagator after amputation of the external photons $A^\mu(x) \to J^\mu(x)$, where only the hadronic part $J^\mu_{\rm had}(x)$ is of interest here. The hadronic "blob" b) of Fig. 8.3 thus is given by the hadronic current correlator

$$\begin{aligned}\hat\Pi^{\mu\nu}(q) &= \mathrm{i}\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}\,qx}\langle 0|J^\mu_{\rm had}(x)J^\nu_{\rm had}(0)|0\rangle \\ &= -\left(g^{\mu\nu}\,q^2 - q^\mu q^\nu\right)\,\hat\Pi'_\gamma(q^2)\ . \end{aligned} \tag{8.9}$$

Note that $\hat\Pi'_\gamma(q^2)$ times e^2 is the usual photon self-energy. We have used transversality which is a consequence of the current conservation. The hadronic photon vacuum polarization thus is given by a single scalar amplitude $\Pi'_\gamma(q^2) = e^2\,\hat\Pi'_\gamma(q^2)$.

The physical thresholds of hadron production in $e^+e^- \to X$ are

$$X = \pi^+\pi^-, \rho, \pi^0\pi^+\pi^-, \omega, K^+K^-, K^0\overline{K^0}, \phi, \cdots$$

with lowest threshold at $s = 4\,m^2_{\pi^\pm}$[33]. The vector mesons ρ, ω and ϕ correspond to $\pi^+\pi^-$, $\pi^0\pi^+\pi^-$ and $K\bar K$ resonance peaks, respectively.

We now may continue our cross–section calculation using

$$(2\pi)^4\,\delta^{(4)}(p_+ + p_- - p_X)\,h^{\mu\nu} \;=\; 2\,\mathrm{Im}\,\hat\Pi^{\mu\nu}(q) = 2\,\left(q^\mu q^\nu - q^2\,g^{\mu\nu}\right)\,\mathrm{Im}\,\hat\Pi'_\gamma(q^2)\,,$$

such that

$$\sigma_h(s) = \frac{\alpha^2}{s^3}\,\frac{8\pi^2}{\sqrt{1-4m_e^2/s}}\,\left\{q_\mu q_\nu - q^2\,g_{\mu\nu} - (p_+ - p_-)_\mu(p_+ - p_-)_\nu\right\}\,\left(q^\mu q^\nu - q^2\,g^{\mu\nu}\right)\,\mathrm{Im}\,\hat\Pi'_\gamma(q^2)\ .$$

Contracting the tensors and working out the scalar products: with $q = p_+ + p_-$, $p_+^2 = m_e^2$, $p_-^2 = m_e^2$, $q\cdot(p_+ - p_-) = p_+^2 - p_-^2 = 0$, $q^2 = (p_+ + p_-)^2 = 2m_e^2 + 2p_+p_-$ or $2p_+p_- = q^2 - 2m_e^2$ and $(p_+ - p_-)^2 = 2m_e^2 - 2p_+p_- = 4m_e^2 - q^2$ we get

$$\ell_{\mu\nu}\cdot 2\,\left(q^\mu q^\nu - q^2\,g^{\mu\nu}\right) = 2\,q^2\left(1 + \frac{2m_e^2}{q^2}\right)$$

and hence

$$\begin{aligned}\sigma_h(s) &= \frac{\alpha^2}{s}\,\frac{16\pi^2}{\sqrt{1-\frac{4m_e^2}{s}}}\,\left(1+\frac{2m_e^2}{s}\right)\,\mathrm{Im}\,\hat\Pi'_\gamma(s) \\ &\simeq \frac{16\pi^2\alpha^2}{s}\,\mathrm{Im}\,\hat\Pi'_\gamma(s) \quad \text{since} \quad s \geq 4m_\pi^2 \gg m_e^2\ . \end{aligned}$$

[33] If we would include higher order electromagnetic interaction, like final state radiation (FSR) from hadrons, the lowest possible state exhibiting a hadron would be $\pi^0\gamma$ with threshold at $s = m^2_{\pi^0}$. In our counting this is a $O(\alpha^3)$ effect not included in our leading order consideration.

We may compare this result with the Born level muon–pair production cross–section, in the high energy approximation given by

$$\sigma(e^+e^- \to \mu^+\mu^-) \simeq \frac{4\pi\alpha^2}{3\,s} \ .$$

Exercise: Calculate the exact Born cross section for $e^+e^- \to \mu^+\mu^-$, by replacing the hadronic tensor $h_{\mu\nu}$ by the leptonic one $\ell_{\mu\nu}$ of the muon. What is the difference to $e^+e^- \to e^+e^-$?

It is common use to represent the hadronic production cross–section $\sigma_h(s)$ in units of the leptonic point cross section. This leads to the definition of the so called R–ratio function

$$\begin{aligned} R(s) &\doteq \frac{\sigma_{\rm tot}(e^+e^- \to \text{hadrons})}{\sigma_0(e^+e^- \to \mu^+\mu^-)} \\ &= \frac{\sigma_h(s)}{\left(\frac{4\pi\alpha^2}{3s}\right)} = \frac{16\pi^2\alpha^2}{4\pi\alpha^2} \cdot 3\,s\,\frac{1}{s}\,\mathrm{Im}\,\hat\Pi'_\gamma(s) \\ &= 12\pi \mathrm{Im}\,\hat\Pi'_\gamma(s) = 12\pi^2\,\rho_1(s) \end{aligned} \tag{8.10}$$

where $\rho_1(s) = \frac{1}{\pi}\,\mathrm{Im}\,\hat\Pi'_\gamma(s)$ has the meaning of a spectral density (see below).

As we will see below the function $R(s)$ has the asymptotic property

$$R(s) \to \ \text{constant} \ \ ; \ \ s \to \infty$$

which means that the imaginary part of the vacuum polarization function $\hat\Pi'_\gamma(s)$ tends to a non–vanishing constant at time–like infinity.

Exploiting analyticity:

Causality[34] together with unitarity imply analyticity $\hat\Pi'_\gamma(s)$ in the complex s–plane except for the cut along the positive real axis starting at $s \geq 4m^2$, where m is the lightest state which can be pair–produced. Cauchy's integral theorem tells us that the contour integral, for the contour $\mathcal{C}$ shown in Fig. 8.4, satisfies

$$F_i(s) = \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{C}} \frac{\mathrm{d}s' F(s')}{s'-s} \ .$$

We first note that by the causal iε–prescription of the time–ordered Green functions the imaginary parts of the analytic amplitudes change sign when $s \to s^*$. Therefore $F(s^*) = F^*(s)$, which is the Schwarz reflection principle. Since $F^*(s) = F(s^*)$ the contribution along the cut may be written as

$$\lim_{\varepsilon\to 0}\,(F(s+\mathrm{i}\varepsilon) - F(s-\mathrm{i}\varepsilon)) = 2\,\mathrm{i}\,\mathrm{Im}\,F(s)\ ; \ \ s \ \ \text{real}\ ,\ s>0 \tag{8.11}$$

and hence for $R \to \infty$

$$F(s) = \lim_{\varepsilon\to 0} F(s+\mathrm{i}\varepsilon) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int\limits_{4m^2}^{\infty} \mathrm{d}s'\,\frac{\mathrm{Im}\,F(s')}{s'-s-\mathrm{i}\varepsilon} + \mathcal{C}_\infty \ . \tag{8.12}$$

[34] Causality is implemented in a QFT by the iε prescription required by the time–ordered Green functions, which relate to the S–matrix elements via the LSZ formulae.

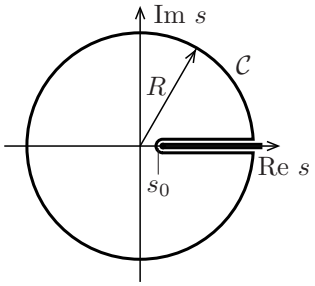


Figure 8.4: Analyticity domain and Cauchy contour $\mathcal{C}$ for the lepton and quark form–factors (vacuum polarization). $\mathcal{C}$ is a circle of radius R with a cut along the positive real axis for $s > s_0 = 4m^2$ where m is the mass of the lightest particles which can be pair–produced.

In all cases where $F(s)$ falls off sufficiently rapidly as $|s| \to \infty$ the boundary term $\mathcal{C}_\infty$ vanishes and the integral converges. This may be checked order by order in perturbation theory. In this case the "un–subtracted" dispersion relation (DR)

$$F(s) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int\limits_{4m^2}^{\infty} \mathrm{d}s' \, \frac{\mathrm{Im}\, F(s')}{s' - s - \mathrm{i}\varepsilon}$$

uniquely determines the function by its imaginary part. A technique based on DRs is frequently used for the calculation of Feynman integrals, because the calculation of the imaginary part is simpler in general. The real part which actually is the object to be calculated is given by the principal value ($\mathcal{P}$) integral

$$\mathrm{Re}\, F(s) = \frac{1}{\pi} \mathcal{P\!\!\!\!\!\!\int}\limits_{4m^2}^{\infty} \mathrm{d}s' \, \frac{\mathrm{Im}\, F(s')}{s' - s} \ ,$$

which is also known under the name Hilbert transform.

In summary: analyticity implies the validity of a dispersion relation (DR). Using the unitarity properties, this DR determines the real part of the vacuum polarization amplitude in terms of its imaginary part. In fact the electromagnetic current correlator exhibits a logarithmic UV singularity and thus requires one subtraction such in place of the unsubtracted DR

$$\hat{\Pi}'_\gamma(q^2) = \frac{1}{\pi} \int\limits_0^{\infty} \mathrm{d}s \, \frac{\mathrm{Im}\, \hat{\Pi}'_\gamma(s)}{s - q^2 - \mathrm{i}\varepsilon} \ , \tag{8.13}$$

we find

$$\hat{\Pi}'_\gamma(q^2) - \hat{\Pi}'_\gamma(0) = \frac{q^2}{\pi} \int\limits_0^{\infty} \mathrm{d}s \, \frac{\mathrm{Im}\, \hat{\Pi}'_\gamma(s)}{s\,(s - q^2 - \mathrm{i}\varepsilon)} \ . \tag{8.14}$$

In a renormalizable QFT UV divergencies in physical amplitudes are at most quadratic. Therefore at most two subtractions (mass– and wavefunction–renormalization, or vertex renormalization) are needed to get any DR finite. In low energy effective theories the number of subtractions needed corresponds to the number of low energy constants at the given order of the expansion (powers in p/Λ; p process specific momentum, Λ, typically Λ_{QCD} or M_p the proton mass).

Exercise: Verify that the subtraction changes the DR of the form (8.13), assumed to converge or supplemented with a cut–off, to (8.14), which we call a subtracted DR.

Physical interpretation (and an application):

As we know the full photon propagator is given by the geometrical progression of self–energy insertions $-\mathrm{i}\Pi_\gamma(q^2)$. The corresponding Dyson summation implies that the free propagator is replaced by the dressed one

$$\mathrm{i}D_\gamma^{\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2+\mathrm{i}\varepsilon} \to \mathrm{i}D_\gamma^{\prime\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2+\Pi_\gamma(q^2)+\mathrm{i}\varepsilon}$$

modulo unphysical gauge dependent terms. By $U(1)_{\rm em}$ gauge invariance the photon remains massless and hence we have $\Pi_\gamma(q^2) = \Pi_\gamma(0) + q^2\,\Pi'_\gamma(q^2)$ with $\Pi_\gamma(0) \equiv 0$. As a result we obtain

$$\mathrm{i}D_\gamma^{\prime\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}}{q^2\,(1+\Pi'_\gamma(q^2))} + \text{ gauge terms}$$

where the "gauge terms" will not contribute to gauge invariant physical quantities, and need not be considered further.

Including a factor e^2 and considering the renormalized propagator (wave function renormalization factor Z_γ) we have

$$\mathrm{i}\,e^2\,D_\gamma^{\prime\mu\nu}(q) = \frac{-\mathrm{i}g^{\mu\nu}\,e^2\,Z_\gamma}{q^2\,\left(1+\Pi'_\gamma(q^2)\right)} + \text{ gauge terms}$$

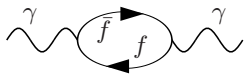
which in effect means that the charge has to be replaced by a **running charge**

$$e^2 \to e^2(q^2) = \frac{e^2 Z_\gamma}{1+\Pi'_\gamma(q^2)} \quad .$$

The wave function renormalization factor Z_γ is fixed by the condition that at $q^2 \to 0$ one obtains the classical charge (charge renormalization in the classical *Thomson limit*). Thus the renormalized charge is

$$e^2 \to e^2(q^2) = \frac{e^2}{1+(\Pi'_\gamma(q^2)-\Pi'_\gamma(0))} \tag{8.15}$$

where the lowest order diagram in perturbation theory which contributes to $\Pi'_\gamma(q^2)$ is



and describes the virtual creation and re–absorption of fermion pairs $\gamma^* \to e^+e^-, \mu^+\mu^-, \tau^+\tau^-, u\bar{u}, d\bar{d}, \cdots\to\ \gamma^*$.

In terms of the fine structure constant $\alpha = \frac{e^2}{4\pi}$ Eq. (8.15) reads

$$\alpha(q^2) = \frac{\alpha}{1-\Delta\alpha} \quad ; \quad \Delta\alpha = -\mathrm{Re}\,\left(\Pi'_\gamma(q^2)-\Pi'_\gamma(0)\right) \quad .$$

The various contributions to the shift in the fine structure constant come from the leptons (lep = e, μ and τ) the 5 light quarks (u, d, s, c, and b and the corresponding hadrons = had) and from the top quark:

$$\Delta\alpha = \Delta\alpha_{\rm lep} + \Delta^{(5)}\alpha_{\rm had} + \Delta\alpha_{\rm top} + \cdots$$

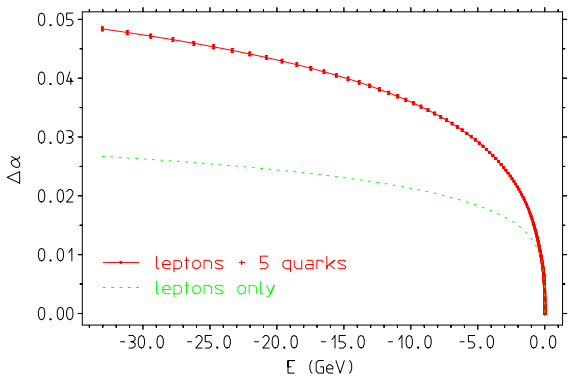


Figure 8.5: Shift of the effective fine structure constant $\Delta\alpha$ as a function of the energy scale in the space–like region $q^2 < 0$ ($E = -\sqrt{-q^2}$). In the sum "lepton+quarks" the band indicates the hadronic uncertainty.

Also W–pairs contribute at $q^2 > M_W^2$. While the other contributions can be calculated order by order in perturbation theory the hadronic contribution $\Delta^{(5)}\alpha_{\rm had}$ exhibits low energy strong interaction effects and hence cannot be calculated by perturbative means. Here the dispersion relations play a key role as discussed already.

The leptonic contributions are calculable in perturbation theory. Using our result Eq. (7.15) for the gluon self–energy, we easily obtain the photon self–energy by the replacements: $g \to eQ_\ell$, $C_2(G) = 0$ (photon is Abelian) and $T(R) = 1$ (QED is Abelian)[35]. As a result, the free lepton loops yield

$$\begin{aligned}
&\Delta\alpha_{\rm lep}(q^2) = \\
&\quad = \sum_{\ell=e,\mu,\tau} \tfrac{\alpha}{3\pi}\left[-\tfrac{5}{3} - y_\ell + (1+\tfrac{y_\ell}{2})\sqrt{1-y_\ell}\,\ln\left(\left|\tfrac{\sqrt{1-y_\ell}+1}{\sqrt{1-y_\ell}-1}\right|\right)\right] \\
&\quad = \sum_{\ell=e,\mu,\tau} \tfrac{\alpha}{3\pi}\left[-\tfrac{8}{3} + \beta_\ell^2 + \tfrac{1}{2}\beta_\ell(3-\beta_\ell^2)\ln\left(\left|\tfrac{1+\beta_\ell}{1-\beta_\ell}\right|\right)\right] \\
&\quad = \sum_{\ell=e,\mu,\tau} \tfrac{\alpha}{3\pi}\left[\ln\left(|q^2|/m_\ell^2\right) - \tfrac{5}{3} + O\left(m_\ell^2/q^2\right)\right] \;\text{ for } |q^2| \gg m_\ell^2 \\
&\quad \simeq \; 0.03142 \text{ for } q^2 = M_Z^2
\end{aligned}$$

where $y_\ell = 4m_\ell^2/q^2$ and $\beta_\ell = \sqrt{1-y_\ell}$ are the lepton velocities. This leading contribution is affected by small electromagnetic corrections only in the next to leading order. The leptonic contribution is actually known to three loops [4, 5] at which it takes the value

$$\Delta\alpha_{\rm leptons}(M_Z^2) \;\simeq\; 314.98 \;\times\; 10^{-4}.$$

As already mentioned, in contrast, the corresponding free quark loop contribution gets substantially modified by low energy strong interaction effects, which cannot be calculated reliably by perturbative QCD. The hadronic contribution to the shift in the fine structure constant one has

[35] The final result for the renormalized photon vacuum polarization for a lepton of mass m then reads

$$\Pi'_{\gamma\,\rm ren}(q^2) = \frac{\alpha}{3\pi}\left\{\frac{5}{3} + y - 2\,(1+\frac{y}{2})\,(1-y)\,G(y)\right\}$$

with $y = 4m^2/q^2$ and $G(y) = \frac{1}{2\sqrt{1-y}}\,\ln\frac{\sqrt{1-y}+1}{\sqrt{1-y}-1}$.

to evaluate by means of the DR

$$\Delta\alpha_{\rm had}(q^2) = -{\rm Re}\,\Pi'_\gamma(q^2) + \Pi'_\gamma(0) = -\frac{e^2\,q^2}{12\pi^2}\,\mathop{\rlap{\rm P}\int}_{4m_\pi^2}^{\infty}\,{\rm d}s\,\frac{R(s)}{s\,(s-q^2-{\rm i}\varepsilon)}\ ,$$

using the experimental $e^+e^- \to$ hadorns data in the non-perturbative regions. Within experimental errors the hadronic shift is uniquely determined by $\sigma_h(s)$, which has been measured directly up to about 40 GeV. At higher energies large $\gamma - Z$ mixing effects come into play. In regions where pQCD applies, in particular for the experimentally not explored high energy tail, $R(s)$ can be safely calculated in pQCD.

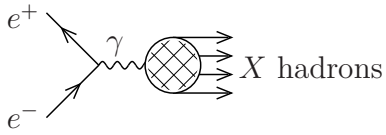
Vacuum polarization effects are large when large scale changes are involved (large logarithms) and because of the large number of light fermionic degrees of freedom as we infer from the asymptotic form in perturbation theory

$$\Delta\alpha^{\rm pert}(q^2) \;\simeq\; \frac{\alpha}{3\pi}\sum_f Q_f^2 N_{cf}\left(\ln\frac{|q^2|}{m_f^2} - \frac{5}{3}\right)\ \ ;\ \ |q^2| \gg m_f^2\ .$$

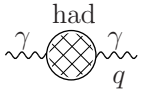
Fig. 8.5 illustrates the running of the effective charges at lower energies in the space–like region. Typical values are $\Delta\alpha(5{\rm GeV}) \sim 3\%$ and $\Delta\alpha(M_Z) \sim 6\%$, where about $\sim 50\%$ of the contribution comes from leptons and about $\sim 50\%$ from hadrons. Note the sharp increase of the screening correction in Fig. 8.5 which implies large corrections at relatively low energies.

Summary: In the one photon exchange approximation, i.e. at $O(\alpha^2)$, the total hadronic production cross–section in electron–positron annihilation

$\sigma_h(e^+e^- \to \gamma^* \to {\rm hadrons})$:



determines the imaginary part of the photon vacuum polarization amplitude



$$:\quad \hat\Pi^{\mu\nu}(q) = \left(q^\mu q^\nu - q^2\,g^{\mu\nu}\right)\,\hat\Pi'_{\gamma\,{\rm had}}(q^2)$$

$$\sigma_h = \frac{16\pi^2\alpha^2}{s}\,{\rm Im}\,\hat\Pi'_{\gamma\,{\rm had}}\ .$$

Usually one represents the $e^+e^- \to$ hadrons data in terms of

$$R(s) = \frac{\sigma_h(s)}{\sigma_0(e^+e^- \to \mu^+\mu^-)} = 12\,\pi\,{\rm Im}\,\hat\Pi'_{\gamma\,{\rm had}}\ .$$

The hadronic shift of the effective fine structure constant at scale M is given by

$$\Delta\alpha_{\rm had}(M^2) = -\frac{\alpha\,M^2}{3\pi}\,{\rm Re}\int_{4m_\pi^2}^{\infty}\,\frac{{\rm d}s\,R(s)^{e^+e^-\to\gamma^*\to{\rm had}}}{s\,(s-M^2-{\rm i}\varepsilon)}\ .$$

8.12.1 Current correlators and the Källen-Lehmann spectral representation

We have seen that dispersion relations play a key role in the attempt to understand relationships between physical quantities in a non–perturbative framework. The Källen-Lehmann spectral representation sheds more light on how general QFT properties can be exploited in this context. We therefore present a short digression on the subject here.

We consider a current correlator[36]

$$\langle 0|J^\mu(x)\,J^\nu(0)|0\rangle = \sum_\alpha \int \mathrm{d}^4p\,\langle 0|J^\mu(x)|p,\alpha\rangle\langle p,\alpha|J^\nu(0)|0\rangle$$

and have applied the completeness relation

$$\sum_\alpha \int \mathrm{d}^4p|p,\alpha\rangle\langle p,\alpha| = \mathbf{1}\ ,$$

where p is the total four–momentum and α denotes an appropriate set of quantum numbers which characterize the state $|p,\alpha\rangle$. In addition, as usual, we use translational invariance $J^\mu(x) = \mathrm{e}^{\mathrm{i}\,Px}\,J^\mu(0)\,\mathrm{e}^{-\mathrm{i}\,Px}$ together with the fact that $|0\rangle$ and $|p,\alpha\rangle$ are eigenstates of the total energy-momentum operator P in order to simplify the two–point function.

Next we define the spectral function tensor

$$\tilde\rho^{\mu\nu} \doteq (2\pi)^3 \sum_\alpha \langle 0|J^\mu(0)|p,\alpha\rangle\langle p,\alpha|J^\nu(0)|0\rangle$$

which has the following properties:
1) Relativistic covariance implies

$$\tilde\rho^{\mu\nu} = \hat\rho_1(p^2)\,g^{\mu\nu} + \hat\rho_2(p^2)\,p^\mu p^\nu\ .$$

2) The spectral condition (physical state must have $p^0 \geq 0$ and $p^2 \geq 0$ and, unless the state is the vacuum, $p = (p^0,\vec p) \neq (0,\vec 0)$) requires

$$\tilde\rho^{\mu\nu} = \Theta(p^0)\,\Theta(p^2)\,\cdots = \Theta(p^0)\int_0^\infty \mathrm{d}s\,\delta(s-p^2)\,\cdots$$

which implies

$$\begin{aligned}\tilde\rho^{\mu\nu} &= \Theta(p^0)\int_0^\infty \mathrm{d}s\,\delta(s-p^2)\,\{\hat\rho_1(s)\,g^{\mu\nu} + \hat\rho_2(s)\,p^\mu p^\nu\}\\ &= \int_0^\infty \mathrm{d}s\,\Theta(p^0)\,\delta(p^2-s)\,\left\{\left(p^\mu p^\nu - p^2\,g^{\mu\nu}\right)\,\rho_1(s) + p^\mu p^\nu\,\rho_0(s)\right\}\ ,\end{aligned}$$

with

$$\rho_1 = -\frac{\hat\rho_1(s)}{s}\ ;\ \ \rho_0 = \hat\rho_2(s) + \frac{\hat\rho_1(s)}{s}\ .$$

The scalar functions $\rho_1(s)$ and $\rho_0(s)$ denote the transversal and the longitudinal spectral functions, respectively.
3) Current conservation (in case it applies)

$$p_\mu\,\tilde\rho^{\mu\nu} = 0 \quad\bowtie\quad \rho_0(s) \equiv 0\ .$$

[36]Specifically, the object we consider here is called the positive frequency part of the current correlator function.

We furthermore can prove the positivity of the spectral function $\rho_1(s)$:

$$\begin{aligned}
\tilde{\rho}^{\mu\nu} &= (2\pi)^3 \sum_\alpha \langle 0|J^\mu(0)|p,\alpha\rangle\langle p,\alpha|J^\nu(0)|0\rangle \\
&= \Theta(p^0) \int_0^\infty \mathrm{d}s\, \delta(p^2-s)\, \left(p^\mu p^\nu - p^2\, g^{\mu\nu}\right)\, \rho_1(s) \\
\tilde{\rho}^{00} &= (2\pi)^3 \sum_\alpha \langle 0|J^0(0)|p,\alpha\rangle\langle p,\alpha|J^0(0)|0\rangle \\
&= (2\pi)^3 \sum_\alpha \left|\langle 0|J^0(0)|p,\alpha\rangle\right|^2 \geq 0 \\
&= \Theta(p^0) \int_0^\infty \mathrm{d}s\, \delta(p^2-s)\, \vec{p}^{\,2}\, \rho_1(s) = \Theta(p^0)\,\Theta(p^2)\,\vec{p}^{\,2}\,\rho_1(p^2)
\end{aligned}$$

This implies $\rho_1(p^2) \geq 0$ and real.

Exercise: Discuss the positivity properties of the functional $\dot{P}(s)$ defined as the derivative of

$$P(s) = \fint_{s_0}^\infty \mathrm{d}s'\, \frac{R(s')}{s'(s'-s)}\ ,$$

where $R(s)$ is a positive ordinary function. As a distribution (or linear functional) $P(s)$ has a derivative. Formally,

$$\dot{P}(s) = \frac{\mathrm{d}P(s)}{\mathrm{d}s} = \fint \mathrm{d}s'\, \frac{R(s')}{(s'-s)^2}\ .$$

With these general properties we may write

$$\begin{aligned}
&\langle 0|J^\mu(x)J^\nu(0)|0\rangle = \int \mathrm{d}^4p\, \mathrm{e}^{-\mathrm{i}\,px}\, \frac{1}{(2\pi)^3}\, \tilde{\rho}^{\mu\nu}(p) \\
&= \int_0^\infty \mathrm{d}s\, \{-g^{\mu\nu}\,\rho_1(s) + \partial^\mu\partial^\nu\, (\rho_1(s)+\rho_0(s))\,\} \times \underbrace{\frac{1}{(2\pi)^3}\int \mathrm{d}^4p\, \mathrm{e}^{-\mathrm{i}\,px}\,\Theta(p^0)\,\delta(p^2-s)}_{+\mathrm{i}\,\Delta^+(x;s)} \\
&= -\mathrm{i}\int \mathrm{d}s\, \left\{(\partial^\mu\partial^\nu - g^{\mu\nu})\, \Delta^+(x;s)\,\rho_1(s) + \partial^\mu\partial^\nu\,\Delta^+(x;s)\,\rho_0(s)\right\}\ ,
\end{aligned}$$

the Källen-Lehmann representation for the positive frequency correlator in coordinate space. Note that $\mathrm{i}\,\Delta^+(x;m^2)$ is the positive frequency part of a scalar free field propagator of mass m. The current commutator function $\langle 0|\,[J^\mu(x)J^\nu(0)]\,|0\rangle$ and the time–ordered Green function $\langle 0|T\,\{J^\mu(x)J^\nu(0)\}\,|0\rangle$ are then given by the same representation (same spectral functions) with the replacements $\Delta^+(x;s) \to \Delta(x;s)$, the free scalar commutator function, and $\Delta^+(x;s) \to \Delta_F(x;s)$, the free scalar Feynman propagator, respectively.

The application to the hadronic vacuum polarization is the following: the Källen-Lehmann representation is given by

$$\begin{aligned}
\hat{\Pi}^{\mu\nu}(q) &= \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}\,qx}\, \langle 0|T\,\{J^\mu(x)J^\nu(0)\}\,|0\rangle \\
&= -\int_0^\infty \mathrm{d}s\, \left(q^\mu q^\nu - q^2\, g^{\mu\nu}\right)\, \rho_1(s)\, \frac{1}{q^2-s+\mathrm{i}\varepsilon}\ .
\end{aligned}$$

Using

$$\frac{1}{q^2-s+\mathrm{i}\varepsilon} = \mathcal{P}\frac{1}{q^2-s} - \mathrm{i}\,\pi\,\delta(q^2-s)\ ,$$

where $\mathcal{P}$ denotes the principle value prescription, we obtain

$$2\,\mathrm{Im}\,\hat{\Pi}^{\mu\nu}(q) = 2\pi \int_0^\infty \mathrm{d}s\, \left(q^\mu q^\nu - q^2\, g^{\mu\nu}\right)\, \rho_1(s)\, \delta(q^2 - s)\ .$$

This means

$$\frac{1}{\pi}\,\mathrm{Im}\,\hat{\Pi}^{\mu\nu}(q) = \left(q^\mu q^\nu - q^2\, g^{\mu\nu}\right)\, \rho_1(q^2)$$

or

$$\mathrm{Im}\,\hat{\Pi}'_\gamma(q^2) = \pi\, \rho_1(q^2)\ .$$

A last remark here: the positivity of $\rho_1(s)$ discussed above implies that $\hat{\Pi}'_\gamma(s)$ is a real analytic function with the property: for real $q^2 < 0$ we get $\mathrm{Im}\,\hat{\Pi}'_\gamma(s) = 0$, for real $q^2 < 4m^2$ the amplitude $\hat{\Pi}'_\gamma(q^2)$ is real $\hat{\Pi}'_\gamma(q^2) = \hat{\Pi}'^*_\gamma(q^2)$ and more generally: $\hat{\Pi}'^*(q^2) = \hat{\Pi}'_\gamma(q^{2*})$. This is an alternative derivation of the properties usually inferred using T–matrix element unitarity relations including the optical theorem.

8.13 High energy hadron-production in e^+e^-–annihilation

At sufficiently high energy $E = \sqrt{s}$ we expect to be able to perform a perturbative expansion in the effective strong coupling constant $\alpha_s(E)$, as, due to *asymptotic freedom*, the latter decreases with increasing energy. The photon at short distances directly couples to the quarks $q = u, d, s, c, b, t$ and by applying perturbation theory we actually calculate the production of colored $q\bar{q}$–pairs, which we know do not exist as free asymptotic states. Since only hadrons like pions, Kaons, vector mesons etc are produced in reality, what we calculate in pQCD is therefore not precisely what we would like to know. For appropriate observables, like the total inclusive cross–section, it is expected or at least hopped that the hadronization of the quarks into hadrons does not affect the quark production cross section too much. Since what primary is produced in any case is a virtual quark–antiquark pair (up to higher orders in α) which has to hadronize in any case, we expect the quark production rate should rather well account for what we get when we sum over any type of hadron state observed. However, we know that the existence of sharp and pretty huge resonance excitations (see later) tells us that the hoped for "quark–hadron duality" cannot be local (i.e. hold bin by bin in the energy), because the quark production cross–section is a smooth non-resonant function up to the known threshold "steps". In any case *quark hadron duality*[37] [7, 8] is the bridge between pQCD based theory and the QCD state space explored in experiments. The precise formulation states that for sufficiently large s the average non–perturbative hadron cross–section equals the perturbative quark cross–section:

$$\overline{\sigma(e^+e^- \to \mathrm{hadrons})}(s) \simeq \sum\nolimits_q \sigma(e^+e^- \to q\bar{q})(s)\ , \tag{8.16}$$

where the averaging extends from threshold up to the given s value which must lie far enough above a threshold (global duality). Approximately, such duality relations then would hold for energy intervals which start just below the last threshold passed up to s. Qualitatively, such a behavior is clearly visible in the data. However, for precise reliable predictions it has not yet been possible to quantify the accuracy of the duality conjecture. A quantitative check would require much more precise cross–section measurements than the ones available today. As we will

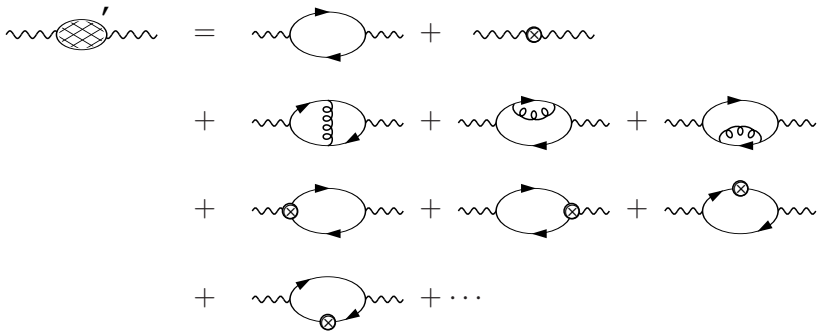
[37] Quark–hadron duality was first observed phenomenologically for the structure function in deep inelastic electron–proton scattering [6].

see quark–hadron duality is obviously there, however, at present, it is not known to what extend it can safely be used as a tool for precision physics.

In the following e will try to quantify these ideas. We first are going to discuss the calculations of $R(s)$ in perturbative QCD. This provides one of the most important testing grounds for pQCD. Practically, what we have to do is to calculate the hadronic current correlators in perturbation theory as a power series in α_s/π. According to the general analysis presented above, the object of interest is

$$R(s) = 12\,\pi\,\mathrm{Im}\,\hat{\Pi}'_{\gamma\,\mathrm{had}}(s) \;\; ; \quad \hat{\Pi}_{\gamma\,\mathrm{had}}(q): \quad , \tag{8.17}$$

where the "blob" is the hadronic one, in pQCD represented by valence quark loops dressed in all possible ways by gluons and possible sea quark loops. The leading plus next to leading QCD perturbation expansion diagrammatically is given by



(8.18)

Lines ~~~ show external photons, ⟶ propagating quarks/antiquarks and ~~~ propagating gluons. See Fig. (5.1) for the Feynman rules of QCD. The vertices ⊗ are marking renormalization counter term insertions. They correspond to subtraction terms which render the divergent integrals finite.

In QED (the above diagrams with gluons replaced by photons) the phenomenon of vacuum polarization was discussed first by Dirac [9] and finalized at the one–loop level by Schwinger [10] and Feynman [11]. Soon later Jost and Luttinger [12] presented the first two–loop calculation.

In 0th order in the strong coupling α_s we have

$$2\,\mathrm{Im} \quad = \left| \quad \right|^2 \tag{8.19}$$

which is proportional to the the free quark–antiquark production cross–section [13]. This approximation corresponds to the so called *Quark Parton Model* (QPM), describing free quarks with the strong interaction turned off. The QPM is expected to provide a good approximation in the high energy limit of QCD. The leading order result can be obtained in two ways from earlier calculations. In Sect. 7.2 we calculated the gluon self-energy. From the latter the photon self-energy may be obtained by taking into account only the last of the four diagrams, the one with the quark loop, now with $T(R)$ replaced by unity: $T(R) \to 1$ and $g \to eQ_q$ with Q_q the charge of the quark in units of the positron charge e. In Eq. (7.15) we replace $g \to eQ_q$, $C_2(G) = 0$ (photon is Abelian) and $T(R) = 1$ (QED is Abelian). Furthermore we need the imaginary part only, which is finite such that $f(d) = 4$, in any case. The self-energy amplitude $\Pi(k^2)$ of (7.15) corresponds to $e^2\,\hat{\Pi}'_{\gamma\,\mathrm{had}}(q^2)$ considered here, since $\hat{\Pi}'_{\gamma\,\mathrm{had}}$ is defined as a current correlator without coupling factor e from the QED–vertex $e\,J^\mu A_\mu$. Also note that by definition the common factor e^2 cancels in the cross section ratio $R(s)$ according to the definition (8.10). In addition we have to sum over the color of the quarks which yields a factor $N_c = 3$ and the quark flavors denoted by f in the

following. We then obtain

$$\mathrm{Im}\,\hat{\Pi}'_{\gamma\,\mathrm{had}}(q^2) = N_c \sum_f \frac{Q_f^2}{12\pi^2}\,\left(1+2m_f^2/q^2\right)\,\mathrm{Im}\,B_0(m_f,m_f;q^2)$$

and the imaginary part of B_0 is given by Eq. (6.41): $\mathrm{Im}\,B_0 = \pi\sqrt{1-4m_f^2/q^2}$. The mass dependent threshold factor is a function of the center of mass quark velocity

$$v_f = \left(1-\frac{4m_f^2}{s}\right)^{1/2}$$

and, using (8.17), we may write

$$\begin{aligned} R(s) &= N_c \sum_f \frac{Q_f^2}{12\pi^2}\,12\pi^2\,\left(1+2m_f^2/s\right)\,\sqrt{1-4m_f^2/s} \quad [s \geq 4m_f^2] \\ &= N_c \sum_f Q_f^2\,\frac{v_f}{2}\,\left(3-v_f^2\right)\,\Theta(s-4m_f^2)\ . \end{aligned}$$

The other possibility to get this result is to calculate the Born cross section for the production of a quark-antiquark pair $\sigma(e^+e^- \to \bar{q}q)$, directly, by taking free quarks $X = \bar{q}q$ in the matrix element (8.4). We then have

$$\langle X|J^\mu_{\mathrm{had}}(0)|0\rangle = Q_q\,\bar{u}_q(p_q,s_q)\,\gamma^\mu\,v_q(p_{\bar{q}},s_{\bar{q}}) \tag{8.20}$$

and $h^{\mu\nu}$ in (8.5) is given by an expression like $\ell_{\mu\nu}$, with q unchanged (photon momentum) and $p_+ \to p_{\bar{q}}$ and $p_- \to p_q$ as outgoing momenta. Up to different charge and masses and a color summation, the cross section $\sigma(e^+e^- \to \bar{q}q)$ agrees formally with the lepto–production cross section $\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}\,\frac{v_\mu}{2}\,(3-v_\mu^2)$ with v_μ the muon velocity (corresponding to $R_\mu(s) = \frac{v_\mu}{2}\,(3-v_\mu^2)$ with $R_\mu(s) \to 1$ as $s \gg m_\mu^2$).

A first assessment of the lowest order result, which is $O(\alpha_s^0)$, i.e. strong interaction switched off! is in order here. The result in any case should make sense at high enough energy. What is high enough, actually was the big question at the time when QCD became the candidate strong interaction theory. Besides $\alpha_s \ll 1$ we also require $4m_f^2/s \ll 1$ to be away from resonances which usually show up in the threshold regions. We then are dealing with the quark–parton model which, with $N_c = 3$, $Q_{u,c,t} = \frac{2}{3}$ and $Q_{d,s,b} = -\frac{1}{3}$, predicts the following constant R–values:

N_f	3	4	5	6
quarks	*uds*	*udsc*	*udscb*	*udscbt*
R	2	$3\,\frac{1}{2}$	$3\,\frac{2}{3}$	5
range	1.8 - 3.73 GeV	4.8 - 10.52 GeV	11.20 - ($2\,m_t$-10.0) GeV	($2\,m_t$+10.0) GeV - ∞

The ranges refer to regions where the cross sections vary slowly, except for the lowest (usually 3) very narrow resonances, which appear the lie below the corresponding flavor thresholds. The gaps are regions of strongly varying data, where applying pQCD makes little sense. In the "perturbative regions" the lowest order pattern (the step wise increase) may clearly be identified in the data as seen in Fig. 8.6. Most remarkably, the color factor 3 is unambiguously confirmed by the data (it resolved the unitarity crisis of 1974! at the end). Quark-hadron duality seems

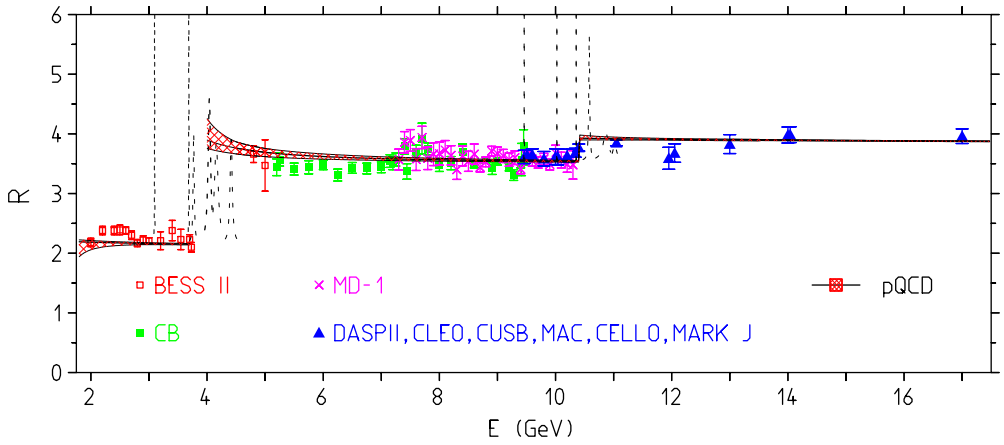


Figure 8.6: Test of pQCD prediction of R with some more recent data in the non-resonant regions. The spikes show the sharp J/ψ ($\bar{c}c$) and Υ ($\bar{b}b$) resonances.

well at work. QCD corrections are fairly small, which means that the effective strong coupling constant is relatively small. We consider these corrections next.

Including the presently known higher order terms the perturbative result for $R(s)$ is given by [14–16]

$$\begin{aligned} R(s)^{\rm pert} &= N_c \sum_f Q_f^2 \, \frac{v_f}{2} \, \left(3 - v_f^2\right) \, \Theta(s - 4m_f^2) \\ &\quad \times \left\{1 + a c_1(v_f) + a^2 c_2 + a^3 c_3 + a^4 c_4 + \cdots\right\} \end{aligned} \tag{8.21}$$

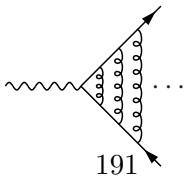
where $a = \alpha_s(s)/\pi$ and, assuming $4m_f^2 \ll s$, i.e. in the massless approximation

$$\begin{aligned} c_1 &= 1 \\ c_2 &= C_2(R) \left\{-\frac{3}{32} C_2(R) - \frac{3}{4}\beta_0 \zeta(3) - \frac{33}{48} N_f + \frac{123}{32} N_c\right\} \\ &= \frac{365}{24} - \frac{11}{12} N_f - \beta_0 \zeta(3) \simeq 1.9857 - 0.1153 N_f \\ c_3 &= -6.6368 - 1.2002 N_f - 0.0052 N_f^2 - 1.2395 \, (\sum_f Q_f)^2/(3 \sum_f Q_f^2) \\ c_4 &= 135.8 - 34.4 \, N_f + 1.88 \, N_f^2 - 0.010 \, N_f^3 - \pi^2 \beta_0^2 \left(1.9857 - 0.1153 \, N_f + \frac{5\beta_1}{6\beta_0}\right) , \end{aligned}$$

with $\beta_0 = (11 - 2/3\, N_f)/4$, $\beta_1 = (102 - 38/3\, N_f)/16$. All results are in the $\overline{MS}$ scheme. $N_f = \sum_{f:4m_f^2 \le s} 1$ is the number of active flavors. There is a mass dependent threshold factor in front of the curly brackets and the exact mass dependence of the first correction term

$$c_1(v) = \frac{2\pi^2}{3v} - (3 + v)\left(\frac{\pi^2}{6} - \frac{1}{4}\right)$$

is singular (Coulomb singularity due to soft gluon final state interaction) at threshold. The singular terms of the n–gluon ladder diagrams



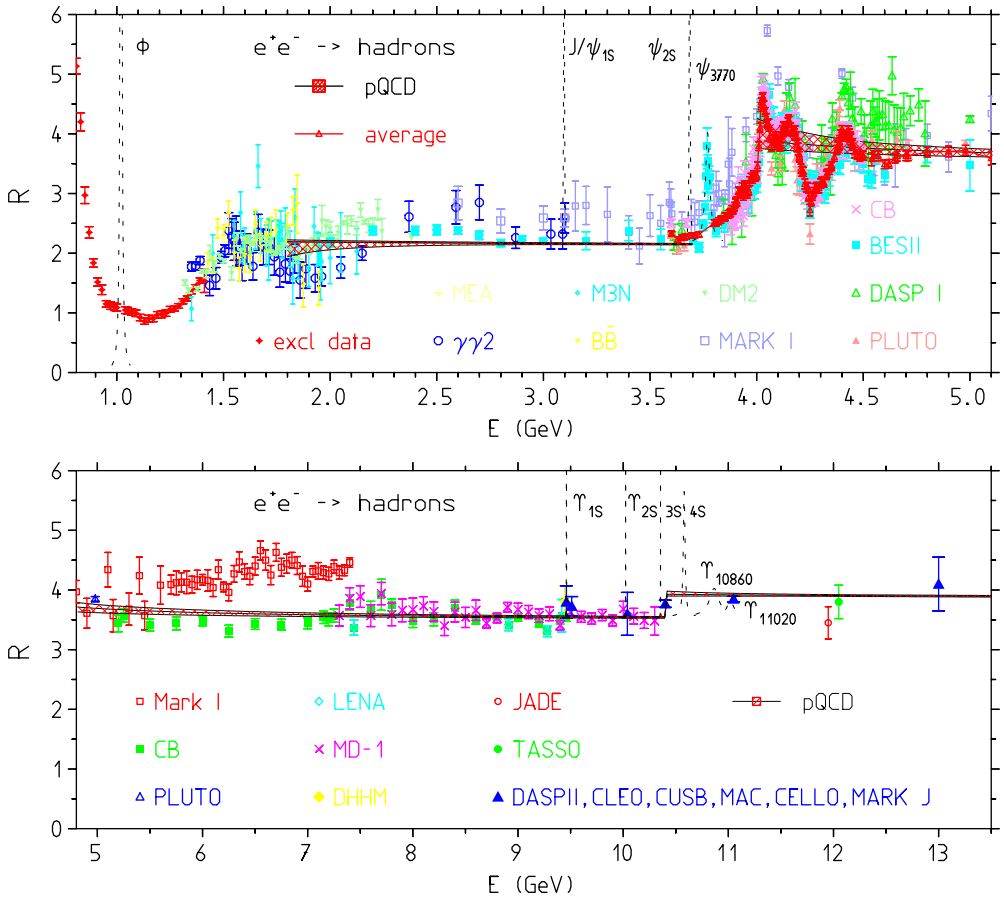


Figure 8.7: Compilation of a fairly complete collection of $e^+e^- \to$ hadrons data [from Ref. [18]] in comparison with up-to-date pQCD predictions for $R(s)$. Only statistical errors of data points are shown. The uncertainty band of the theory is due to the uncertainties in QCD parameters (mainly α_s).

exponentiate and thereby remove the singularity [17]:

$$1 + x \to \frac{2x}{1 - \mathrm{e}^{-2x}} \; ; \quad x = \frac{2\pi\alpha_s}{3v}$$

$$\left(1 + c_1(v)\,\frac{\alpha_s}{\pi} + \cdots\right) \;\to\; \left(1 + c_1(v)\,\frac{\alpha_s}{\pi} - \frac{2\pi\alpha_s}{3v}\right) \frac{4\pi\alpha_s}{3v}\,\frac{1}{1 - \exp\left\{-\frac{4\pi\alpha_s}{3v}\right\}} \; . \tag{8.22}$$

Since pQCD in any case should not be applied near threshold, the above remark is somewhat academic. In contrast to QED, where Coulomb interaction problems are there and have to be cured appropriately.

Mandatory is applying renormalization group improvement of pQCD calculations: the coupling α_s and the masses m_q have to be understood as running parameters

$$R\left(\frac{m_{0f}^2}{s_0}, \alpha_s(s_0)\right) = R\left(\frac{m_f^2(\mu^2)}{s}, \alpha_s(\mu^2)\right) \; ; \quad \mu = \sqrt{s} \; . \tag{8.23}$$

where $\sqrt{s_0}$ is a reference energy. RG resummation dramatically improves the convergence of perturbative approximations. Mass effects are important once one approaches a threshold from the

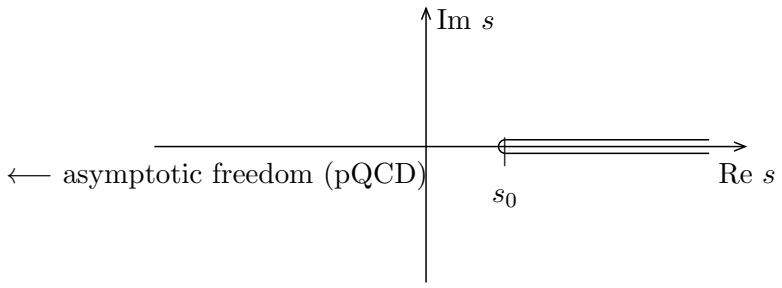


Figure 8.8: Analyticity domain for the photon vacuum polarization function. In the complex s–plane there is a cut along the positive real axis for $s > s_0 = 4m^2$ where m is the mass of the lightest particles which can be pair–produced.

perturbatively save region sufficiently far above the thresholds where mass effects may be safely neglected. They have been calculated up to three loops by Chetyrkin, Kühn and collaborators [19] and have been implemented in the FORTRAN routine `RHAD` by Harlander and Steinhauser [20].

In Fig. 8.7 we have collected the e^+e^-–annihilation data into hadrons and compare the data with state of the art pQCD calculations of $R(s)$ (for the experimental data [**?**, 21, 24–29]). The agreement between theory and experiment is indeed remarkable in view of the conceptual problems with the hadronization. It should be mentioned that statistical errors only are shown in the plot. Typically, pQCD fails at low energy and in quark threshold regions, where hadronic resonances show up. The latter usually are parametrized by Breit-Wigner resonance shapes (see below).

Where would we expect that we can trust the perturbative result? Perturbative QCD is supposed to work best in the deep Euclidean region[38] away from the physical region characterized by the cut in the analyticity plane Fig. 8.8. Fortunately, the physical region to a large extent is accessible to pQCD as well as we have seen, provided the energy scale is sufficiently large and one looks for the appropriate observable, which in our case is the inclusive total hadronic cross–section.

As we have shown earlier the total cross–section represented by the imaginary part of the vacuum polarization function $\hat{\Pi}(q^2)$ corresponds to the jump across the cut. On the cut we have the thresholds of the physical states, with lowest lying channels: $\pi^+\pi^-$, $\pi^0\pi^+\pi^-$, $\cdots$ and resonances ρ, ω, ϕ, $J/\psi\cdots$, $\Upsilon\cdots$, $\cdots$. QCD is confining the quarks (a final proof of confinement is yet missing) in hadrons. In any case the quarks *hadronize* (see Fig. 8.9), a highly non–perturbative phenomenon which is poorly understood in detail.

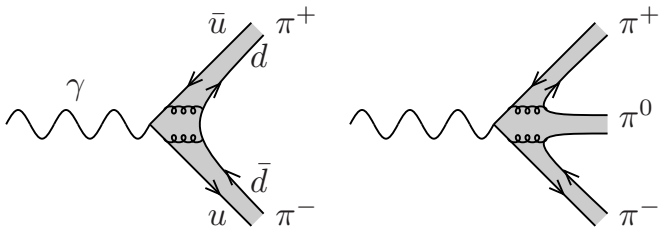


Figure 8.9: Hadron production in low energy e^+e^-–annihilation: the primarily created quarks must hadronize. The shaded zone indicates strong interactions via gluons which confine the quarks inside hadrons.

While at higher energies pQCD works sufficiently far away from thresholds and resonances, i.e. in

[38] This is directly accessible in Deep Inelastic electron–proton Scattering (DIS), which we will learn about later.

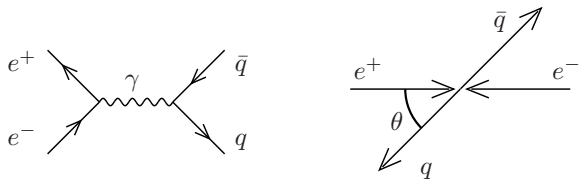


Figure 8.10: Fermion pair production in e^+e^-–annihilation. The lowest order Feynman diagram (left) and the same process in the c.m. frame (right). The arrows represent the spatial momentum vectors and θ is the production angle of the quark relative to the electron in the c.m. frame.

regions where $R(s)$ is a slowly varying monotonic function, neither the physical *thresholds* nor the *resonances* are obtained with perturbation theory! Not to talk about the pions as quasi-Goldstone modes of the spontaneous breakdown of chiral symmetry. We note that the perturbative quark–pair thresholds in (8.21) do not nearly approximate the physical thresholds for the low energy region below about 2 GeV, as we will see. Less problematic is the space–like (Euclidean) region $-q^2 \to \infty$, since it is away from thresholds and resonances. The best monitor for a comparison between theory and experiment has been proposed by Adler [30] long time ago: the so called *Adler–function*, up to a normalization factor, the derivative of the vacuum polarization function in the space–like region (see Sect. 8.14.1 below). In any case only the data presently can tell us to what extent we understand strong interaction physics and in many cases a semi–phenomenological approach is very successful.

Let us elaborate a little bit more about "parton physics" in e^+e^-–annihilation. At higher energies highly energetic *partons*, quarks and/or gluons, are produced and we learned how pQCD is able to give the correct prediction of $\sigma_{\rm tot}(e^+e^- \to \gamma^* \to \text{hadrons})$ in non–resonant regions, in the sense of quark–hadron duality (8.16). However, the consequences of the validity of pQCD are more far–reaching. According to perturbation theory the production of hadrons in e^+e^-–annihilation proceeds via the primary creation of a quark–antiquark pair which must hadronize as shown in Fig. 8.9, in the simplest cases. The elementary hard process tells us that in a high energy collision of positrons and electrons (in the center of mass frame) q and $\bar{q}$ are produced with high momentum in opposite directions (back–to–back), as illustrated in Fig. 8.10. The differential cross–section, up to a color factor the same as for $e^+e^- \to \mu^+\mu^-$, reads

$$\frac{d\sigma}{d\Omega}(e^+e^- \to q\bar{q}) = \frac{3}{4}\frac{\alpha_s^2}{s}\sum Q_f^2 \,\left(1+\cos^2\theta\right) \tag{8.24}$$

typical for an angular distribution of a spin 1/2 particle. Indeed, the quark and the antiquark seemingly hadronize individually in that they form *jets* [31]. Jets are bunches of hadrons which concentrate in a relatively narrow angular cone. This in spite of the fact that the quarks have unphysical charge and color. True physical states only can have integer charge and must be color singlets. Apparently, while charge and color have enough time to recombine into color singlets of integer charge, the momentum apparently has not sufficient time to distribute isotropically in the hadronization process. The extra quarks needed to form physical states are one or more virtual pairs pulled out of the vacuum and carried along by the primary quarks (see Fig. 8.9). As a rule pQCD is applicable to the extent that "hard partons", quarks or gluons, may be interpreted as jets. Fig. 8.11 illustrates such $q\bar{q}$ (two–jet event) and $q\bar{q}g$ (three–jet event) jets. Three jet events produced with the electron positron storage ring PETRA at DESY in 1979 revealed the existence of the gluon. The higher the energy the narrower the jets, quite opposite to expectations at pre QCD times when most people believed events with increasing energy will be more and more isotropic multi–hadron states.

It is one of the spectacular and at the beginning completely unexpected predictions of QCD,

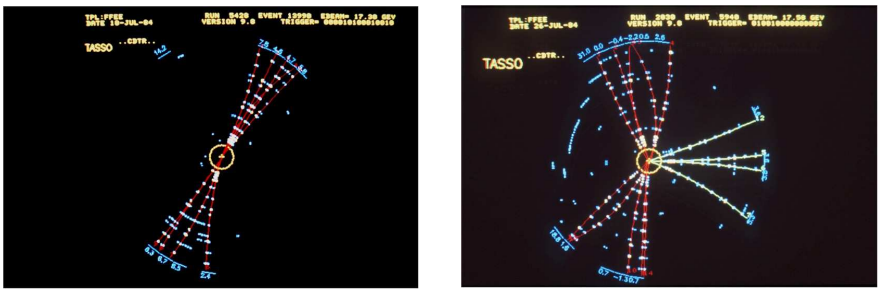


Figure 8.11: Two and three jet event first seen by TASSO at DESY in 1979.

that the energy get collimated more and more into narrower and narrower jets as the enegy increases. However, as the energy increases also the number of primary partons increases such that the energy distributes among a larger number of jest, which then brings back a more isotropic distribution.

8.14 Non–Perturbative Effects, Operator Product Expansion

Here we have to address the question: what are the contributions to $R(s)$ from the quark condensates which are the order parameters of the spontaneous breakdown of chiral symmetry. Such non–perturbative (NP) effects are parametrized as prescribed by the Operator Product Expansion (OPE)[39] of the electromagnetic current correlator [34]

$$\begin{aligned}\hat{\Pi}'^{\rm NP}_{\gamma}(Q^2) &= \frac{4\pi\alpha}{3}\sum_{q=u,d,s} Q_q^2 N_{cq}\cdot\left[\frac{1}{12}\left(1-\frac{11}{18}a\right)\frac{\langle\frac{\alpha_s}{\pi}GG\rangle}{Q^4}\right.\\ &+ 2\left(1+\frac{a}{3}+\left(\frac{11}{2}-\frac{3}{4}l_{q\mu}\right)a^2\right)\frac{\langle m_q\bar{q}q\rangle}{Q^4}\\ &+ \left.\left(\frac{4}{27}a+\left(\frac{4}{3}\zeta_3-\frac{257}{486}-\frac{1}{3}l_{q\mu}\right)a^2\right)\sum_{q'=u,d,s}\frac{\langle m_{q'}\bar{q'}q'\rangle}{Q^4}\right]+\cdots\end{aligned} \tag{8.28}$$

where $a \equiv \alpha_s(\mu^2)/\pi$ and $l_{q\mu} \equiv \ln(Q^2/\mu^2)$. $\langle\frac{\alpha_s}{\pi}GG\rangle$ and $\langle m_q\bar{q}q\rangle$ are the scale–invariantly defined condensates. Sum rule estimates of the condensates yield typically (large uncertainties) $\langle\frac{\alpha_s}{\pi}GG\rangle \sim \ (0.389\ \text{GeV})^4$, $\langle m_q\bar{q}q\rangle \sim -(0.098\ \text{GeV})^4$ for $q = u, d$, and $\langle m_q\bar{q}q\rangle \sim -(0.218\ \text{GeV})^4$

[39]The operator product expansion (Wilson short distance expansion) [32] is a formal expansion of the product of two local field operators $A(x)\,B(y)$ in powers of the distance $(x-y)\to 0$ in terms of singular coefficient functions and regular composite operators:

$$A(x)\,B(y) \simeq \sum_i C_i(x-y)\,\mathcal{O}_i(\frac{x+y}{2}) \tag{8.25}$$

where the operators $\mathcal{O}_i(\frac{x+y}{2})$ represent a complete system of local operators of increasing dimensions. The coefficients may be calculated formally by normal perturbation theory by looking at the Green functions

$$\langle 0|TA(x)\,B(y)\,X|0\rangle = \sum_{i=0}^{N} C_i(x-y)\,\langle 0|T\mathcal{O}_i(\frac{x+y}{2})\,X|0\rangle + R_N(x,y) \tag{8.26}$$

constructed such that

$$R_N \ \to\ 0 \ \text{ as } \ (x-y)^{a_N}\ ;\ (x-y)^2<0\ ,\quad a_N < a_{N+1}\ \forall\ N \tag{8.27}$$

(asymptotic expansion). By X we denoted any product of fields suitable to define a physical state $|X\rangle$ via the LSZ reduction formula. A detailed explanation will be given in Sect. **??** later (see also [33])

for $q = s$. Note that the above expansion is just a parametrization of the high energy tail of NP effects associated with the existence of non–vanishing condensates. The dilemma with the OPE in our context is that it works for large enough Q^2 only and in this form fails do describe NP physics at lower Q^2. Once it starts to be numerically relevant pQCD starts to fail because of the growth of the strong coupling constant. In $R(s)$ NP effects as parametrized by (8.28) have been shown to be small in [35–37]. Note that the quark condensate, the vacuum expectation value (VEV) $\langle \mathcal{O}_q \rangle$ of the dimension 3 operator $\mathcal{O}_q \doteq \bar{q}q$, is a well defined non–vanishing order parameter in the chiral limit of QCD. In pQCD it is vanishing to all orders. In contrast the VEV of the dimension 4 operator $\mathcal{O}_G \doteq \frac{\alpha_s}{\pi} GG$ is non–vanishing in pQCD but ill–defined at first as it diverges like Λ^4 in the UV cut–off. $\mathcal{O}_G$ contributes to the trace of the energy momentum tensor[40] [38–40]

$$\Theta^{\mu}_{\ \mu} = \frac{\beta(g_s)}{2g_s}\, GG + (1+\gamma(g_s))\left\{m_u\, \bar{u}u + m_d\, \bar{d}d + \cdots\right\} \tag{8.31}$$

where $\beta(g_s)$ and $\gamma(g_s)$ are the RG coefficients (8.5) and in the chiral limit

$$\varepsilon_{\rm vac} = -\left\{\frac{\beta_0}{32} + O(\alpha_s)\right\}\, \langle \mathcal{O}_G \rangle \tag{8.32}$$

represents the vacuum energy density which is not a bona fide observable in a continuum QFT. In the Shifman-Vainshtein-Zakharov (SVZ) approach [34] it is treated to represent the soft part with respect to the renormalization scale μ, while the corresponding OPE coefficient comprises the hard physics from scales above μ.

Note that in the chiral limit $m_q \to 0$ the trace (8.31) does not vanish as expected on the classical level. Thus scale invariance (more generally conformal invariance) is broken in any QFT unless the β–function has a zero. This is another renormalization anomaly, which is a quantum effect not existing in a classical field theory. The renormalization group is another form of encoding the broken dilatation Ward identity. It's role for the description of the asymptotic behavior of the theory under dilatations (scale transformations) has been discussed in Sect. 8, where it was shown that under dilatations the effective coupling is driven into a zero of the β–function. For an asymptotically free theory like QCD we reach the scaling limit in the high energy limit. At finite energies we always have scaling violations, as they are well known from deep inelastic electron nucleon scattering. In e^+e^-–annihilation the scaling violation are responsible for the energy dependence (via the running coupling) of $R(s)$ in regions where mass effects are negligible.

[40] In a QFT a symmetric energy momentum tensor $\Theta_{\mu\nu}(x)$ should exist such that the generators of the Poincaré group are represented by (see cf. [1])

$$P_\mu = \int \mathrm{d}^3x\, \Theta_{0\mu}(x)\ , M_{\mu\nu} = \int \mathrm{d}^3x\ \left(x_\mu\, \Theta_{0\nu} - x_\nu\, \Theta_{0\mu}\right)(x)\ . \tag{8.29}$$

This corresponds to Noether's theorem (see (1.5.2)) for the Poincaré group. In a strictly renormalizable massless QFT which exhibits only dimensionless couplings classically one would expect the theory to be conformally invariant. The energy momentum tensor then would also implement infinitesimal dilatations and special conformal transformations. That is, the currents

$$D_\mu(x) = x^\rho\, \Theta_{\mu\rho}\ ;\ \ K_{\mu\nu} = 2\, x^\rho\, x_\nu\, \Theta_{\mu\rho} - x^2 \Theta_{\mu\nu} \tag{8.30}$$

ought to be conserved, which requires the trace of the energy momentum tensor to vanish $\Theta^{\mu}_{\ \mu} = 0$.

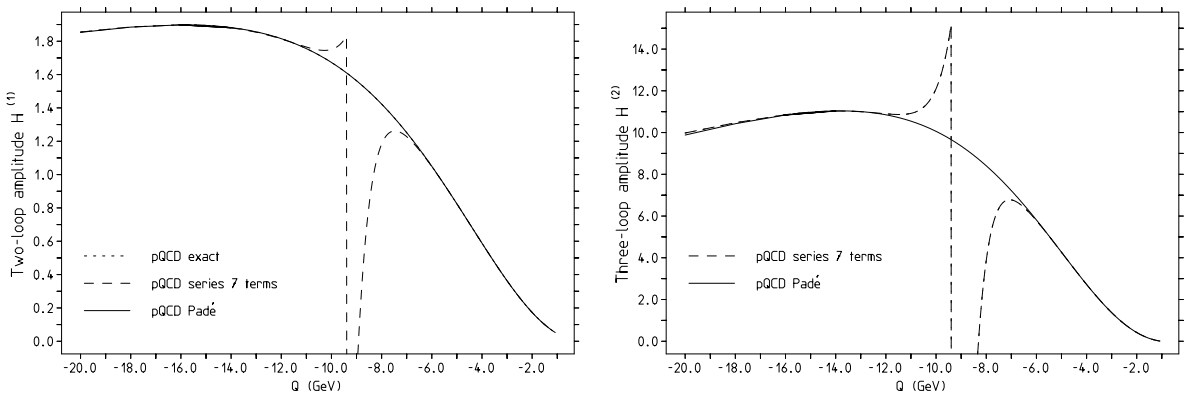


Figure 8.12: Perturbative coefficient functions H_i for the Adler function in full QCD at 2 [left] and 3 [right] loops. The curvature is entirely due to the non-zero masses. In the massless limit the H_i's are normalized to unity. From [37].

8.14.1 Testing non–perturbative hadronic effects via the Adler function

The non-perturbative Adler function related to the photon vacuum polarization can be calculated in terms of experimental e^+e^- annihilation data by the dispersion integral

$$\begin{aligned} D(Q^2) &= Q^2 \left(\int_{4m_\pi^2}^{E_{\rm cut}^2} \frac{R^{\rm data}(s)}{(s+Q^2)^2}\,{\rm d}s \right. \\ &\quad \left. + \int_{E_{\rm cut}^2}^{\infty} \frac{R^{\rm pQCD}(s)}{(s+Q^2)^2}\,{\rm d}s \right) . \end{aligned} \tag{8.33}$$

Here $Q^2 = -q^2$ is the squared Euclidean momentum transfer and s the center of mass energy squared for hadron production in e^+e^-–annihilation. Formally the Adler function is defined as the derivative of the shift in the fine structure constant

$$\frac{D(Q^2)}{Q^2} = (12\pi^2)\,\frac{{\rm d}\hat{\Pi}'_\gamma\,(q^2)}{{\rm d}q^2} = -\frac{3\pi}{\alpha}\frac{\rm d}{{\rm d}q^2}\Delta\alpha_{\rm had}(q^2)\;,$$

evaluated in the Euclidean at $Q^2 = -q^2$. $\hat{\Pi}'_\gamma\,(q^2)$ is the photon vacuum polarization amplitude defined by Eq. (8.9). The perturbative result is given in [37]. Crucial for this prediction are known full massive QCD results [41–43]. Note that the main Q^2 dependence of $D(Q^2)$ is due to the quark masses m_c and m_b. Without mass effects, up to small effects from the running of α_s, $D(Q^2) = 3\sum_f Q_f^2\,(1 + O(\alpha_s))$ is a constant depending on the number of active flavors.

We write the contributions as a loop expansion

$$D(Q^2) = D^{(0)}(Q^2) + D^{(1)}(Q^2) + D^{(2)}(Q^2) + \cdots + D^{\rm NP}(Q^2)$$

At one loop, taking the derivative of (8.16) with appropriate coefficient, we obtain

$$D^{(0)}(Q^2) = \sum_f Q_f^2 N_{cf} H^{(0)}$$

with $H \equiv (12\pi^2)\,\dot{\hat{\Pi}}'_V$, $\dot{\hat{\Pi}}'_V \equiv -s\,d\hat{\Pi}'_V/ds$, in terms of the vector current amplitude $\hat{\Pi}'_V$. Explicitly,

$$H^{(0)} = 1 + \frac{3y}{2} - \frac{3y^2}{4}\frac{1}{\sqrt{1-y}}\ln\xi \quad y = 4m_f^2/s \ , \quad \xi = \frac{\sqrt{1-y}-1}{\sqrt{1-y}+1}$$

where ξ is taking values $0 \le \xi \le 1$ for $s \le 0$.

Asymptotically we have the expansion

$$H^{(0)} \to \begin{cases} \frac{1}{5}\frac{Q^2}{m_f^2} - \frac{3}{70}\left(\frac{Q^2}{m_f^2}\right)^2 + \frac{1}{105}\left(\frac{Q^2}{m_f^2}\right)^3 + \cdots & Q^2 \ll m_f^2 \\ 1 - 6\frac{m_f^2}{Q^2} - 12\left(\frac{m_f^2}{Q^2}\right)^2 \ln\frac{m_f^2}{Q^2} + 24\left(\frac{m_f^2}{Q^2}\right)^3\left(\ln\frac{m_f^2}{Q^2}+1\right) + \cdots & Q^2 \gg m_f^2 \end{cases} \tag{8.34}$$

and this behavior determines the quark parton model (QPM) (leading order QCD) property of the Adler function: heavy quarks ($m_f^2 \gg Q^2$) decouple like Q^2/m_f^2 while light modes ($m_f^2 \ll Q^2$) contribute $Q_f^2 N_{cf}$ to $D^{(0)}$.

At two loops we have the known analytic result (Broadhurst 85 and others)

$$D^{(1)}(Q^2) = \frac{\alpha_s(Q^2)}{\pi}\sum_f Q_f^2 N_{cf}\, H^{(1)}$$

where $H^{(1)} = (12\pi^2)\,\dot{\hat{\Pi}}'^{(2)}_V(-Q^2, m_f^2)$. At three loops the result is known as low and large momentum expansion (Chetyrkin, Harlander, Kühn, Steinhauser 96/97)

$$D^{(2)}(Q^2) = \left(\frac{\alpha_s(Q^2)}{\pi}\right)^2 \sum_f Q_f^2 N_{cf}\, H^{(2)}$$

where $H^{(2)} = (12\pi^2)\,\dot{\hat{\Pi}}'^{(3)}_V(-Q^2, m_f^2)$. Both series expansions diverge at the boundary of the circle of convergence $Q^2 = 4m^2$, which means that we have a problem in the region where mass effects are of the order of unity in the Euclidean region. Here we apply a conformal mapping (Schwinger 48)

$$y^{-1} = \frac{-Q^2}{4m^2} \to \omega = \frac{1-\sqrt{1-1/y}}{1+\sqrt{1-1/y}}$$

from the complex negative q^2 half–plane to the interior of the unit circle $|\omega| < 1$ together with Padé resummation (Fleischer and Tarasov 94). The Padé approximant provides a good estimation to much higher values of $1/y$ up to about $1/y \sim 4$. This is displayed in Fig. 8.12. Padé improvement allows us to obtain reliable results also in the relevant Euclidean "threshold region", around $y = 1$.

We also include the 4–loop [44,45] and 5–loop [46] contributions in the high energy limit (massless approximation)

$$D(Q^2) \simeq 3\sum_f Q_f^2 \left(1 + a + d_2 a^2 + d_3 a^3 + d_4 a^4 + \cdots\right) \tag{8.35}$$

with $a = \alpha_s(Q^2)/\pi$,

$$\begin{aligned} d_2 &= 1.9857 - 0.1153\, N_f \,, \\ d_3 &= 18.2428 - 4.2159\, N_f + 0.0862\, N_f^2 - 1.2395\,(\textstyle\sum Q_f)^2/(3\sum Q_f^2)\,, \\ d_4 &= -0.010\, N_f^3 \,+\, 1.88\, N_f^2 \,-\, 34.4\, N_f \,+\, 135.8 \ . \end{aligned}$$

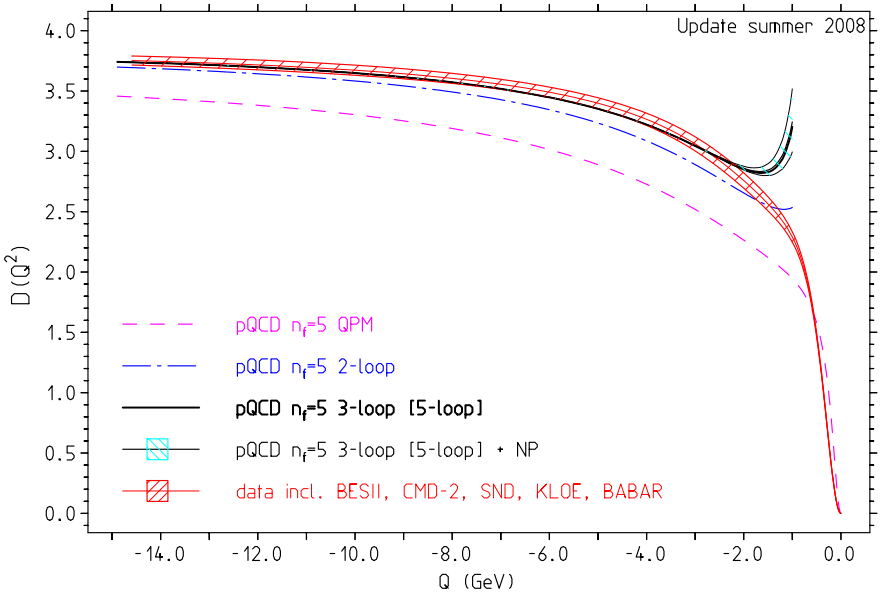


Figure 8.13: The "experimental" non-perturbative Adler–function versus theory (pQCD + NP). The error includes statistical + systematic here (in contrast to most R-plots showing statistical errors only!). "[5-loop]" indicates that 4- and 5-loop contribution in the massless limit are taken into account. For more details see Ref. [37].

The corresponding formula for $R(s)$ only differs at the 4–loop and 5–loop level due to the effect from the analytic continuation from the Euclidean to the Minkowski region which yields $r_3^R = d_3 - \pi^2\beta_0^2\frac{d_1}{3}$ with $\beta_0 = (11 - 2/3\,N_f)/4$, $d_1 = 1$ and $r_4^R = d_4 - \pi^2\beta_0^2\left(d_2 + \frac{5\beta_1}{6\beta_0}\,d_1\right)$ with $\beta_1 = (102 - 38/3\,N_f)/16$. Numerically the 4–loop term proportional to d_3 amounts to -0.0036% at 100 GeV and increases to about 0.32% at 2.5 GeV. The higher order massless results only improve the perturbative high energy tail (see Fig. 8.13). Towards low Q^2 we also approach the Landau pole of $\alpha_s(Q^2)$, present typically in $\overline{\rm MS}$ type schemes, and pQCD ceases to "converge".

The Adler–function is a good monitor to compare the pQCD as well as the NP results with experimental data. Fig. 8.13 shows that pQCD in the Euclidean region works very well for $\sqrt{Q^2}\gtrsim$2.5 GeV [37]. The NP effects just start to be numerically significant where pQCD starts to fail. Thus, no significant NP effects can be established from this plot.

Note that what we used here is not the $\overline{\rm MS}$ scheme but full massive QCD in the background MOM scheme [43]. The proper curvature is the key point in this analysis and the curvature is primarily due to the mass dependence. In the Euclidean region the latter is smooth ans momotonic unlike in the time–like region where we have step like behavior at thersholds. The gross feature shape is obtained in the QPM, but numerically it fails badly. The first correction of $O(\alpha_s)$ improves the situation dramatically, however, starts to fail at rather high energies (below about 20 GeV) already, and approaching the non-perturbative regime below about 2 GeV starts to diverge (nearby Landau pole). At 3–loops for the first time we get a surprisingly good perturbative description of the data, down to not far above 2 GeV, where again the breakdown of pQCD is hitting us. In view of the fairly large differences between the different perturbative orders, a massive 4–loop calculation is expected to provide further substantial improvement. Needless to say, the good agreement betwen theory and experiment for the Adler function as a test observable also strenghtens our confidence in pQCD results for other onservables.

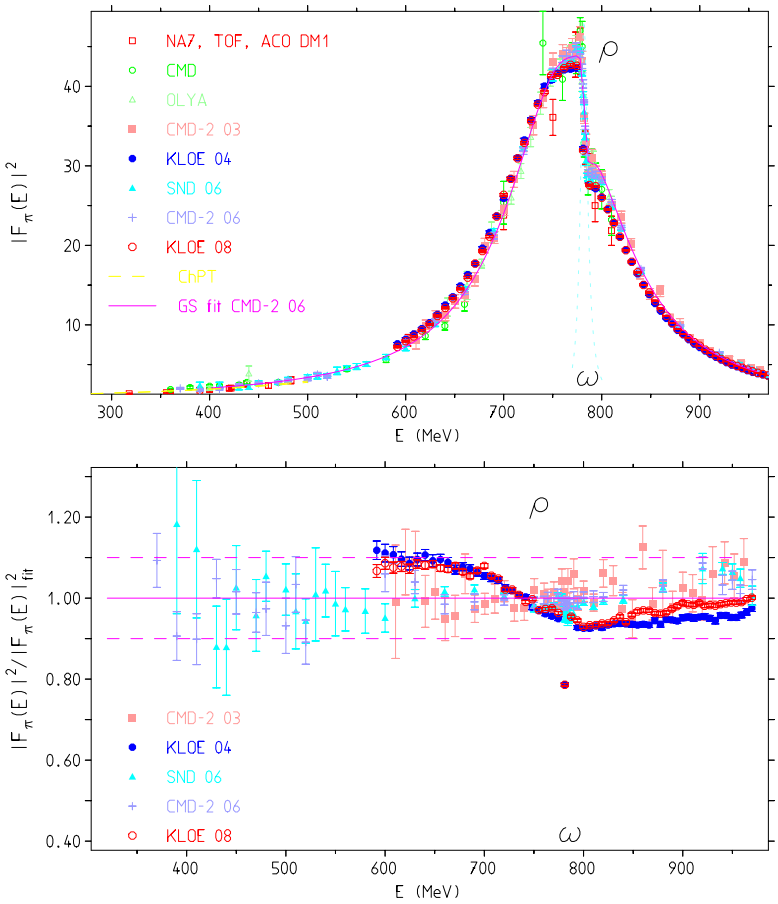


Figure 8.14: The modulus square of pion form factor $|F_\pi|^2$ (which is proportional to cross–section $\sigma_h(s)$) as a function of $E = \sqrt{s}$. The low energy domain is dominated by the channel $e^+e^- \to \pi^+\pi^-$, which developes the ρ–resonance. The $\rho - \omega$ mixing, due to isospin breaking by $m_u \neq m_d$, is distorting the ideal Breit-Wigner resonance shape of the ρ. The ratio $|F_\pi(E)|^2/|F_\pi(E)|^2_{\rm fit}$ shows the fairly good compatibility of the newer measurements relative to a CMD-2 fit. Dashed horizontal lines mark ± 10%.

8.15 Some considerations on hadron production at low energy

Certainly, quark-antiquark pair production is far from describing experimental cross–section data at low energies, where "infrared slavery" (confinement) is the dominating feature. In Fig. 8.14 we show a compilation of the low energy data in terms of the pion form factor. The latter is defined by

$$\sigma_{\pi\pi}(s) = \frac{\pi\alpha^2}{3s}\,(\beta_\pi)^3 |F_\pi(s)|^2 \ . \tag{8.36}$$

with $\beta_\pi = (1 - 4m_\pi^2/s)^{1/2}$ the pion veclocity in the CM frame. Expressed in terms of $R(s)$, the form factor is given by

$$|F_\pi(s)|^2 = 4\,R_{\pi\pi}(s)\,(\beta_\pi)^{-3} \ . \tag{8.37}$$

Low energy hadro-production is dominated by the lightest hadrons, the isospin $SU(2)$ triplet (π^+, π^0, π^-) of pions, pseudoscalar spin 0 mesons of masses: $m_{\pi^\pm} = 139.75018(35)$ MeV, $m_{\pi^0} = 134.9766(6)$ MeV. As elaborated earlier, spontaneous breakdown of the nearby chiral symmetry

of QCD, an intrinsically non–perturbative phenomenon, shapes the hadron physics in the low energy tail. Pions as quark–antiquark color singlet bound states exhibit electromagnetic and weak interactions via the quarks. This is particularly pronounced in the case of the neutral π^0 which decays electromagnetically via $\pi^0 \to \gamma\gamma$ and has a much shorter life time $\tau_{\pi^0} = 8.4(6) \cdot 10^{-17}$ sec than the charged partners which can decay by weak interaction only, according to $\pi^+ \to \mu^+\nu_\mu$, and hence live longer by almost 10 orders of magnitude $\tau_{\pi^\pm} = 2.6033(5) \cdot 10^{-8}$ sec.

As perturbation theory fails, chiral perturbation theory is limited to the very low energy tail only and lattice QCD so far is not able to control time-like processes, one has to resort to models which are inspired by known properties of QCD as well as known phenomenological facts. As an example, electromagnetic interactions of charged pions treated as point–particles would be described by scalar QED, as a first step in the sense of a low energy expansion. In fact, in the threshold region $e^+e^- \to \pi^+\pi^-$ is the dominating process. However, at slightly higher energies, the ρ meson represents a prominent $\pi^+\pi^-$–resonance peak, showing a resonance enhancement by about a factor 50. Here, it looks natural to apply a vector–meson dominance (VMD) like model (see below). Note that in photon–hadron interactions the photon mixes with hadronic vector–mesons like the ρ^0. But also nearby resonances like ρ^0 and ω are mixing and thereby destorting undisturbed resonance patterns (see Fig. 8.14). In the following we are going to discuss some of the relevant models in some detail.

8.15.1 Pions and Scalar QED

Pions seen in a particle detector in may respects behave like point-particles , Wigner states in the sense of relativistic quantum mechanics. Such states and their associated LSZ interpolating fields were intruduced in Sect. 1.5. The effective Lagrangian for the electromagnetic interaction of a charged point–like pion described by a complex scalar field φ follows from the free Lagrangian

$$\mathcal{L}_\pi^{(0)} \;=\; (\partial_\mu\varphi)(\partial^\mu\varphi)^* - m_\pi^2\varphi\varphi^* \tag{8.38}$$

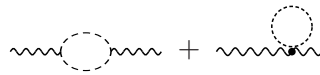
via *minimal substitution* $\partial_\mu\varphi \to D_\mu\varphi = (\partial_\mu + \mathrm{i}eA_\mu(x))\,\varphi$, which replaces the ordinary by the *covariant derivative*, and which implies the scalar QED (sQED) Lagrangian

$$\mathcal{L}_\pi^{\mathrm{sQED}} \;=\; \mathcal{L}_\pi^{(0)} - \mathrm{i}e(\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^*)A^\mu + e^2 g_{\mu\nu}\varphi\varphi^* A^\mu A^\nu \;.$$

Thus gauge invariance implies that the pions must couple via two different vertices to the electromagnetic field, and the corresponding Feynman rules are given in Fig. 8.15.

In sQED the contribution of a pion loop to the photon VP is given by

$$-\mathrm{i}\,\Pi_\gamma^{\mu\nu\,(\pi)}(q) =$$



(8.39)

The bare result for the transversal part, defined as in (8.9), reads

$$\Pi_\gamma^{(\pi)}(q^2) = \frac{e^2}{48\pi^2}\left\{B_0(m,m;q^2)\;\left(q^2-4m^2\right) - 4\,A_0(m) - 4m^2 + \frac{2}{3}q^2\right\} \tag{8.40}$$

with $\Pi_\gamma(0) = 0$. We again calculate the renormalized transversal self–energy $\Pi_\gamma^{'}(q^2) = \Pi_\gamma(q^2)/q^2$ which is given by $\Pi_{\gamma\mathrm{ren}}^{'}(q^2) = \Pi_\gamma^{'}(q^2) - \Pi_\gamma^{'}(0)$. The subtraction term

$$\Pi_\gamma^{'(\pi)}(0) = \frac{-e^2}{48\pi^2}\;\left\{\frac{A_0(m)}{m^2} + 1\right\}$$

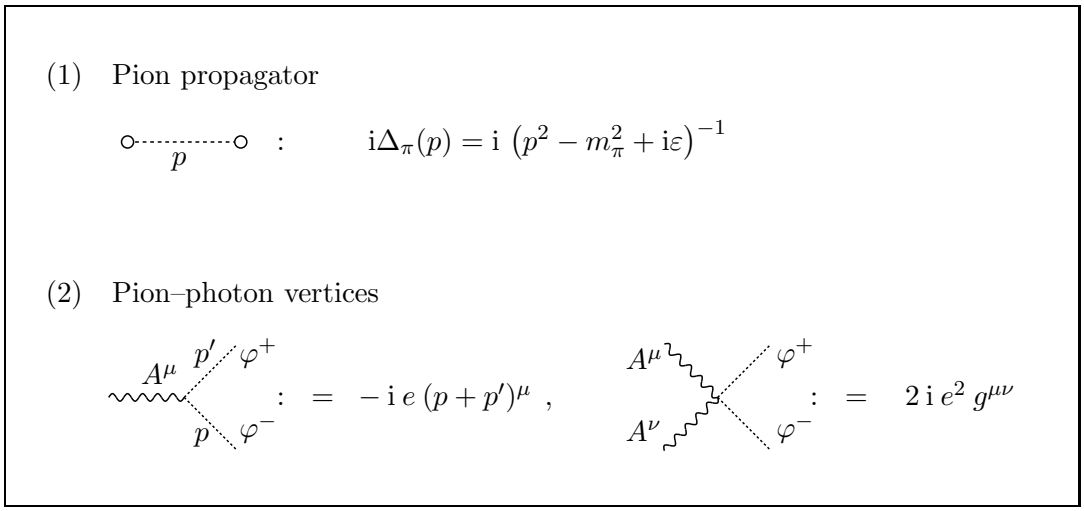


Figure 8.15: Feynman rules for sQED. p is incoming, p' outgoing.

is the $\pi^\pm$ contribution to the photon wavefunction renormalization and the renormalized transversal photon self–energy reads

$$\Pi'^{\,(\pi)}_{\gamma\mathrm{ren}}(q^2) = \frac{\alpha}{6\pi}\left\{\frac{1}{3} + (1-y) - (1-y)^2\,G(y)\right\}$$

where $y = 4m^2/q^2$ and $G(y)$ given by (7.16). For $q^2 > 4m^2$ there is an imaginary or absorptive part given by substituting

$$G(y) \to \mathrm{Im}\,G(y) = -\frac{\pi}{2\sqrt{1-y}}$$

such that

$$\mathrm{Im}\,\Pi'^{\,(\pi)}_{\gamma}(q^2) = \frac{\alpha}{12}\,(1-y)^{3/2}$$

and for large q^2 is 1/4 of the corresponding value for a lepton. According to the *optical theorem* the absorptive part may be written in terms of the $e^+e^- \to \gamma^* \to \pi^+\pi^-$ production cross–section $\sigma_{\pi^+\pi^-}(s)$ as

$$\mathrm{Im}\,\Pi'^{\,\mathrm{had}}_{\gamma}(s) = \frac{s}{4\pi\alpha}\,\sigma_{\mathrm{had}}(s)$$

which hence we can read off to be

$$\sigma_{\pi^+\pi^-}(s) = \frac{\pi\alpha^2}{3s}\,\beta_\pi^3$$

again with $\beta_\pi = \sqrt{(1-4m_\pi^2/s)}$ the pion velocity. This means that sQED predicts

$$F_\pi(s) = 1\ , \tag{8.41}$$

in view of Fig. 8.14 a complete failure up to 50 : 1. Therefore sQED in general only can make sense if we replace $e \to e\,F_\pi(q^2)$, $e^2 \to e^2\,|F_\pi(q^2)|^2$ and take the form factor from experiment or from some more realistic model like the VMD model and improvements thereof. It means that the bound state nature of the charged pion is taken care off by introducing a non-trivial

q^2–dependent pion form factor. Note that in the low energy limit (classical limit) electromagnetic current conservation implies

$$F_\pi(0) = 1 \ , \tag{8.42}$$

a mandatory constraint.

Often one has to resort to sQED in particular in connection with the soft photon radiation problem of charged particles, where sQED provides a good description of the problem. However, in calculating the pion contribution the photon vacuum polarization, we would include hard photons when we apply it in the region of interest above the $\pi^+\pi^-$ production threshold to about 1 GeV or even higher. In conclusion: sQED in the classical limit povides the exact describtion of photon radiation, in particular for what concerns the IR problem (Bloch-Nordsieck prescription). The hard end of the radiation spectrum is also known exactly, as given by the quark–photon interaction (QED of quarks). An experimental "proof" that the latter is correct is the reasonably good QPM prediction of $R^{(0)}(s)$, discussed before. The intermediate range is presently not well under control, however.

8.15.2 The Vector Meson Dominance Model

As strict sQED fails badly as soon as the ρ meson, with mass about 776 MeV, is approacched, another phenomenologically motivated mechanism comes into play. Actually Fig. 8.14 suggests that the photon itself has a hadronic component described in leading approximation by the ρ^0 via a direct coupling, i.e., by $\gamma - \rho^0$–mixing. Indeed in photon–hadron interactions the photon mixes with hadronic vector–mesons like the ρ^0. The naive VMD model attempts to take into account this hadronic dressing by replacing the photon propagator as

$$\frac{\mathrm{i}\, g^{\mu\nu}}{q^2} + \cdots \to \frac{\mathrm{i}\, g^{\mu\nu}}{q^2} + \cdots - \frac{\mathrm{i}\,(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2})}{q^2 - m_\rho^2} = \frac{\mathrm{i}\, g^{\mu\nu}}{q^2}\ \frac{m_\rho^2}{m_\rho^2 - q^2} + \cdots \ ,$$

where the ellipses stand for the gauge terms. Of course real photons $q^2 \to 0$ in any case remain undressed and the dressing would go away for $m_\rho^2 \to \infty$. The main effect is that it provides a damping at high energies with the ρ mass as an effective cut–off (physical version of a Pauli-Villars cut–off). However, the naive VMD model does not respect chiral symmetry properties. More precisely, the hypothesis of *vector–meson dominance* [47] relates the matrix element of the hadronic part of the electromagnetic current $j_\mu^{\rm had}(x)$ to the matrix element of the source density $J^{(\rho)}(x)$ of the neutral vector meson ρ^0 by

$$\langle B|j_\mu^{\rm had}(0)|A\rangle = -\frac{M_\rho^2}{2\gamma_\rho}\ \frac{1}{q^2 - M_\rho^2}\ \langle B|J_\mu^{(\rho)}(0)|A\rangle$$

where $q = p_B - p_A$, p_A and p_B the four momenta of the hadronic states A and B, respectively, M_ρ is the mass of the ρ meson. So far our VMD ansatz only accounts for the isovector part, but the isoscalar contributions mediated by the ω and the ϕ mesons may be included in exactly the same manner, as shown in Fig. 8.16. The key idea is to treat the vector meson resonances like the ρ as elementary fields in a first approximation. Free massive spin 1 vector bosons are described by a Proca field $V_\mu(x)$ satisfying the Proca equation $(\Box + M_V^2)\, V_\mu(x) - \partial_\mu\, (\partial_\nu V^\nu) = 0$, which is designed such that it satisfies the Klein-Gordon equation and at the same time eliminates the unwanted spin 0 component: $\partial_\nu V^\nu = 0$. In the interacting case this equation is replaced by a *current–field identity* (CFI) [47]

$$(\Box + M_V^2)\, V_\mu(x) - \partial_\mu\, (\partial_\nu V^\nu) = g_V\ J_\mu^{(V)}(x)$$

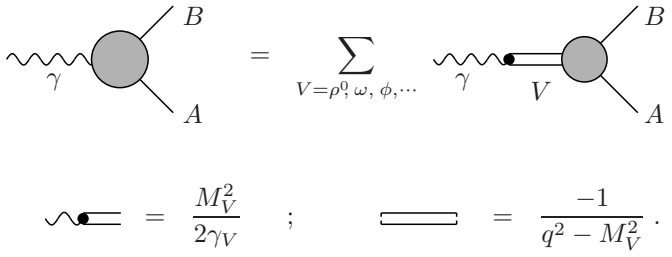


Figure 8.16: The vector meson dominance model. A and B denote hadronic states.

where the r.h.s. is the source mediating the interaction of the vector meson and g_V the coupling strength. The current should be conserved $\partial^\mu J_\mu^{(V)}(x) = 0$. The CFI then implies

$$\langle B|V_\mu(0)|A\rangle = -\frac{g_V}{q^2 - M_V^2}\,\langle B|J_\mu^{(V)}(0)|A\rangle \tag{8.43}$$

where terms proportional to q^μ have dropped due to current conservation. The VMD assumes that the hadronic electromagnetic current is saturated by vector meson resonances[41]

$$j_\mu^{\rm had}(x) = \sum_{V=\rho^0,\,\omega,\,\phi,\cdots} \frac{M_V^2}{2\gamma_V}\,V_\mu(x)$$

such that, e.g.

$$\langle \rho(p)|j_\mu^{\rm had}(0)|0\rangle = \varepsilon(p,\lambda)_\mu\,\frac{M_\rho^2}{2\gamma_\rho}\ ,\quad p^2 = M_\rho^2\ . \tag{8.45}$$

The mass–dependent factor M_V^2 must be there for dimensional reasons, γ_V is a coupling constant introduced in this form by convention. The VMD relation (8.43) thus derives from the CFI and ansatz (8.45). The VMD model is known to describe the gross features of the electromagnetic properties of hadrons quite well, most prominent example are the nucleon form factors.

A way to incorporate vector–mesons ρ, ω, ϕ, ... in accordance with the basic symmetries of QCD is the Resonance Lagrangian Approach (RLA) [52,53], an extended version of CHPT [54,55] (see Sect. 3.4), which also implements VMD in a consistent manner. In the flavor $SU(3)$ sector, similar to the pseudoscalar field $\Phi(x)$ (3.16), the $SU(3)$ gauge bosons conveniently may be written as a 3×3 matrix field

$$V_\mu(x) \;=\; \sum_i T_i V_{\mu i} = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & \rho^+ & K^{*+} \\ \rho^- & \frac{-\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & K^{*0} \\ K^{*-} & \overline{K}^{*0} & -2\,\frac{\omega_8}{\sqrt{6}} \end{pmatrix}_\mu \tag{8.46}$$

in order to keep track of the appropriate $SU(3)$ weight factors.

[41] In large N_c QCD [48–50] all hadrons become infinitely narrow, since all widths are suppressed by powers of $1/N_c$, and the VMD model becomes exact with an infinite number of narrow vector meson states. The large-N_c expansion attempts to approach QCD ($N_c = 3$) by an expansion in $1/N_c$. In leading approximation in the $SU(\infty)$ theory $R(s)$ would have the form [51]

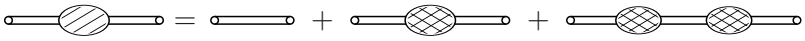
$$R(s) = \frac{9\pi}{\alpha^2}\sum_{i=0}^{\infty}\Gamma_i^{ee} M_i\,\delta(s - M_i^2)\ . \tag{8.44}$$

8.15.3 Mass and Width of Vector Mesons

All hadronic vector mesons are unstable, usually decaying predominantly into hadrons, like $\rho^0 \to \pi^+\pi^-$ or $\omega \to \pi^+\pi^-\pi^0$, and therefore the zero width approximation breaks down near resonance. For a more realistic describtion of the vector mesons their finite width has to be taken into account. We may understand the finite width as a self-energy effect, via the Dyson summation of the self-energy insertions. The procedure is very similar to the one discussed for the gluon propagator Sect. 7.2, except that we are dealing with a massive spin 1 boson here. As usual we decompose the spin 1 vector boson self-energy $\Pi_{\mu\nu}(q^2, m^2, \cdots)$ into a transverse $\Pi(q^2)$ and a longitudinal $L(q^2)$ part

$$\Pi_{\mu\nu}(q^2, m^2, \cdots) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Pi(q^2) + \frac{q_\mu q_\nu}{q^2} L(q^2), \tag{8.47}$$

where for the definition of the mass and width only the tranverse part is relevant. We then only consider the transversal part, and suppress the transversal projector, which is an overall factor. The full or dressed propagator is then given by the geometrical progression



$$\begin{aligned}
&= \frac{-\mathrm{i}}{q^2-m^2} + \frac{-\mathrm{i}}{q^2-m^2}(-\mathrm{i}\,\Pi)\frac{-\mathrm{i}}{q^2-m^2} + \frac{-\mathrm{i}}{q^2-m^2}(-\mathrm{i}\,\Pi)\frac{-\mathrm{i}}{q^2-m^2}(-\mathrm{i}\,\Pi)\frac{-\mathrm{i}}{q^2-m^2} + \cdots \\
&= \frac{-\mathrm{i}}{q^2-m^2}\left\{1 + \left(\frac{-\Pi}{q^2-m^2}\right) + \left(\frac{-\Pi}{q^2-m^2}\right)^2 + \cdots\right\} \\
&= \frac{-\mathrm{i}}{q^2-m^2}\left\{\frac{1}{1+\frac{\Pi}{q^2-m^2}}\right\} = \frac{-\mathrm{i}}{q^2-m^2+\Pi(q^2)} \equiv -\mathrm{i}\,D(q^2)\ .
\end{aligned} \tag{8.48}$$

The dressed propagator then reads

$$D_{\mu\nu}(q) = \frac{-\mathrm{i}}{q^2-m^2+\Pi(q^2)}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{q_\mu q_\nu}{q^2}\ \cdots\ . \tag{8.49}$$

The position of the pole $\tilde{M}^2$ of the propagator of a massive spin 1 boson in a quantum field theory is a solution for q^2 at which the inverse of the connected full propagator equals zero, i.e.,

$$\tilde{M}^2 - m^2 + \Pi(\tilde{M}^2, m^2, \cdots) = 0, \tag{8.50}$$

where $\Pi(q^2, \cdots)$ is the transversal part of the one-particle irreducible self-energy. The latter depends on all parameters of the theory but, in order to the keep notation simple, we have indicated explicitly only the dependence on the external momentum q and in some cases also m, where m is the mass of the particle under consideration. This can be either the bare mass m_0 or the renormalized mass defined in some particular renormalization scheme.

Generally, the pole $\tilde{M}^2$ is located in the complex plane of q^2 and has a real and an imaginary part. We write

$$\tilde{M}^2 \equiv M^2 - \mathrm{i}M\Gamma. \tag{8.51}$$

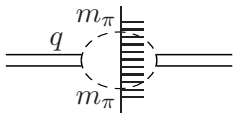
The real part of (8.51) defines M which we call the pole mass, while the imaginary part is related to the width Γ of the particle. This is the natural generalization of the physical mass of a stable particle, which is defined by the mass of its asymptotic scattering state. M always denotes the pole mass ($\equiv$ on-shell mass).

We note the following properties of Π_ρ:
1) $\Pi_\rho(q^2)$ is complex

$$\Pi_\rho = \text{Re } \Pi_\rho + \text{i Im } \Pi_\rho$$

when $q^2 > 4m_\pi^2$, taking into account only the strong interaction under which ρ decays into pion–pairs. The relevant physical parameters are $M_\rho = (775.8 \pm 0.5)$ MeV, $m_{\pi^\pm} = (139.57018 \pm 0.00035)$ MeV and $\Gamma_\rho = (145.14 \pm 1.50)$ MeV. There is also a tiny direct electromagnetic $\rho \to e^+e^-$ channel which has a branching fraction $\text{Br}(\rho \to e^+e^-) = (4.67 \pm 0.09) \cdot 10^{-5}$, which is not taken into account here.

As disussed earlier, the imaginary part corresponds to the cut diagram



with physical intermediate state. Consequently, the ρ is an unstable particle and on the mass-shell $q^2 = M_\rho^2$ of the ρ we have

$$\text{Im } \Pi_\rho(q^2 = \tilde{M}_\rho^2) = M_\rho \, \Gamma_\rho \neq 0 \tag{8.52}$$

defining the finite width Γ_ρ of the ρ-meson. The real part Re Π_ρ is UV-divergent and requires renormalization: The bare propagator

$$D_{\rho\, b} = \frac{1}{q^2 - M_{\rho\, b}^2}$$

with $M_{\rho\, b}$ the bare ρ mass in modified by self–energy effect to

$$D_{\rho\, b} = \frac{1}{q^2 - M_{\rho\, b}^2 + \Pi_\rho(q^2)}$$

and required mass and wave–function renormalization:

$$M_{\rho b}^2 = M_\rho^2 + \delta M_\rho^2$$

where δM_ρ^2 is the mass counter-term fixed by the condition:

$$\begin{gathered} \text{Re } \left[q^2 - M_\rho^2 - \delta M_\rho^2 + \Pi_\rho(q^2)\right] \Big|_{q^2 = \tilde{M}_\rho^2} = 0 \\ \Updownarrow \\ \delta M_\rho^2 = \text{Re}\, \Pi_\rho(\tilde{M}_\rho^2) \end{gathered} \tag{8.53}$$

this removes the quadratically divergent term from the ρ self-energy, and defines the physical mass by

$$M_\rho^2 = \text{Re}\, \tilde{M}_\rho^2 \ . \tag{8.54}$$

The remaining logarithic UV divergence is removed by the condition that

- the real part of the residue of the propagator pole must be normalized to one.

If we expand the self-energy at the pole

$$\Pi_\rho(q^2) \simeq \Pi_\rho(\tilde{M}_\rho^2) + (q^2 - \tilde{M}_\rho^2)\, \frac{d\Pi_\rho}{\mathrm{d}q^2}(\tilde{M}_\rho^2) + \cdots \; ; \; q^2 \to \tilde{M}_\rho^2$$

we obtain, using $\delta M_\rho^2 = \mathrm{Re}\,\Pi(\tilde{M}_\rho^2)$, $M_\rho \Gamma_\rho = \mathrm{Im}\,\Pi(\tilde{M}_\rho^2)$,

$$\begin{aligned} D_\rho &= \frac{1}{q^2 - M_\rho^2 + \Big(\Pi_\rho(q^2) - \mathrm{Re}\,\Pi_\rho(\tilde{M}_\rho^2)\Big)} \\ &= \frac{1}{q^2 - \tilde{M}_\rho^2}\, \frac{1}{1 + \frac{d\Pi_\rho}{\mathrm{d}q^2}(\tilde{M}_\rho^2)} + O(q^2 - \tilde{M}_\rho^2) \end{aligned}$$

and the residue of the pole can be read off. If we now perform the field renormalization and consider the propagator of the renormalized field $D_{\rho\, ren} = Z_\rho^{-1}\, D_{\rho\, bare}$ i. e.

$$\frac{1}{q^2 - M_\rho^2 + \Pi_{\rho\, ren}(q^2)} = \frac{1}{Z_\rho} \cdot \frac{1}{q^2 - M_\rho^2 + \Big(\Pi_\rho(q^2) - \mathrm{Re}\,\Pi_\rho(\tilde{M}_\rho^2)\Big)} \tag{8.55}$$

which is required to have residue one and thus

$$Z_\rho = \mathrm{Re}\;\; \left[1 + \frac{d\Pi_\rho}{\mathrm{d}q^2}(\tilde{M}_\rho^2)\right]^{-1} \quad . \tag{8.56}$$

All together we have

$$\begin{aligned} D_{\rho\,\mathrm{ren}} &= \frac{1}{q^2 - M_\rho^2 + \Pi_{\rho\, ren}(q^2)} \\ &= \frac{1}{q^2 - \tilde{M}_\rho^2} + O(q^2 - \tilde{M}_\rho^2) \end{aligned}$$

which is he result we were looking for. Note that off-pole corrections may be substantial, in paricular, because $\Gamma_\rho / M_\rho \sim 0.2$ is not small. Also notice that we have not used here that the self–energy is a perturbative correction, as usual in perturbative problems. Neither pQCD nor CHPT do apply here.

In summary: the unstable spin 1 vector meson is described by a propagator exhibiting a pole in the complex q^2-plane. The pole

$$\tilde{M}_\rho^2 \equiv \big(q^2\big)_{\mathrm{pole}} = M_\rho^2 - \mathrm{i}\, M_\rho\, \Gamma_\rho \tag{8.57}$$

is characterized by the correspondence

$$\begin{aligned} \text{physical mass} &\iff \text{real part of location of propagator pole} \\ \text{width} &\iff \text{imaginary part of the location of the pole .} \end{aligned}$$

8.15.4 $\rho^0 - \gamma$–Mixing

The simple relation between the full propagator and the irreducible self-energy only holds if there is no mixing, like for the charged $\rho^\pm$. In the neutral sector, because of $\gamma - \rho^0$ mixing, we cannot

consider the ρ^0 and γ propagators separately. They form a 2×2 matrix propagator, so that (8.50) is modified into[42]

$$s_P - m^2_{\rho^0} - \Pi_{\rho^0\rho^0}(s_P) - \frac{\Pi^2_{\gamma\rho^0}(s_P)}{s_P - \Pi_{\gamma\gamma}(s_P)} = 0 \ , \tag{8.60}$$

with $s_P = \tilde{M}^2_{\rho^0}$. The above considereatio remains true if we denote the self-energies by Π_V ($V = \rho^0, \rho^\pm$) with

$$\Pi_{\rho^\pm}(p^2, \cdots) = \Pi_{\rho^+\rho^-}(p^2, \cdots)$$

and

$$\Pi_{\rho^0}(p^2, \cdots) = \Pi_{\rho^0\rho^0}(p^2, \cdots) + \frac{\Pi^2_{\gamma\rho^0}(p^2, \cdots)}{p^2 - \Pi_{\gamma\gamma}(p^2, \cdots)} \ .$$

Thus, formally, the form (8.50) applies for both the $\rho^\pm$ and the ρ^0.

8.15.5 Vector–Meson Production Cross–Sections

The cross–section for spin 1 boson production in e^+e^-–annihilation is described in full deail in Sect. 14 of [1]. We have to replace (8.20) by (8.45) and use the completeness relation

$$\sum_\lambda \varepsilon(p,\lambda)_\mu \, \varepsilon^*(p,\lambda)_\nu = \left(-g_{\mu\nu} - \frac{p_\mu p_\nu}{M^2_\rho}\right)$$

for the massive spin 1 boson polarization vectors $\varepsilon(p,\lambda)$, to calculate the hadronic tensor

$$h_{\mu\nu} = \frac{1}{3}\sum_\lambda \varepsilon(p,\lambda)_\mu \, \frac{M^2_\rho}{2\gamma_\rho}\varepsilon^*(p,\lambda)_\nu \, \frac{M^2_\rho}{2\gamma_\rho} = \left(-g_{\mu\nu} - \frac{p_\mu p_\nu}{M^2_\rho}\right) \frac{M^4_\rho}{2\gamma^2_\rho} \ ,$$

needed the caculate the spin average $\overline{|T|^2}$ in (8.5).

The result for the $e^+e^- \to \rho^0$ cross–section and some useful approximations are the following:

[42]The simplest way to treat this problem is to start from the inverse propagator given by the irreducible self-energies (sum of 1pi diagrams). Again we restrict ourselves to a discussion of the transverse part and we take out a trivial factor $-$i $g_{\mu\nu}$ in order to keep notation as simple as possible. With this convention we have for the inverse $\gamma - \rho$ propagator the symmetric matrix

$$\hat{D}^{-1} = \begin{pmatrix} k^2 + \Pi_{\gamma\gamma}(k^2) & \Pi_{\gamma\rho}(k^2) \\ \Pi_{\gamma\rho}(k^2) & k^2 - M^2_\rho + \Pi_{\rho\rho}(k^2) \end{pmatrix} \tag{8.58}$$

Using 2×2 matrix inversion

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Rightarrow M^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

we find for the propagators

$$\begin{aligned} D_{\gamma\gamma} &= \frac{1}{k^2 + \Pi_{\gamma\gamma}(k^2) - \frac{\Pi^2_{\gamma\rho}(k^2)}{k^2 - M^2_\rho + \Pi_{\rho\rho}(k^2)}} \simeq \frac{1}{k^2 + \Pi_{\gamma\gamma}(k^2)} \\ D_{\gamma\rho} &= \frac{-\Pi_{\gamma\rho}(k^2)}{(k^2 + \Pi_{\gamma\gamma}(k^2))(k^2 - M^2_\rho + \Pi_{\rho\rho}(k^2)) - \Pi^2_{\gamma\rho}(k^2)} \simeq \frac{-\Pi_{\gamma\rho}(k^2)}{k^2\,(k^2 - M^2_\rho)} \\ D_{\rho\rho} &= \frac{1}{k^2 - M^2_\rho + \Pi_{\rho\rho}(k^2) - \frac{\Pi^2_{\gamma\rho}(k^2)}{k^2 + \Pi_{\gamma\gamma}(k^2)}} \simeq \frac{1}{k^2 - M^2_\rho + \Pi_{\rho\rho}(k^2)} \ . \end{aligned} \tag{8.59}$$

These expressions sum correctly all the reducible bubbles. The approximations indicated are the one-loop results. The extra terms are higher order contributions.

- **Breit-Wigner resonance: field theory version**

The field theoretic form of a Breit-Wigner resonance obtained by the Dyson summation of a massive spin 1 transversal part of the propagator in the approximation that the imaginary part of the self–energy yields the width by $\mathrm{Im}\,\Pi_V(M_V^2) = M_V\Gamma_V$ near resonance.

$$\sigma_{BW}(s) = \frac{12\pi}{M_R^2}\frac{\Gamma_{e^+e^-}}{\Gamma}\frac{s\Gamma^2}{(s-M_R^2)^2 + M_R^2\Gamma^2} \ .$$

- **Breit-Wigner resonance**

The resonance cross–section from a classical non–relativistic Breit-Wigner resonance is given by

$$\sigma_{BW}(s) = \frac{3\pi}{s}\frac{\Gamma\Gamma_{e^+e^-}}{(\sqrt{s}-M_R)^2 + \frac{\Gamma^2}{4}} \ .$$

- **Narrow width resonance**

The narrow width approximaion for a zero width resonance reads

$$\sigma_{\mathrm{NW}}(s) = \frac{12\pi^2}{M_R}\Gamma_{e^+e^-}\delta(s-M_R^2) \ .$$

Exercise: Derive the non-relativistic and the narrow width approximation from the field theory version of the resonance cross–section.

For broad resonances the different parametrizations of the resonance in general yield very different results. Therefore, it is important to know how a resonance was parametrized to get the resonance parameters like M_R and Γ. For narrow resonances, which we will have to deal with later, results are not affected in a relevant way by using different parametrizations. Note that for the broad non–relativistic ρ meson the classical BW parametrization works well, not however a narrow width approximation.

Due to isospin breaking of the strong interactions ($m_d - m_u$ mass difference as well as electromagnetic effects $Q_u = 2/3 \neq Q_d = -1/3$) the ρ and ω mix and more sophisticated parametrizations must be applied, like the Gounaris-Sakurai parametrization [56] based on the vector meson dominance (VMD) model. More appropriate is a parametrization which relies on first principle concepts only, the description by unitarity, analyticity and constrained by chiral perturbation theory. In the following we will quantify these issues in more detail.

8.15.6 $\rho^0 - \omega$–Mixing

The electromagnetic form factor of the pion $F_\pi(s)$ usually is defined in an idealized world of strong interactions with two quark flavors (u and d) only, and electroweak interactions switched off. In $\gamma^* \to \pi^+\pi^-$ the pion form factor $F_\pi(s)$ is defined as a matrix element of the external electromagnetic current and has an isovector part $I = 1$ as well as an isoscalar part $I = 0$. The latter is due to isospin breaking by the mass difference of the u and d quarks: $m_u - m_s \neq 0$, which leads to $\rho - \omega$ mixing:

$$|\rho\rangle = |\rho_0\rangle - \varepsilon|\omega_0\rangle \ , |\omega\rangle = |\omega_0\rangle + \varepsilon|\rho_0\rangle \ , \tag{8.61}$$

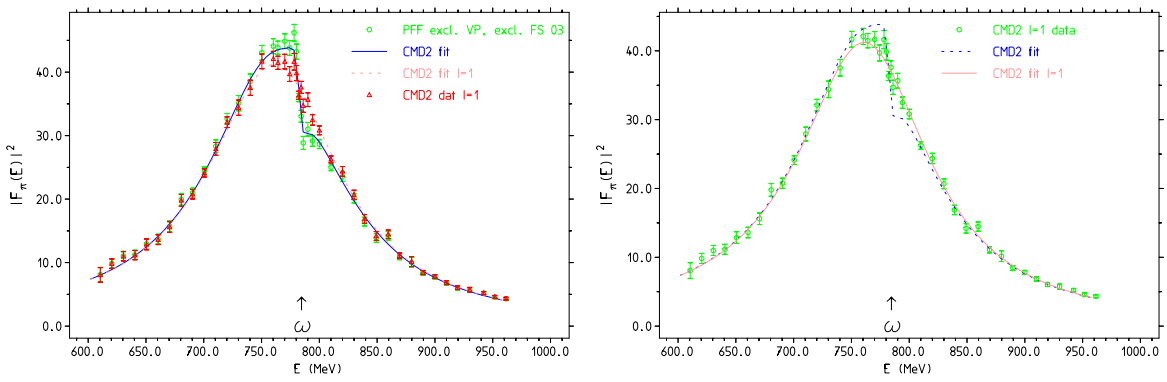


Figure 8.17: CMD-2 data for $|F_\pi|^2$ in $\rho-\omega$ region together with Gounaris-Sakurai fit. Left before subtraction right after subtraction of the ω.

where $|\omega_0\rangle$ and $|\rho_0\rangle$ are the pure isoscalar and isovector states, respectively, and ε is the $\rho-\omega$ mixing parameter. Then, in the energy region close to the $\rho(776)$– and $\omega(783)$–meson masses, the form factor can be written as

$$\begin{aligned} F_\pi(s) &\simeq \left[\frac{F_\rho}{s-M_\rho^2}+\varepsilon\,\frac{F_\omega}{s-M_\omega^2}\right]\left[\frac{F_\rho}{-M_\rho^2}+\varepsilon\,\frac{F_\omega}{-M_\omega^2}\right]^{-1} \\ &\approx -\frac{M_\rho^2}{s-M_\rho^2}\left[1+\varepsilon\,\frac{F_\omega(M_\omega^2-M_\rho^2)s}{F_\rho M_\omega^2(s-M_\omega^2)}\right], \end{aligned} \tag{8.62}$$

where we only keep the terms linear in ε. The quantities M_ω and M_ρ are complex and contain the corresponding widths.

The mixing is responsible for the typical distortion of the ρ–resonance (see Figs. 8.17 and 8.14), which originally would be a pure isospin $I=1$ Breit-Wigner type resonance. The pion form factor (8.62) is the basic ansatz for the Gounaris-Sakurai formula [56] which is often used to represent experimental data by a phenomenological fit (see e.g. [24]).

8.15.7 Gounaris-Sakurai Parametrization of $F_\pi(s)$

Based on VMD ideas and $\rho-\omega$–mixing, Gounaris and Sakurai [56] proposed as a parameterization of F_π (see [25, 56] for details) (FSR not included; BW=Breit-Wigner; $\mathrm{BW}^{\mathrm{GS}}$=Gounaris-Sakurai modified Breit-Wigner):

$$F_\pi(s)=\frac{\mathrm{BW}^{\mathrm{GS}}_{\rho(770)}(s)\cdot\left(\frac{1+\delta\,\mathrm{BW}_{\omega(783)}(s)}{1+\delta}\right)+\beta\,\mathrm{BW}^{\mathrm{GS}}_{\rho(1450)}(s)+\gamma\,\mathrm{BW}^{\mathrm{GS}}_{\rho(1700)}(s)}{1+\beta+\gamma}, \tag{8.63}$$

where we included also the higher resonances ρ' and ρ''. This formula often is used to represent experimental pion–pair production data. We may also use this formula to extract the isovector part of the square of the pion form factor $|F_\pi|^{2\,I=1}_{(e^+e^-)}(s)$ by setting the mixing parameter $\delta=0$. The result is displayed in Fig. 8.17, and illustrates the effect of $\rho-\omega$ mixing at the ρ–resonance. For the $\rho(770)$, $\rho(1450)$ and $\rho(1700)$ the GS parametrization is used [56]:

$$\mathrm{BW}^{\mathrm{GS}}_{\rho(\mathrm{M}_\rho)}=\frac{\mathrm{M}_\rho^2\,(1+d\cdot\Gamma_\rho/\mathrm{M}_\rho)}{\mathrm{M}_\rho^2-s+f(s)-\mathrm{i}\mathrm{M}_\rho\Gamma_\rho(s)},\quad \text{where}$$

where $\mathrm{M}_\rho, \Gamma_\rho$ denote mass and width of the ρ–meson and

$$f(s) = \Gamma_\rho \frac{\mathrm{M}_\rho^2}{p_\pi^3(\mathrm{M}_\rho^2)} \left[p_\pi^2(s) \left(h(s) - h(\mathrm{M}_\rho^2) \right) + (\mathrm{M}_\rho^2 - s)\, p_\pi^2(\mathrm{M}_\rho^2) \left. \frac{dh}{ds} \right|_{s=\mathrm{M}_\rho^2} \right] \ . \tag{8.64}$$

Here $p_\pi(s) = \frac{1}{2}\sqrt{s}\beta_\pi(s)$ denotes the pion momentum with $\beta_\pi = \sqrt{1 - 4m_\pi^2/s}$ the pion velocity in the c.m. frame,

$$h(s) = \frac{2}{\pi} \frac{p_\pi(s)}{\sqrt{s}} \ln \frac{\sqrt{s} + 2p_\pi(s)}{2m_\pi} \ , \tag{8.65}$$

such that

$$\left. \frac{\mathrm{d}h}{\mathrm{d}s} \right|_{s=M_\rho^2} = h(M_\rho^2) \left[(8p_\pi^2(\mathrm{M}_\rho^2))^{-1} - (2\mathrm{M}_\rho^2)^{-1} \right] + (2\pi \mathrm{M}_\rho^2)^{-1} \ . \tag{8.66}$$

The constant d is chosen to satisfy $\mathrm{BW}^{\mathrm{GS}}_{\rho(\mathrm{M}_\rho)}(0) = 1$:

$$d = \frac{3}{\pi} \frac{m_\pi^2}{p_\pi^2(\mathrm{M}_\rho^2)} \ln \frac{\mathrm{M}_\rho + 2p_\pi(\mathrm{M}_\rho^2)}{2m_\pi} + \frac{\mathrm{M}_\rho}{2\pi p_\pi(\mathrm{M}_\rho^2)} - \frac{m_\pi^2 \mathrm{M}_\rho}{\pi p_\pi^3(\mathrm{M}_\rho^2)} \ . \tag{8.67}$$

For the energy dependence of the $\rho(770)$ width, the P-wave phase space is taken:

$$\Gamma_\rho(s) = \Gamma_\rho \left[\frac{p_\pi(s)}{p_\pi(\mathrm{M}_\rho^2)} \right]^3 \left[\frac{\mathrm{M}_\rho^2}{s} \right]^{1/2} \ . \tag{8.68}$$

For the $\omega(782)$-meson contribution the simple Breit-Wigner parametrization with a constant width is used. We further note that the leptonic width is given by

$$\Gamma_{\rho \to e^+e^-} = \frac{2\alpha^2 p_\pi^3 \left(\mathrm{M}_\rho^2 \right)}{9\mathrm{M}_\rho \Gamma_\rho} \frac{(1 + d \cdot \Gamma_\rho/\mathrm{M}_\rho)^2}{(1+\beta)^2}. \tag{8.69}$$

and the branching ratio of the $\omega \to \pi^+\pi^-$ decay by

$$Br(\omega \to \pi^+\pi^-) = \frac{2\alpha^2 p_\pi^3(\mathrm{M}_\omega^2)}{9\mathrm{M}_\omega \Gamma_{\omega \to e^+e^-} \Gamma_\omega} \left| \mathrm{BW}^{\mathrm{GS}}_{\rho(770)}(\mathrm{M}_\omega^2) \right|^2 \frac{|\delta|^2}{(1+\beta)^2} \ . \tag{8.70}$$

8.15.8 The Theory of the Pion Form Factor in the Threshold Region

We first concentrate on the threshold behavior of $F_\pi(s)$. Experimental data are poor below about 400 MeV because the cross section is suppressed near the threshold by phase space. Here the low energy structure of QCD as represented by chiral perturbation theory (CHPT), which we have outlined in Sect. 3.4, provides at appropriate "from first principles" framework. Applications to the the pion form factor have been discussed in [55, 57, 58].

We are interested in low energy pion production matrix elements of the form

$$\langle \pi^{i_1}(p_1) \ \cdots \ \pi^{i_n}(p_n) \ \mathrm{out} | I_\mu^i(0) | 0 \rangle \tag{8.71}$$

where the current $I_\mu^i = V_\mu^i = \bar{q}\, T^i\, \gamma_\mu q$ when n is even and $I_\mu^i = A_\mu^i = \bar{q}\, T^i\, \gamma_\mu \gamma_5 q$ when n is odd. V_μ^i and A_μ^i are the vector and axial vector currents, respectively, belonging to the chiral flavor group $SU(2)_V \otimes SU(2)_A$, which is exact only in the chiral limit when quark masses are taken to be zero. The $SU(2)$ generators are given by $T^i = \frac{\sigma^i}{2}$ in terms of the Pauli matrices σ^i. In

the chiral limit the chiral group is spontaneously broken down to $SU(2)_V$ and according to the Goldstone theorem three massless Goldstone bosons must exist: the pions: $\pi^\pm,\ \pi^0$. The broken phase is characterized by the light quark condensates $\langle\bar{q}q\rangle \neq 0 \quad q = u, d$ as order parameters. If quark masses are switched on the pions acquire a mass, which however remains small because the u and d quarks turn out to be surprisingly light in the real world (see (3.14)). At physical quark masses the value of the condensate is estimated to be $\langle m_q\bar{q}q\rangle \sim -(0.098\ \text{GeV})^4$ for $q = u, d$.

The isospin symmetry limit corresponds to $m_u = m_d = \hat{m} = \frac{1}{2}(m_u + m_d)$ which in most practical cases is a rather good approximation as $m_u - m_d$ is small. Since the quark charges are $Q_u = 2/3$ and $Q_d = -1/3$, isospin is broken by electromagnetism at $O(\alpha)$. More precisely, in the SM the charge is given by the Gell-Mann–Nishijima relation $Q = T_3 + \frac{Y}{2}$, and thus the electromagnetic current is of the form

$$j^{\mu\,\text{had}}_{\text{em}} = \bar{\psi}\gamma^\mu\,\frac{Y + \sigma_3}{2}\,\psi$$

with $\psi = \binom{u}{d}$ the 1st family quark isospin doublet. The form shows that it is a mixture of an isoscalar part proportional to the hypercharge Y and an isovector part proportional to the 3rd component of an isovector. Similarly, the difference in the weak isospin charges $T_{3u} = 1/2$ and $T_{3d} = -1/2$ leads to isospin breaking by the weak interactions. Typical isospin violation effects are the $m_{\pi^\pm} - m_{\pi^0}$ electromagnetic mass difference or $\rho - \omega$–mixing, discussed below.

a) The vector form factor in CHPT:
The hadronic matrix element of interest in pion pair production is

$$\langle\pi^i(p_1)\pi^j(p_2)\,\text{out}|V^k_\mu(0)|0\rangle = \mathrm{i}\varepsilon^{ijk}\,(p_1 - p_2)_\mu\,F_V(s) \tag{8.72}$$

with $s = (p_1 + p_2)^2$. The vector form factor $F_V(s)$ has been calculated in CHPT in [55] (one–loop), [57] (two–loop numerical) and [58] (two–loop analytical). The last reference gives a compact analytical result

$$F_V(\,s\,) = 1 + \frac{1}{6}\langle r^2\rangle^\pi_V\,s\, + c^\pi_V\,s^2 + f^{\rm U}_V\left(\frac{s}{m_\pi^2}\right) \tag{8.73}$$

$$\begin{aligned}
f^{\rm U}_V(\,x\,) &= \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left\{\frac{x}{9}\left(1 + 24\pi^2\sigma^2\bar{J}(x)\right) - \frac{x^2}{60}\right\} \\
&+ \left(\frac{m_\pi^2}{16\pi^2 f_\pi^2}\right)^2\left\{\left[\bar{l}_2 - \bar{l}_1 + \frac{\bar{l}_6}{2} + \frac{6\bar{l}_4}{x}\right]\frac{x^2}{27}\left(1 + 24\pi^2\sigma^2\bar{J}(x)\right) - \frac{x^2}{30}\bar{l}_4\right. \\
&+ \frac{3191}{6480}\,x^2 + \frac{223}{216}\,x - \frac{16}{9} - \frac{\pi^2\,x}{540}\,(\,37\,x + 15\,) \\
&+ \frac{4\pi^2}{27}\,(\,7\,x^2 - 151\,x + 99\,)\,\bar{J}(x) + \frac{2\pi^2}{9\,x}\,(\,x^3 - 30\,x^2 + 78\,x - 128\,)\bar{K}_1(x) \\
&+ \left.8\pi^2\left(x^2 - \frac{13}{3}\,x - 2\right)\,\bar{K}_4(x)\right\}.
\end{aligned} \tag{8.74}$$

Where we have used the following functions:

$$\begin{aligned}
\bar{J}(\,x\,) &= \frac{1}{16\pi^2}\,(\,F(\,x\,) + 2\,)\ , \\
\bar{K}_1(\,x\,) &= \frac{1}{16\pi^2}\frac{F^2(x)}{\sigma^2}\ , \\
\bar{K}_4(\,x\,) &= \frac{1}{16\pi^2}\frac{F(x)}{x\,\sigma^2} + \frac{1}{32\pi^2}\frac{1}{x\,\sigma^2}\left[\frac{F^2(x)}{\sigma^2} + \pi^2\right] \\
&+ \frac{1}{48\pi^2}\frac{1}{x\,\sigma^2}\left\{\frac{1}{x\,\sigma^4}\left[F^3(x) + \pi^2\sigma^2F(x)\right] - \pi^2\right\} + \frac{1}{192} - \frac{1}{32\pi^2}\ ,
\end{aligned} \tag{8.75}$$

with

$$F(\,x\,) = \sigma \ln \frac{\sigma - 1}{\sigma + 1}\ , \quad \sigma = \sqrt{1 - 4/x}\ .$$

The functions $\bar{K}_i(\,x\,)$ and $\bar{J}(\,x\,)$ are analytic everywhere apart from a branch cut from 4 to ∞, and go to zero as $x \to 0$.

All the constants which occur in $f_V^{\rm U}(\,s/m_\pi^2\,)$ are known. The subtraction constants $\langle r^2\rangle_V^\pi$ and c_V^π are calculable in CHPT and can be expressed in terms of the low energy constants $l_i^r(\mu)$, chiral logs, and the new low energy constants which appear in $\mathcal{L}_6$. With this representation of the subtraction constants given by CHPT one automatically satisfies the relevant Ward identities, up to the order at which one is working. We do not give this explicit representation here because up to now there is no information on the numerical value of the new $\mathcal{L}_6$ low energy constants. In the future, with more accurate data on various low energy processes, and more two loop calculation available, one could try to pin down at least some of them, but this will require a considerable amount of work and it is beyond the scope of this introduction. A fit to the space–like NA7 data [59] with the expression (8.73) leaving $\langle r^2\rangle_V^\pi$ and c_V^π as free parameters, and including the theoretical error, leads to

$$\begin{aligned} \langle r^2\rangle_V^\pi &= 0.431 \pm 0.020 \pm 0.016\,{\rm fm}^2 \\ c_V^\pi &= 3.2 \pm 0.5\ \pm 0.9\ {\rm GeV}^{-4} \end{aligned} \tag{8.76}$$

where the first and second errors indicate the statistical and theoretical uncertainties, respectively. The central value of c_V^π is rather close to the value obtained by resonance saturation, $c_V^\pi = 4.1\,{\rm GeV}^{-4}$ [57].

The crucial point here is that the threshold behavior is severely constrained by the chiral structure of QCD via the rather precise data for the pion form factor in the space–like region. The space–like fit provides a good description of the data in the time-like region. Pure chiral perturbation theory is able to make predictions only for the low energy tail of the form factor. However, theory in this case can do much more, by exploiting systematically *analyticity*, *unitarity* and the properties of the *chiral limit*. This relates space–like data, $\pi\pi$–scattering phase shifts and time–like data in a manner leading to severe theoretical constraints! This we will consider next.

b) The pion form factor à la Omnès–Muskhelishvili:
Some basic concepts we need for a "from first principles" study of the pion form factor we already encountered in the first part of this Section. In particular, the role of analyticity and unitarity we have discussed in some detail above. First attempts and ideas to model $F_\pi(s)$ followed in Sects. 8.15.1 and 8.15.2. The electromagnetic form factor of the pion $F_\pi(s)$ in first place is defined in a world with strong interactions and two quark flavors (u and d) only, electroweak interactions are switched off. $F_\pi(s)$ has an isovector part $I = 1$ as well as an isoscalar part $I = 0$. The latter is due to isospin breaking by the mass difference of the u and d quarks: $m_u - m_s \neq 0$, which leads to $\rho - \omega$ mixing outlined before in Sect. 8.15.6. The mixing is responsible for the typical distortion of the ρ–resonance (see Fig. 8.14), which originally would be a pure isospin $I = 1$ Breit-Wigner type resonance. The isospin symmetry limit is also the starting point of our theoretical consideration.

The corresponding electromagnetic vector current form factor $F_\pi(s)$ has the following properties [60, 61]:
i) $F_\pi(s)$ is an analytic function of s in the whole complex s–plane, except for a cut on the positive real axis for $4m_\pi^2 \leq s < \infty$. If we approach the cut from above $s \to s + {\rm i}\varepsilon$, $\varepsilon > 0$, $\varepsilon \to 0$ the form factor remains complex and is characterized by two real functions, the modulus and the phase

$$F_\pi(s) = |F_\pi(s)|\, {\rm e}^{\,{\rm i}\,\delta(s)}\ ; \tag{8.77}$$

ii) analyticity relates Re $F_\pi(s)$ and Im $F_\pi(s)$ by a DR, which may be expressed as a relation between modulus and phase $\delta(s) = \arctan(\mathrm{Im}\, F_\pi(s)/\mathrm{Re}\, F_\pi(s))$, known as the Omnès representation [62]

$$F_\pi(s) = P(s)\, \exp\left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s'\, \frac{\delta(s')}{s'\,(s'-s)} \right\} \tag{8.78}$$

where $P(s)$ is a polynomial, which determines the behavior at infinity, or, equivalently, the number and position of the zeros;
iii) charge conservation $F_\pi(0) = 1$, which fixes $P(0) = 1$;
iv) $F_\pi(s)$ is real below the 2 pion threshold ($-\infty < s < 4m_\pi^2$), which implies that $P(s)$ must be a polynomial with real coefficients;
v) the inelastic threshold is $s_{\rm inel} = 16m_\pi^2$.

The representation (8.78) tells us that once we know the phase on the cut and the location of the zeros of $P(s)$ the form factor is calculable in the entire s–plane. In the elastic region $s \leq s_{\rm inel}$ Watson's theorem[43], exploiting unitarity, relates the phase of the form factor to the P wave phase shift of the $\pi\pi$ scattering amplitude with the same quantum numbers, $I = 1,\ J = 1$:

$$\delta(s) = \delta_1^1(s) \quad \text{for} \quad s \leq s_{\rm inel} = 16m_\pi^2\ . \tag{8.82}$$

However, it is an experimental fact that the inelasticity is negligible until the quasi two–body channels $\omega\pi$, $a_1\pi$, are open, thus in practice one can take (8.82) as an excellent approximation up to about 1 GeV (while $\sqrt{s_{\rm inel}} \simeq 0.56$ GeV). Actually, the phase difference (8.82) satisfies the bound [63]

$$\sin^2(\delta(s) - \delta_1^1(s)) \leq \frac{1}{2}\,[1 - \sqrt{1 - r^2(s)}\,] \quad , \quad r(s) = \frac{\sigma^{\rm I=1}_{\rm non-2\pi}}{\sigma_{e^+e^-\to\pi^+\pi^-}}$$

and $\eta_1 \leq (1-r)/(1+r)$, provided $r < 1$, which holds true below 1.13 GeV (below 1 GeV $r < 0.143 \pm 0.024$, or $\delta - \delta_1^1 \lesssim 6^\circ$, strongly decreasing towards lower energies).

The $\pi\pi$ scattering phase shift is due to elastic re-scattering of the pions in the final state (*final state interaction*) as illustrated by Fig. 8.18

The $\pi\pi$ scattering phase shift has been studied recently in the framework of the Roy equations (see [64]), first with no extra input [65], and also exploiting in addition chiral symmetry [66–69].

[43] The pion isovector form factor is defined by the matrix element

$$\langle out\ \pi^+(p_+)\pi^-(p_-)|j_\mu(0)|0\rangle = -i(p_+ - p_-)_\mu\, F_\pi(s)\ , \tag{8.79}$$

where $j_\mu(x)$ is the electromagnetic vector current and $s = (p_+ + p_-)^2$. The $\pi^+\pi^-$ state in this matrix element, in order not to vanish, must be in a $I = 1,\ J = 1$ (P wave) state, J the angular momentum. If we look at the charge density j_0, time-reversal (T) invariance tells us that

$$\langle out\ \pi^+\pi^-|j_0(0)|0\rangle = \langle in\ \pi^+\pi^-|j_0(0)|0\rangle^*\ , \tag{8.80}$$

as for fixed J only "in" and "out" get interchanged. The complex conjugation follows from the fact that T must be implemented by an anti-unitary transformation. Now, with S the unitary scattering operator, which transforms in and out scattering states according to $|X\ out\rangle = S^+|X\ in\rangle$ (X the label of the state) we have

$$\begin{aligned} \langle out\ \pi^+\pi^-|j_0(0)|0\rangle &= \langle in\ \pi^+\pi^-|Sj_0(0)|0\rangle \\ &= \mathrm{e}^{2\mathrm{i}\delta_{\pi\pi}}\ \langle in\ \pi^+\pi^-|j_0(0)|0\rangle \\ &= \mathrm{e}^{2\mathrm{i}\delta_{\pi\pi}}\ \langle out\ \pi^+\pi^-|j_0(0)|0\rangle^* \end{aligned} \tag{8.81}$$

which implies $F_\pi(s) = \mathrm{e}^{2\mathrm{i}\delta_{\pi\pi}}\ F_\pi^*(s)$. As two pions below the inelastic thresholds may scatter elastically only, by unitarity the S-matrix must be a pure phase in this case. The factor 2 is a convention, $\delta_{\pi\pi}(s)$ is the $\pi\pi$–scattering phase shift.

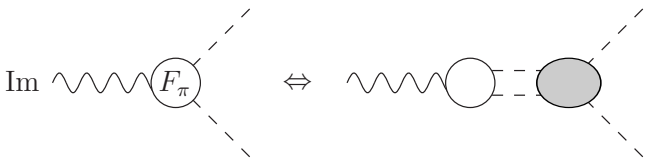


Figure 8.18: Final state interaction due to $\pi\pi \to \pi\pi$ scattering

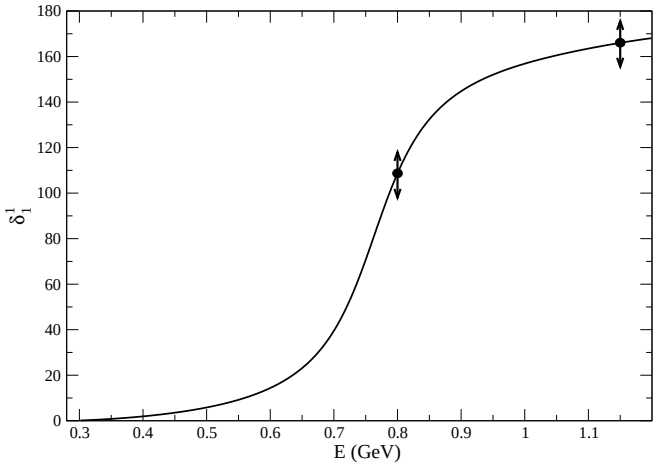


Figure 8.19: The $\pi\pi$ phase shift δ_1^1: the values of the phase at the two points (marked by ⇕) are free; the Roy equation and chiral symmetry completely fix the solution (from [61])

As a result it turns out that $\delta_1^1(s)$

is constrained to a remarkable degree of accuracy up to about $E_0 = 0.8$ GeV (matching point). The behavior of $\delta_1^1(s)$ in the region below the matching point is controlled by three parameters: two S–wave scattering length a_0^0, a_0^2 and by the boundary value $\phi \equiv \delta_1^1(E_0)$. One may treat ϕ as a free parameter and rely on the very accurate predictions for a_0^0, a_0^2 from chiral perturbation theory. This information may be used to improve the accuracy of the pion form factor and thus to reduce the uncertainty of the hadronic contribution to the muon g-2. Similar approaches, however, not using the constraints from CHPT, have been followed in [70, 71]. Before we can apply the outlined method to reality, we have to take into account the isospin breaking. For this purpose we separate off an other factor which takes into account the $I = 0$ contribution due to $\rho - \omega$–mixing. Accordingly we now write the form factor in the form [60, 61]

$$F_\pi(s) = G_1(s) \cdot G_\omega(s) \cdot G_2(s) \ .$$

• $G_1(s)$ is the Omnès-Muskhelishvili factor describing the cut due to the 2π intermediate states:

$$G_1(s) = \exp\left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s' \, \frac{\delta_1^1(s')}{s'\,(s'-s)} \right\} \ . \tag{8.83}$$

• $G_\omega(s)$ accounts for the ω–pole contribution due to $\rho - \omega$–mixing with mixing amplitude ε:

$$G_\omega(s) = 1 + \varepsilon \frac{s}{s_\omega - s} + \ldots \qquad s_\omega = (M_\omega - \frac{1}{2}\mathrm{i}\Gamma_\omega)^2 \ .$$

In order to get it real below the physical thresholds we use an energy dependent width

$$\begin{aligned}\Gamma_\omega(s) &= \Gamma(\omega\to 3\pi,s)+\Gamma(\omega\to\pi^0\gamma,s)+\Gamma(\omega\to 2\pi,s)\\ &= \frac{s}{M_\omega^2}\Gamma_\omega\left\{ Br(\omega\to 3\pi)\frac{F_{3\pi}(s)}{F_{3\pi}(M_\omega^2)}+Br(\omega\to\pi^0\gamma)\frac{F_{\pi\gamma}(s)}{F_{\pi\gamma}(M_\omega^2)}+Br(\omega\to 2\pi)\frac{F_{2\pi}(s)}{F_{2\pi}(M_\omega^2)}\right\}\end{aligned}$$

where $Br(V\to X)$ denotes the branching fraction for the channel X and $F_X(s)$ is the phase space function for the corresponding channel normalized such that $F_X(s)\to const$ for $s\to\infty$. For the two-body decays $V\to P_1P_2$ we have $F_{P_1P_2}(s)=(1-(m_1+m_2)^2/s)^{3/2}$. The channel $V\to 3\pi$ is dominated by $V\to\rho\pi\to 3\pi$ and this fact is used when calculating $F_{3\pi}(s)$ [72].

Low energy singularities generated by states with 2 or 3 pions are accounted for by the first two factors of the "master equation" above. The remaining function $G_2(s)$ represents the smooth background that contains the curvature generated by the remaining singularities. The 4 π channel opens at $s=16\ m_\pi^2$ but phase space strongly suppresses the strength of the corresponding branch point singularity of the form $(1-s_{\rm inel}/s)^{9/2}$ – a significant inelasticity only manifests itself for $s>s_{\rm in}=(M_\omega+m_\pi)^2$. The conformal mapping

$$z=\frac{\sqrt{s_{\rm in}-s_1}-\sqrt{s_{\rm inel}-s}}{\sqrt{s_{\rm inel}-s_1}+\sqrt{s_{\rm in}-s}}$$

maps the plane cut along $s>s_{\rm inel}$ onto the unit disk in the z–plane. It contains a free parameter s_1 - the value of s which gets maps into the origin. We find that if s_1 is taken negative and sufficiently far from the origin the fit becomes rather insensitive to the details of the parametrization. In the following we set $s_1=-1.0$ GeV^2. We approximate $G_2(s)$ by a polynomial in z:

$$G_2(s)=1+\sum_{i=1}^{n_P}c_i\,(z^i-z_0^i)\ ,$$

where z_0 is the image of $s=0$. The shift of z by $z\to z-z_0$ is required to preserves the charge normalization condition $G_2(0)=1$. The form of the branch point singularity $(1-s_{\rm inel}/s)^{9/2}$ imposes four constraints on the polynomial; a non-trivial contribution from $G_2(s)$ thus requires a polynomial of fifth order at least. Varying the degree n_P of the polynomial and constraining the free coefficients by the data is the only model uncertainty. In this way we let the data fix the location of the zeros in the complex s–plane of the Omnès representation (8.78).

Numerical results:
The above representation of the vector form factor involves a product of three functions each of which has free parameters which are fitted by the data. The free parameters are the following:

- in G_1 we have two free parameters, the values of the phase δ_1^1 at 0.8 GeV and at 1.15 GeV;
- in G_2 we have n_P-4 parameters, where n_P is the degree of the polynomial in the conformal variable z. Varying n_P within a reasonable range $n_P=5,\cdots,8$ will allow us to estimate the model uncertainty;
- in G_ω we have in principle three free parameters: ε_ω, M_ω and Γ_ω – however, since mass and width of the ω are rather well known from other experiments, we take their values from the PDG. Since the ω–resonance is very narrow G_ω is very sensitive to the energy calibration, which therefore we allow to vary within the experimental systematic uncertainty.

Altogether we have $4+(n_P-4)$ free parameters, depending on the degree of the polynomial used to describe the inelastic effects.

For results we refer to [60, 61]. This method provides a systematic way to combine experimental information from $|F_\pi(s)|^2$ measurements with corresponding information of the phase of the pion form factor.

8.15.9 Beyond Chiral Perturbation Theory: Models with Massive Spin 1 Fields

Chiral perturbation theory does not explicitly account for resonances like the ρ mesons, which in a way may be considered as $\pi\pi$ bound states. In order to extend the applicability of CHPT towards higher energies one has to abandon the pure first principles path of low energy QCD and go to an effective theory which explicitly takes into account vector meson resonances and combines CHPT with the Vector Dominance Model (VMD) in a consistent manner. Such a framework has been developed to $\mathcal{O}\left(p^4\right)$ in [52, 53]

$$\begin{aligned}\mathcal{L} &= \frac{f_\pi^2}{4}\mathrm{Tr}(D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &\quad -\frac{1}{2}\mathrm{Tr}(\nabla^\lambda V_{\lambda\mu}\nabla_\nu V^{\nu\mu} - \frac{1}{2}m_\rho^2 V_{\mu\nu}V^{\mu\nu}) - \frac{1}{2}\mathrm{Tr}(\nabla^\lambda A_{\lambda\mu}\nabla_\nu A^{\nu\mu} - \frac{1}{2}m_a^2 A_{\mu\nu}A^{\mu\nu}) \\ &\quad +\frac{F_V}{2\sqrt{2}}\mathrm{Tr}(V_{\mu\nu}f_+^{\mu\nu}) + \frac{iG_V}{\sqrt{2}}\mathrm{Tr}(V_{\mu\nu}u^\mu u^\nu) + \frac{F_A}{2\sqrt{2}}\mathrm{Tr}(A_{\mu\nu}f_-^{\mu\nu}),\end{aligned} \tag{8.84}$$

where $U = \exp(i\sqrt{2}\Phi/f_\pi)$, Φ describes the $SU(3)$ octet of pseudoscalar mesons, $V_{\mu\nu}$ ($A_{\mu\nu}$) is antisymmetric field describing the $SU(3)$ octet of polar-vector (axial-vector) mesons, and $F^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$ is the electromagnetic field strength tensor, where the photon field is denoted by B^μ. Further, $f_\pi,\ F_V, G_V, F_A$ are constants, whose numerical values are specified in [52, 53]. More details concerning definitions and notation the reader may find in the original papers. For variants of resonance Lagrangian approach we refer to [73–76].

The extension to $\mathcal{O}\left(p^6\right)$ including P odd anomalous terms has been worked out in [77]. It reads

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} \\ \mathcal{L}^{(2)} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(\mathrm{i}\not{D} - m\right)\psi + e^2\, C\,\langle \hat{Q}U^+\hat{Q}U\rangle \\ &\quad +\frac{F_0^2}{4}\left(\langle d^\mu U^+ d_\mu U\rangle + 2B_0\langle \mathcal{M}^+U + U^+\mathcal{M}\rangle\right) \\ \mathcal{L}^{(4)} &= \frac{\mathrm{i}\,e^2 N_c}{24\pi^2}\,\varepsilon_{\mu\nu\alpha\beta}\,\partial^\mu a^\nu A^\alpha \langle \hat{Q}^2\partial^\beta U U^+ + \hat{Q}^2 U^+\partial^\beta U \\ &\quad -\frac{1}{2}\hat{Q}U\hat{Q}\partial^\beta U^+ + \frac{1}{2}\hat{Q}U^+\hat{Q}\partial^\beta U\rangle + \cdots \\ &= -\frac{\alpha N_c}{12\pi F_0}\,\varepsilon_{\mu\nu\alpha\beta}\,F^{\mu\nu}A^\alpha\partial^\beta\pi^0 + \cdots \\ \mathcal{L}^{(6)} &= \frac{3\,\mathrm{i}\,\alpha^2}{32\pi^2}\bar{\psi}\gamma_\mu\gamma_5\psi\,\Big\{\chi_1\,\langle\hat{Q}^2\,\left(U^+d^\mu U + d^\mu U U^+\right)\rangle \\ &\quad \chi_2\,\langle \hat{Q}U^+\hat{Q}d^\mu U - \hat{Q}d^\mu U^+\hat{Q}U\rangle\Big\} + \cdots \\ &+ \frac{\alpha^2}{4\pi^2 F_0}\,\chi\,\bar{\psi}\gamma_\mu\gamma_5\psi\,\partial^\mu\pi^0 + \cdots\end{aligned} \tag{8.85}$$

where $\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)$ is the mass matrix, $\hat{Q} = \mathrm{diag}(2/3, -1/3, -1, 3)$ is the charge matrix and $\chi = -(\chi_1 + \chi_2)/4$.

Exercises: Section ??

① Calculate the exact Born cross section for $e^+e^- \to \mu^+\mu^-$, by replacing the hadronic tensor $h_{\mu\nu}$ by the leptonic one $\ell_{\mu\nu}$ of the muon. What is the difference to $e^+e^- \to e^+e^-$?

② Verify that the subtraction changes the DR of the form (8.13), assumed to converge or supplemented with a cut–off, to (8.14), which we call a subtracted DR.

③ Discuss the positivity properties of the functional $\dot{P}(s)$ defined as the derivative of

$$P(s) = \not\int_{s_0}^{\infty} \mathrm{d}s' \, \frac{R(s')}{s'(s'-s)} \ ,$$

where $R(s)$ is a positive ordinary function. As a distribution (or linear functional) $P(s)$ has a derivative. Formally,

$$\dot{P}(s) = \frac{\mathrm{d}P(s)}{\mathrm{d}s} = \not\int \mathrm{d}s' \, \frac{R(s')}{(s'-s)^2} \ .$$

is an integral over an integrand which is almost everywhere positive.

④ Derive the non-relativistic and the narrow width approximation from the field theory version of the vector–meson resonance cross–section.

References

[1] B. Richter, in *Proceedings of the XVII International Conference on High Energy Physics*, London (1974).

[2] J. Ellis, in *Proceedings of the XVII International Conference on High Energy Physics*, London (1974).

[3] H. Fritzsch and H. Leutwyler, *Current Commutators In The Hadronic Vacuum And E+ E- Annihilation*, CALT-68-416.

[4] G. Källén, A. Sabry, K. Dan. Vidensk. Selsk. Mat.-Fys. Medd. **29** (1955) No. 17

[5] M. Steinhauser, Phys. Lett. B**429** (1998) 158

[6] E. D. Bloom, F. J. Gilman, Phys. Rev. Lett. **25** (1970) 1140;
Phys. Rev. D **4** (1971) 2901

[7] E. C. Poggio, H. R. Quinn, S. Weinberg, Phys. Rev. D **13** (1976) 1958

[8] M. A. Shifman, *Quark–hadron duality*, hep-ph/0009131

[9] P. A. M. Dirac, Théorie du positron. In: *Septième Conseile de Physique Solvay: Structure et propriétés des noyaux atomique, October 22-29, 1933*, (Gauthier-Villars, Paris 1934), pp.203-230;
P. A. M. Dirac, Proc. Cambridge. Phil. Soc. **30** (1934) 150

[10] J. Schwinger, Phys. Rev. **75** (1949) 651

[11] R. P. Feynman, Phys. Rev. **76** (1949) 749

[12] R. Jost, J. M. Luttinger, Helv. Phys. Acta **23** (1950) 201

[13] H. Fritzsch, Phys. Rev. D **10** (1974) 1624;
H. Fritzsch, H. Leutwyler, Report CALT-68-416, 1974

[14] T. Appelquist, H. Georgi, Phys. Rev. D **8** (1973) 4000;
A. Zee, Phys. Rev. D **8** (1973) 4038

[15] K. G. Chetyrkin, A. L. Kataev, F. V. Tkachov, Phys. Lett. B **85** (1979) 277;
M. Dine, J. Sapirstein, Phys. Rev. Lett. **43** (1979) 668; W. Celmaster, R. J. Gonsalves, Phys. Rev. Lett. **44** (1980) 560;
K. G. Chetyrkin, Phys. Lett. B **391** (1997) 402

[16] S. G. Gorishnii, A. L. Kataev, S. A. Larin, Phys. Lett. B **259** (1991) 144;
L. R. Surguladze, M. A. Samuel, Phys. Rev. Lett. **66** (1991) 560 [Erratum-ibid. **66** (1991) 2416]

[17] V. A. Novikov, L. B. Okun, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, V. I. Zakharov, Phys. Rept. **41** (1978) 1

[18] F. Jegerlehner, Nucl. Phys. Proc. Suppl. **162** (2006) 22; Nucl. Phys. Proc. Suppl. **181-182** (2008) 135 [arXiv:0807.4206 [hep-ph]].

[19] K. G. Chetyrkin, J. H. Kühn, Phys. Lett. B **342** (1995) 356;
K. G. Chetyrkin, J. H. Kühn, A. Kwiatkowski, Phys. Rept. **277** (1996) 189;
K. G. Chetyrkin, J. H. Kühn, M. Steinhauser, Phys. Lett. B **371**, 93 (1996); Nucl. Phys. B **482**, 213 (1996); **505**, 40 (1997);
K.G. Chetyrkin, R. Harlander, J.H. Kühn, M. Steinhauser, Nucl. Phys. B **503**, 339 (1997)
K. G. Chetyrkin, R. V. Harlander, J. H. Kühn, Nucl. Phys. B **586** (2000) 56 [Erratum-ibid. B **634** (2002) 413]

[20] R. V. Harlander, M. Steinhauser, Comput. Phys. Commun. **153** (2003) 244

[21] M. R. Whalley, J. Phys. G **29** (2003) A1

[22] A. E. Blinov et al., Z. Phys. C **70** (1996) 31

[23] J. Z. Bai et al. [BES Collaboration], Phys. Rev. Lett. **84** (2000) 594; Phys. Rev. Lett. **88** (2002) 101802

[24] R. R. Akhmetshin et al. [CMD-2 Collaboration], Phys. Lett. B **578** (2004) 285

[25] R. R. Akhmetshin et al [CMD-2 Collaboration], Phys. Lett. B **527**, 161 (2002)

[26] V. M. Aulchenko et al. [CMD-2 Collaboration], JETP Lett. **82** (2005) 743 [Pisma Zh. Eksp. Teor. Fiz. **82** (2005) 841]; R. R. Akhmetshin et al., JETP Lett. **84** (2006) 413 [Pisma Zh. Eksp. Teor. Fiz. **84** (2006) 491]; Phys. Lett. B **648** (2007) 28

[27] A. Aloisio et al. [KLOE Collaboration], Phys. Lett. B **606** (2005) 12

[28] M. N. Achasov et al. [SND Collaboration], J. Exp. Theor. Phys. **103** (2006) 380 [Zh. Eksp. Teor. Fiz. **130** (2006) 437]

[29] B. Aubert et al. [BABAR Collaboration], Phys. Rev. D **70** (2004) 072004; **71** (2005) 052001; **73** (2006) 012005; **73** (2006) 052003

[30] S. L. Adler, Phys. Rev. D **10** (1974) 3714;
A. De Rujula, H. Georgi, Phys. Rev. D **13** (1976) 1296

[31] G. Sterman, S. Weinberg, Phys. Rev. Lett. **39** (1977) 1436

[32] K. G. Wilson, Phys. Rev. **179** (1969) 1499;
K. G. Wilson, J. B. Kogut, Phys. Rept. **12** (1974) 75;
V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. B **249**, 445 (1985) [Yad. Fiz. **41**, 1063 (1985)]

[33] M. A. Shifman, World Sci. Lect. Notes Phys. **62** (1999) 1 (Chap. 1,2)

[34] M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. B **147** (1979) 385

[35] F. Jegerlehner, Z. Phys. C **32** (1986) 195

[36] M. Davier, A. Höcker, Phys. Lett. B **419** (1998) 419; Phys. Lett. B **435** (1998) 427

[37] S. Eidelman, F. Jegerlehner, A. L. Kataev, O. Veretin, Phys. Lett. B **454** (1999) 369

[38] R. J. Crewther, Phys. Rev. Lett. **28** (1972) 1421

[39] M. S. Chanowitz, J. R. Ellis, Phys. Rev. D **7** (1973) 2490

[40] S. L. Adler, J. C. Collins, A. Duncan, Phys. Rev. D **15** (1977) 1712;
J. C. Collins, A. Duncan, S. D. Joglekar, Phys. Rev. D **16** (1977) 438

[41] K. G. Chetyrkin, J. H. Kühn, M. Steinhauser, Nucl. Phys. B **482** (1996) 213; Nucl. Phys. B **505** (1997) 40.

[42] K. G. Chetyrkin, R. Harlander, J. H. Kühn, M. Steinhauser, Nucl. Phys. B **503** (1997) 339.

[43] F. Jegerlehner, O. V. Tarasov, Nucl. Phys. B **549** (1999) 481.

[44] S. G. Gorishnii, A. L. Kataev, S. A. Larin, Phys. Lett. B **259** (1991) 144.

[45] L. R. Surguladze, M. A. Samuel, Phys. Rev. Lett. **66** (1991) 560 [Erratum-ibid. **66** (1991) 2416].

[46] J. H. Kühn, these proceedings; P. A. Baikov, K. G. Chetyrkin, J. H. Kühn, Phys. Rev. Lett. **88** (2002) 012001; Phys. Rev. D **67** (2003) 074026; Phys. Lett. B **559** (2003) 245; arXiv:0801.1821 [hep-ph].

[47] J. J. Sakurai, Annals of Phys. (NY) **11** (1960) 1; H. Joos, Acta Phys. Austriaca Suppl. **4** (1967); N. M. Kroll, T. D. Lee, B. Zumino, Phys. Rev. **157** (1967) 1376

[48] G. 't Hooft, Nucl. Phys. B **72** (1974) 461; ibid. **75** (1974) 461; E. Witten, Nucl. Phys. B **160** (1979) 57

[49] A. V. Manohar, Hadrons in the $1/N$ Expansion, In: *At the frontier of Particle Physics*, ed M. Shifman, (World Scientific, Singapore 2001) Vol. 1, pp 507–568

[50] H. Leutwyler, Nucl. Phys. Proc. Suppl. **64** (1998) 223; R. Kaiser, H. Leutwyler, Eur. Phys. J. C **17** (2000) 623

[51] B. V. Geshkenbein, V. L. Morgunov, arXiv:hep-ph/9410252

[52] G. Ecker, J. Gasser, A. Pich, E. de Rafael, Nucl. Phys. B **321** (1989) 311

[53] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, E. de Rafael, Phys. Lett. B **223** (1989) 425

[54] J. Gasser, H. Leutwyler, Ann. of Phys. (NY) **158** (1984) 142

[55] J. Gasser, H. Leutwyler, Nucl. Phys. B **250** (1985) 517

[56] G. J. Gounaris, J. J. Sakurai, Phys. Rev. Lett. **21** (1968) 244

[57] J. Gasser, U.-G. Meißner, Nucl. Phys. B **357** (1991) 90

[58] G. Colangelo, M. Finkemeier, R. Urech, Phys. Rev. D **54** (1996) 4403

[59] S. R. Amendolia et al. [NA7 Collaboration], Nucl. Phys. B **277** (1986) 168

[60] H. Leutwyler, hep-ph/0212324

[61] G. Colangelo, Nucl. Phys. Proc. Suppl. **131** (2004) 185; ibid. **162** (2006) 256

[62] R. Omnès, Nuovo Cim. **8** (1958) 316

[63] L. Łukaszuk, Phys. Lett. B **47** (1973) 51; S. Eidelman, L. Łukaszuk, Phys. Lett. B **582** (2004) 27

[64] B. Hyams et al., Nucl. Phys. B **64** (1973) 134

[65] B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rept. **353** (2001) 207

[66] G. Colangelo, J. Gasser, H. Leutwyler, Nucl. Phys. B **603** (2001) 125

[67] I. Caprini, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rev. D **68** (2003) 074006

[68] B. Ananthanarayan, I. Caprini, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Lett. B **602** (2004) 218

[69] H. Leutwyler, hep-ph/0612111; hep-ph/0612112

[70] J. A. Casas, C. Lopez, F. J. Ynduráin, Phys. Rev. D **32** (1985) 736.

[71] B. V. Geshkenbein, Phys. Rev. D **61** (2000) 033009

[72] F. M. Renard, Nucl. Phys. B **82** (1974) 1; N. N. Achasov, N. M. Budnev, A. A. Kozhevnikov, G. N. Shestakov, Sov. J. Nucl. Phys. **23** (1976) 320; N. N. Achasov, A. A. Kozhevnikov, M. S. Dubrovin, V. N. Ivanchenko, E. V. Pakhtusova, Int. J. Mod. Phys. A **7** (1992) 3187

[73] M. Bando, T. Kugo, S. Uehara, K. Yamawaki, T. Yanagida, Phys. Rev. Lett. **54** (1985) 1215; M. Bando, T. Kugo, K. Yamawaki, Phys. Rept. **164** (1988) 217; M. Harada, K. Yamawaki, Phys. Rept. **381** (2003) 1

[74] U. G. Meissner, Phys. Rept. **161** (1988) 213; see also S. Weinberg, Phys. Rev. **166** (1968) 1568

[75] A. Dhar, R. Shankar, S. R. Wadia, Phys. Rev. D **31** (1985) 3256;
D. Ebert, H. Reinhardt, Phys. Lett. B **173** (1986) 453

[76] J. Prades, Z. Phys. **C63**, 491 (1994); Erratum: Eur. Phys. J. **C11**, 571 (1999).

[77] J. Bijnens, L. Girlanda, P. Talavera, Eur. Phys. J. C **23** (2002) 539

9 Quarkonia

Quarkonia name the $q\bar{q}$ meson states of the heavy quarks $q = c$ and b. Correspondingly we have two series of new states the Charmonium and the Bottomonium states. The top quark is too short lived under weak decays that it is able to form corresponding toponium bound states. To classify charmonium and bottomonium states we remind about the corresponding states in the light quark sector.

Normal hadrons made of u and d quarks are the pions, the nucleons and the light vector meson resonances ρ and ω. The latter are $J^{PC} = 1^{--}$ states:

$$\rho^0 = \frac{1}{\sqrt{2}}\,(u\bar{u} - d\bar{d})\;;\;\; \rho^+ = u\bar{d}\;;\;\; \omega = \frac{1}{\sqrt{2}}\,(u\bar{u} + d\bar{d})\;\;.$$

Strange hadrons are composed of an s quark together with u, d or s quarks, the Kaons, hyperons and the vector resonances, which again are $J^{PC} = 1^{--}$ states:

$$\phi = s\bar{s}\;;\;\; K^{*+} = u\bar{s}\;;\;\; K^{*0} = d\bar{s}\;\;.$$

In 1970 Glashow, Illiopoulos and Maiani (GIM) postulated the existence of a heavier weak doublet partner of the s quark, in order to prevent tree level flavor changing neutral currents, and which they named charm c (see Sect. 3.3).

In 1974 Ting [1] and Richter [2] independently discovered a new state called J/ψ in e^+e^-–annihilation, which actually includes a whole family of resonances the J/ψ spectral family. The first results from SPEAR at SLAC are shown in Fig. 9.1. The plot tells us that the new resonances must be very stable. If they would decay easily they would be broad, having a large width. A very much scaled up version in presented in Fig. 9.2. This second plot clearly demonstrates that the hadronic bound states not only decay into hadrons but also directly into lepton pairs. More recent substantially improved charm physics results have been obtained with the BES detector at Beijing.

The new state was interpreted as Charmonium

$$J/\psi = \frac{1}{\sqrt{3}}\sum_a c_a\bar{c}_a$$

where $a = r(\text{ed})$, $g(\text{reen})$, $b(\text{lue})$ denotes the 3 color degrees of freedom.

The calculation of hadronic bound states is a key problem of QCD. For light quarks u, d and s perturbative or semi–perturbative calculation are hopeless, but lattice QCD is making big progress here with numerical simulations based on the QCD path integral. In contrast, for the relatively heavy quarks c and b, α_s is already relatively small because of the relatively large c.m. energy. Since in addition the constituents are heavy, it is suggestive that methods applied in positronium physics or in calculation of hydrogen–atoms could work here as well with appropriate modifications. These methods are based on non–relativistic approximations, i.e., interactions between constituents are approximated by potentials $V(r)$ together with the corresponding Schrödinger equation

$$[T + V]\,\Psi = E\,\Psi\;\;, \tag{9.1}$$

where T is the kinetic energy.

In 1975 Appelquist and Politzer [3] worked out the charmonium spectrum by adapting corresponding positronium calculations. Before going into potential model calculations we present a basic phenomenological overview (for a much more detailed review is [4]).

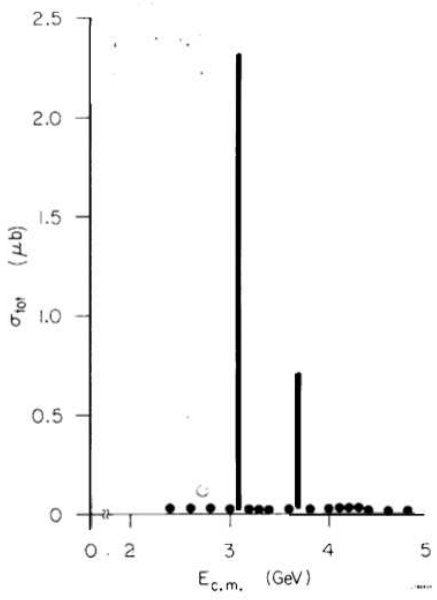


Figure 9.1: The total cross section of hadron production vs. center-of-mass energy. As seen fist by Mark I at SLAC (from B. Richter's Nobel lecture [2]).

9.1 Charmonium

The ground state of charmonium is J/ψ a ${}^3S_1$ state. The spectroscopic notation is: $n^{2S+1}L_J$. n is the principal quantum number and $J = L+S$ the total angular momentum, the sum of angular momentum L and spin S. General properties are

$$\left.\begin{array}{ll} \text{Parity:} & P = (-1)^{L+1} = -1 \\ \text{Charge conjugation:} & C = (-1)^{L+S} = -1 \end{array}\right\} \begin{array}{c} \text{same QN's as photon} \\ J^{PC} = 1^{--} \end{array} ,$$

however, in contrast to the photon they are very heavy: $M_{J/\psi} \sim 3.1$ GeV.

The extraordinary properties of the lowest lying J/ψ states are the large lifetimes! The precise J/ψ characteristics are

$M_{J/\psi}$	=	3096.916 ± 0.011 MeV
$\Gamma_{\text{tot}}^{(J/\psi)}$	=	63.2 ± 2.1 keV
$\Gamma(J/\psi \to e^+e^-)$	=	$5.55 \pm 0.14 \pm 0.02$ keV

Usually, we would expect such a heavy state to be very instable as there is a lot of energy available to produce lighter secondary particles. When comparing strong versus electromagnetic decays:

$$\begin{array}{lll} \Gamma(J/\psi \to \text{hadrons}) & \text{hadronic width} & \text{(strong interaction)} , \\ \Gamma(J/\psi \to \text{leptons}) & \text{leptonic width} & \text{(elmagn. interaction)} , \end{array}$$

we observe a surprisingly small ratio of roughly 10 : 1, between strong and electromagnetic modes, only. The reason for this we will learn to understand in the sequel.

The lowest lying resonance $J/\psi(1S)$ exhibits a number of exited states denoted often by $\psi'(2S)$, ψ'', $\cdots$, where the first exited state has parameters

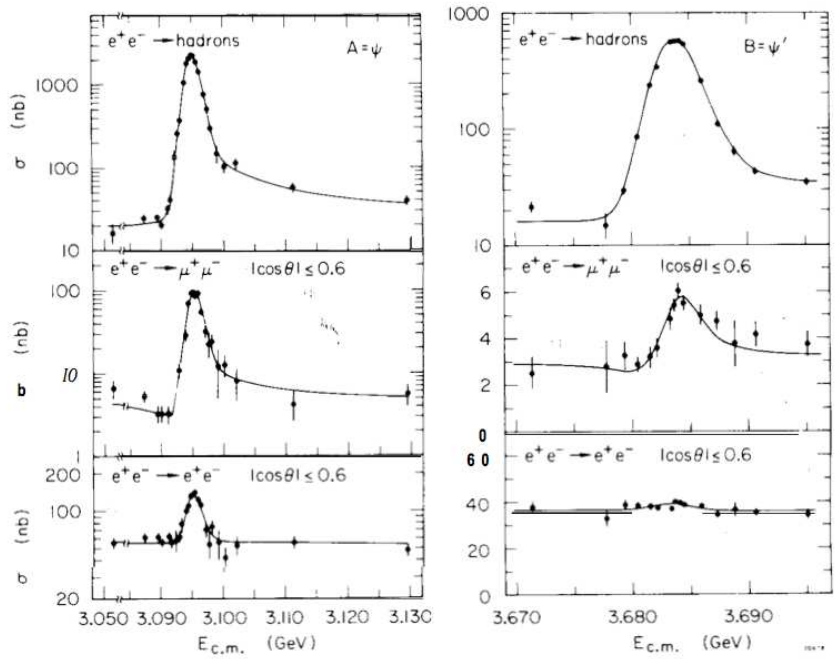


Figure 9.2: Scans of hadron, $\mu^+\mu^-$ and e^+e^- pair production cross section in the regions of the J/ψ and ψ'' (from B. Richter's Nobel lecture [2]).

$M_{\psi'}$	=	3686.09 ± 0.04 MeV
$\Gamma_{\rm tot}^{(\psi')}$	=	317 ± 9 keV
$\Gamma(\psi' \to e^+e^-)$	=	2.38 ± 0.04 keV

About 98% of the decays are hadronic, whereof 57 % of the type $\psi' \to J/\psi + $anything. Exclusive branching fraction typically look like:

$$\text{i)}\ \ \psi' \to e^+e^-, \mu^+\mu^- \quad (0.75\%\ \text{each}) \quad ; \quad \text{ii)}\ \ \psi' \to (\pi' s, K' s) \quad (10\%)$$
$$\text{iii)}\ \ \psi' \to J/\psi\pi\pi \quad (49\%\ ) \quad ; \quad \psi' \to J/\psi\eta \quad (3\%)$$
$$\text{iv)}\ \ \text{electromagn. cascades}$$

The electromagnetic cascades proceed via

$$\psi' \ \to \ \cdots + \gamma \ \begin{cases} \to \ \chi \ \text{mesons} \\ \to \ J/\psi + \gamma \end{cases}$$

and make possible the transition $\psi' \to J/\psi + \gamma\gamma$ while $\psi' \not\to J/\psi + \gamma$ is forbidden by CP parity. In fact, between ψ' and J/ψ additional states with QN's ${}^3P_0$, ${}^3P_1$, ${}^3P_2$ with $J^{PC} = 0^{++},\ 1^{++},\ 2^{++}$ exist, such that decays

$$\psi' \to {}^3P_J + \gamma \ \ \text{and} \ \ {}^3P_J \to J/\psi + \gamma$$

are possible.

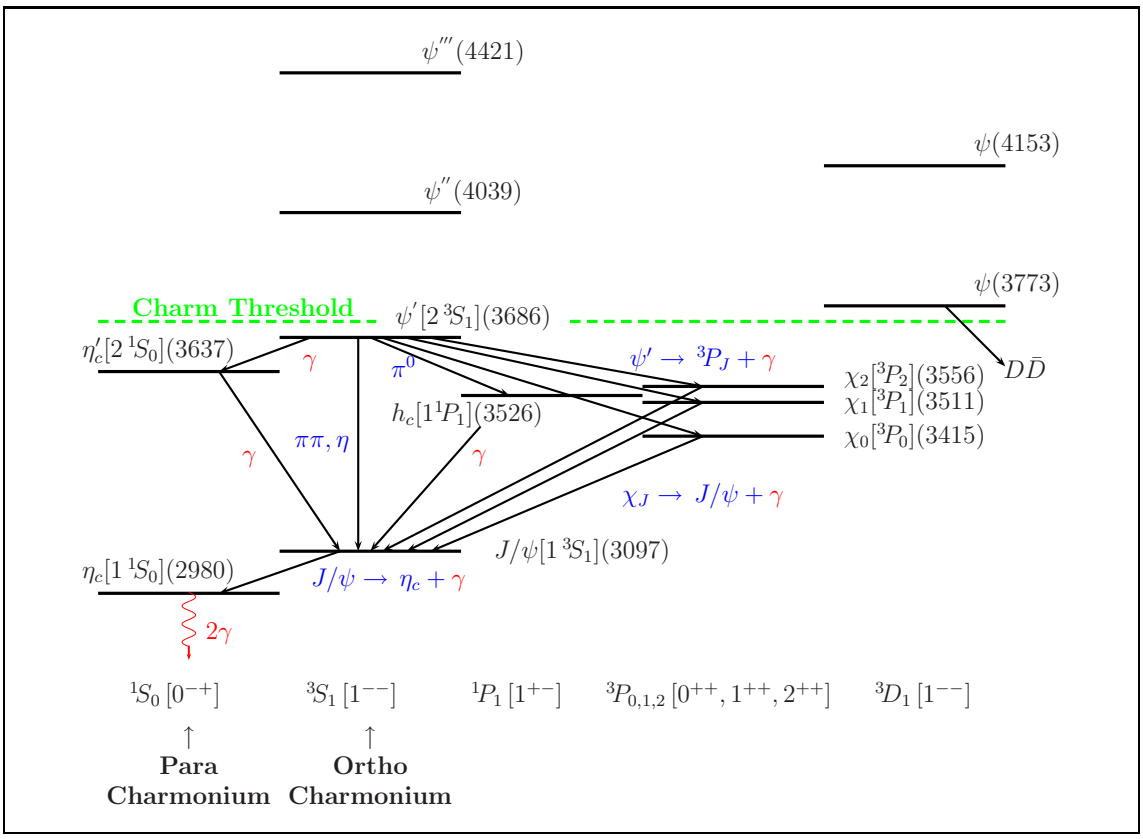


Figure 9.3: Lowest states of the charmonium spectrum and radiative transitions.

In 1975 the DASP Collaboration at DESY [6] and subsequently SPEAR at SLAC [7, 8] first observed the χ states. About 10 years latter the Crystal Ball detector at SPEAR (SLAC) allowed one to perform a very precise spectroscopic investigation of the radative ψ'–decays [5]. A spectrum of this measurement is shown in Fig. 9.4. More recent radiative transition spectra from CLEO are shown in Fig. 9.5. A very rich charmonium spectroscopy emerges. Actual results for masses and widths of the cascade states are:

$$
\begin{aligned}
0^{++} &\;\widehat{=}\; \chi(3415): \quad M = 3414.75 \pm 0.31 \text{ MeV}, \quad \Gamma = 10.2 \pm 0.7 \text{ MeV} \quad \mathcal{P}_{\rm br}{=}9.4\ \% \\
1^{++} &\;\widehat{=}\; \chi(3511): \quad M = 3510.66 \pm 0.07 \text{ MeV}, \quad \Gamma = 0.89 \pm 0.05 \text{ MeV} \quad \mathcal{P}_{\rm br}{=}8.8\ \% \\
2^{++} &\;\widehat{=}\; \chi(3556): \quad M = 3556.20 \pm 0.09 \text{ MeV}, \quad \Gamma = 2.03 \pm 0.12 \text{ MeV} \quad \mathcal{P}_{\rm br}{=}8.3\ \% \quad .
\end{aligned}
$$

By $\mathcal{P}_{\rm br} = \Gamma(\psi' \to \chi\gamma)/\Gamma_{\rm tot}^{(\psi')}$ we denoted the production branching fraction via the radiative decays $\psi' \to \chi_J + \gamma$.

Exclusive hadronic transitions $\psi(2S) \to J/\psi + X$ are $\pi^+\pi^- J/\psi$ [33.5%], $\pi^0\pi^0 J/\psi$ [16.5%], $\eta J/\psi$ [3.3%] and $\pi^0 J/\psi$ [0.13%].

In Fig. 9.3 we show the spectrum which has been established experimentally. Later we will develop the tools which allow us to understand the observed spectral scenery.

Presently there are 6 J/ψ series 1^{--} resonances established. The parameters of the one's not already given above are:

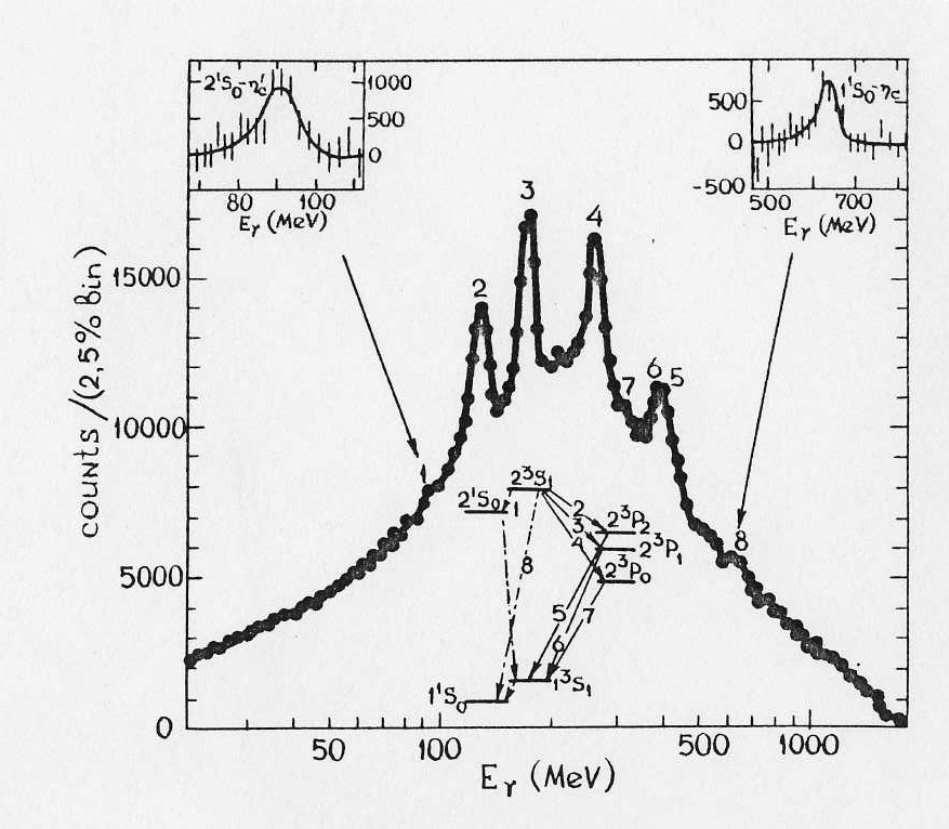


Figure 9.4: Spetrum of ψ' radiatve charmonium decays as measured by the Crystal Ball Collaboration. Shown are the number of photon counts versus the photon energy. Courtesy of H. Kolanoski [5].

State	$\psi(3770)$	$\psi(4040)$	$\psi(4160)$	$\psi(4415)$
Mass MeV	3772.92 ± 0.35	4039 ± 1	4153 ± 3	4421 ± 4
Width MeV	27.3 ± 1.0	80 ± 10	103 ± 8	62 ± 20
$\mathrm{Br}(\psi \to e^+e^-)$	$(0.97 \pm 0.07) \cdot 10^{-5}$	$(1.07 \pm 0.16) \cdot 10^{-5}$	$(0.81 \pm 0.09) \cdot 10^{-5}$	$(0.94 \pm 0.32) \cdot 10^{-5}$

Note the much bigger widths of these states, in comparison to the very narrow two lowest states $J/\psi(1S)$ and $\psi(2S)$. Actually, the next higher state $\psi(3770)$ decays with 85.3% branching ratio as

$$\psi(3770) \to D\bar{D}$$

into a pair of open charm particles (48.7% $D^0\bar{D}^0$, 36.1% D^+D^-). This tells us that we have passed the charm threshold! Of the vector mesons, only the two lowest states have negative binding energy, and this is what makes them so surprisingly stable.

A new state $h_c[1\,^1P_1](3526)$ was discoverd in 2005 bt CLEO. It is produced hadronically via $\psi(2S) \to \pi^0 h_c$ with $h_c \to \eta_c\gamma$. Its precise mass is an important test for $q\bar{q}$ potental models.

A recent $\gamma\gamma \to$ hadrons spectrum of BABAR showing the pseudoscalars η_c and η_c' in the invariant mass distribution of the $K_S^0 K^\pm\pi^\mp$ channel is shown in Fig. 9.6.

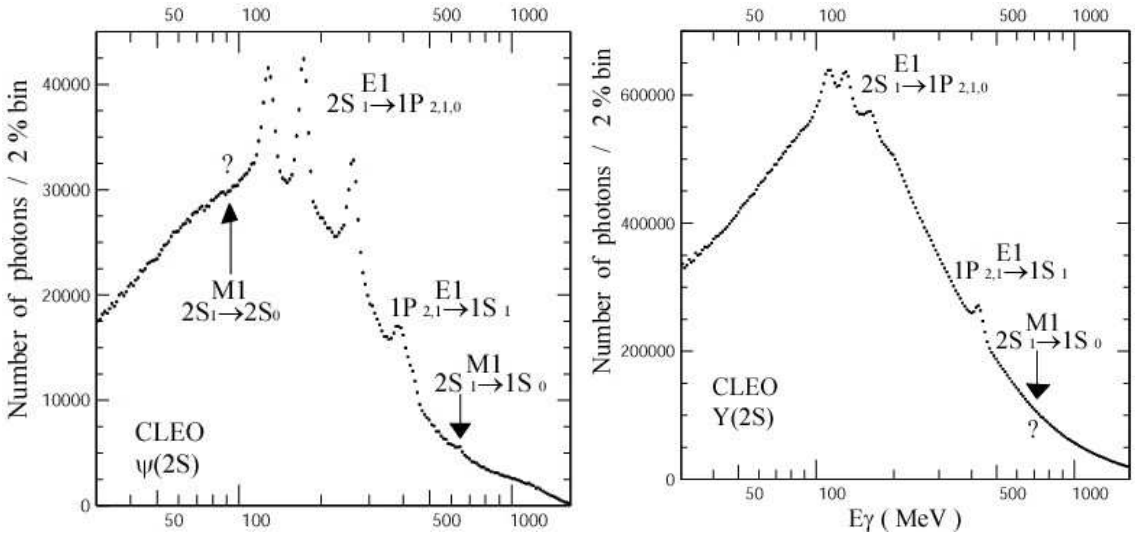


Figure 9.5: Inclusive photon spectra from 2^3S_1 decays f $c\bar{c}$ and $b\bar{b}$ measured with the CLEO detector. From Skwarnicki [9].

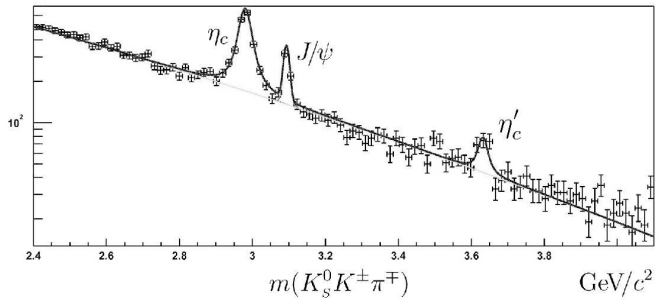


Figure 9.6: Observation of $\gamma\gamma \to \eta_c(1S)$ and $\gamma\gamma \to \eta_c(2S)$ by the BABAR experiment. A peak due to $e^+e^- \to \gamma J/\psi$ is also visible. From Skwarnicki [9].

9.2 Charmed Mesons

While charmonium states are of type $c\bar{c}$, and thus have net charm zero, charmed mesons have open charm and are of type $c\bar{q}$ and $q\bar{c}$, where $q = u, d, s$.

quark content	$c\bar{u}$	$c\bar{d}$	$c\bar{s}$	$u\bar{c}$	$d\bar{c}$	$s\bar{c}$
pseudo scalars $J^P = 0^-$	D^0	D^+	D_s^+	$\bar{D}^0$	D^-	D_s^-
vector mesons $J^P = 1^-$	D^{*0}	D^{*+}	D_s^{*+}	$\bar{D}^{*0}$	D^{*-}	D_s^{*-}

The charmed pseudoscalar mesons D^0 and $D^\pm$ were established in 1976, the $D_s^\pm$ in 1977. Main production mechanism is B($\psi'' \to D\bar{D}$) ($\gtrsim 90\%$). All decays are parity violating (see the Exercises) and hence proceed via weak interactions. The reason is that charm is conserved in strong and electromagnetic interactions. Since the states are heavy there is a plenitude of decay modes and the main channels are typically at the percent level (some reaching close to 15%):

state	mass [MeV]	mean life $\tau \times 10^{15}$	some decays
D^0	1864.48 ± 0.17	$410.1 \pm 1.5\,s$	$K^-\pi^+$[3.89%], $K^-\pi^+\pi^0$[13.9%], $K^-e^+\nu_e$[3.58%]
$D^\pm$	1869.62 ± 0.20	$1040 \pm 7\,s$	$K^-\pi^+\pi^+$[9.2%], $K^0_S\pi^+\pi^0$[6.8%], $\bar{K}^0\mu^+\nu_\mu$[9.3%]
$D^\pm_s$	1968.49 ± 0.34	$500 \pm 7\,s$	$K^+K^-\pi^+$ [5.5%], $K^+K^-\pi^+\pi^0$[5.6%], $\tau^+\nu_\tau$[6.6%]

Note that the pseudo scalar states only decay via weak processes!

Exercises:

1) Draw the relevant Feynman diagrams (within the SM) for the decays $D^0 \to K^-\pi^+$, $K^-e^+\nu_e$ and $K^-\pi^+\pi^0$, based on the quark content of the hadrons.

2) The decay $D^0 \to K^-\pi^+\pi^0$ is observed with a branching fraction 13.9 %. What is the branching fraction of the charge conjugate decay $D^0 \to K^+\pi^-\pi^0$?

3) Explain why $D^+_s \to \mu^+\nu_\mu$ has a branching fraction of $6.2{\cdot}10^{-3}$ % only, while $D^+_s \to \tau^+\nu_\tau$ has one of 6.6 %.

The vector states D^* and D^*_s were detected fist in 1977. They are produced in $\psi(4.028) \to D^*\bar{D}^*$). They are to be considered as excited states, which decay via strong and electromagnetic transitions into the lighter charmed D mesons:

state	mass [MeV]	width	decays
D^{*0}	2006.97 ± 0.19	$\Gamma < 2.1$ MeV	$D^0\pi^0$ [61.9%], $D^0\gamma$ [38.1%]
$D^{*\pm}$	2010.27 ± 0.17	$\Gamma = 96 \pm 22$ keV	$D^0\pi^\pm$ [67.7%], $D^\pm\pi^0$ [30.7%], $D^\pm\gamma$ [1.6%]
$D^{*\pm}_s$	2112.3 ± 0.5	$\Gamma < 1.9$ MeV	$D^\pm_s\gamma$ [94.2%], $D^\pm_s\pi^0$ [5.8%]

A much more detailed experimental study of this rich field of hadron spectroscopy has been performed since and is going on up to day, e.g. with the charm–factory CLEOc at Cornell/USA and a similar $c-\tau$–factory BEPCII/BESIII in Beijing/China. We refer to the Particle Data Tables for more up to date information on this program.

What is most important for us are the facts which established the new quantum number *charm*:

1. Very narrow peaks J/ψ and ψ'
2. Broad peaks above about 4 GeV
3. χ intermediate states
4. D mesons which decay parity violating, i.e., decay via weak interaction. This decay pattern establishes that charm is conserved in strong and electromagnetic processes
5. Exited D^* states
6. The ratio $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ increases strongly at about 4 GeV (the charm threshold) (see Fig. 8.7)

9.3 The Υ resonances

The first observation of a resonance at 9.4 GeV was made in the dimuon spectrum of 400 GeV proton–nucleon collisions [10] at Fermilab. In 1978, after upgrading the DORIS storage ring at DESY, the PLUTO, DASP and DESY-Heidelberg experiments [11–13] clarified the nature of the new resonances in e^+e^-–annihilation: the Υ resonance and the Υ spectral family were to be interpretation as Bottomonium states

$$\Upsilon = \frac{1}{\sqrt{3}} \sum_a b_a \bar{b}_a \ .$$

Later more precise measuerement were performed with the CUSB and CLEO detectors at the CESR e^+e^-–ring at Cornell and with the ARGUS and Crystal Ball detectors at DORIS II ring at DESY. Data from CUSB are shown in Fig. 9.7.

The b (bottom) quark was a new quark, which was expected to exist after the discovery of the τ–lepton in 1975, which was the first fermion beyond the complete second lepton–quark family. In the SM the family structure is required by renormalizability which in turn is possible by family-wise anomaly cancellation. A third family had been proposed in 1973 by Kobayashy and Maskawa as a natural explanation of CP violation which did not require to introduce a new type of "superweak" interaction.

The main properties of the experimentally confirmed Υ states are:

$M_{\Upsilon(1S)} = 9460.30 \pm 0.26$ MeV	$\Gamma_{\Upsilon(1S)} = 54.02 \pm 1.25$ keV	$\Gamma_{\Upsilon(1S)e^+e^-} = 1.340 \pm 0.018$ keV
$M_{\Upsilon(2S)} = 10023.26 \pm 0.31$ MeV	$\Gamma_{\Upsilon(2S)} = 31.98 \pm 2.63$ keV	$\Gamma_{\Upsilon(2S)e^+e^-} = 0.612 \pm 0.011$ keV
$M_{\Upsilon(3S)} = 10355.2 \pm 0.5$ MeV	$\Gamma_{\Upsilon(3S)} = 20.32 \pm 1.85$ keV	$\Gamma_{\Upsilon(3S)e^+e^-} = 0.443 \pm 0.008$ keV
$M_{\Upsilon(4S)} = 10579.4 \pm 1.2$ MeV	$\Gamma_{\Upsilon(4S)} = 20.5 \pm 2.5$ MeV	$\Gamma_{\Upsilon(4S)e^+e^-} = 0.272 \pm 0.029$ keV
$M_{\Upsilon^{(5)}} = 10865 \pm 8$ MeV	$\Gamma_{\Upsilon^{(5)}} = 110 \pm 13$ MeV	$\Gamma_{\Upsilon^{(5)}e^+e^-} = 0.31 \pm 0.07$ keV
$M_{\Upsilon^{(6)}} = 11019 \pm 8$ MeV	$\Gamma_{\Upsilon^{(6)}} = 97 \pm 16$ MeV	$\Gamma_{\Upsilon^{(6)}e^+e^-} = 0.13 \pm 0.03$ keV

The branching fractions into the cascade states are:

$\Upsilon(2S)$	$\to \gamma\chi_{b2}(1P)\,(7.15 \pm 0.35)\%$	$\Upsilon(3S)$	$\to \gamma\chi_{b2}(2P)\,(13.1 \pm 1.6)\%$
	$\to \gamma\chi_{b1}(1P)\,(6.9 \pm 0.4)\%$		$\to \gamma\chi_{b1}(2P)\,(12.6 \pm 1.2)\%$
	$\to \gamma\chi_{b0}(1P)\,(3.8 \pm 0.4)\%$		$\to \gamma\chi_{b0}(2P)\,(5.9 \pm 0.6)\%$

Other major modes are $\Upsilon(2S) \to \Upsilon(1S)\pi^+\pi^-(18.8 \pm 0.6)\%$; $\Upsilon(1S)\pi^0\pi^0(9.0 \pm 0.8)\%$, $\Upsilon(3S) \to \Upsilon(2S)$+anything$(10.6\pm0.8)\%$ [$\Upsilon(2S)\pi^+\pi^-(2.8\pm0.6)\%$; $\Upsilon(2S)\pi^0\pi^0(2.0\pm0.3)\%$; $\Upsilon(2S)\gamma\gamma(5.0\pm 0.7)\%$]. Furthermore, $\Upsilon(3S) \to \Upsilon(1S)\pi^+\pi^-(4.5 \pm 0.2)\%$; $\Upsilon(1S)\pi^0\pi^0(2.1 \pm 0.3)\%$.

Two series of cascade states the $1P$ and $2P$ states are well establised:

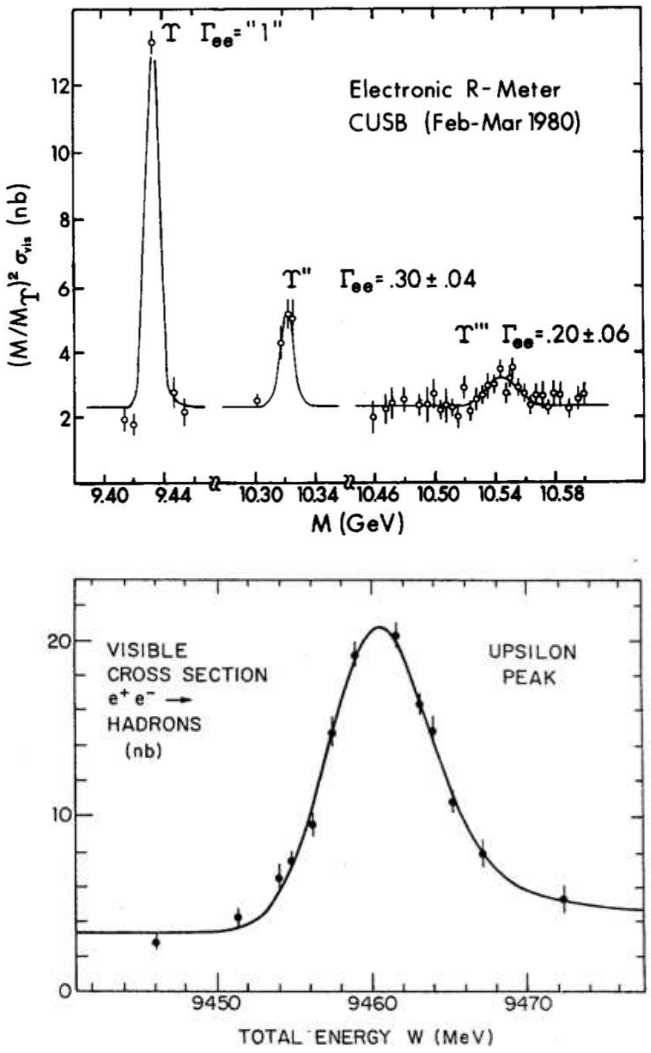


Figure 9.7: A scan of the first 3 Υ resonances from the CUSB detector at the CESR ring at Cornell.

$M_{\chi_{b0(1P)}} = 9859.44 \pm 0.42 \pm 0.31$	$\chi_{b0(1P)} \to \gamma\Upsilon(1S) < 6\%$
$M_{\chi_{b1(1P)}} = 9892.78 \pm 0.26 \pm 0.31$	$\chi_{b1(1P)} \to \gamma\Upsilon(1S)\,(35 \pm 8)\%$
$M_{\chi_{b2(1P)}} = 9912.21 \pm 0.26 \pm 0.31$	$\chi_{b2(1P)} \to \gamma\Upsilon(1S)\,(22 \pm 4)\%$
$M_{\chi_{b0(2P)}} = 10232.50 \pm 0.40 \pm 0.50$	$\chi_{b0(2P)} \to \gamma\Upsilon(2S)\,(4.6 \pm 2.1)\%\,,\ \gamma\Upsilon(1S)\,(0.009 \pm 0.006)\%$
$M_{\chi_{b1(2P)}} = 10255.46 \pm 0.22 \pm 0.50$	$\chi_{b1(2P)} \to \gamma\Upsilon(2S)\,(21 \pm 4)\%\,,\quad \gamma\Upsilon(1S)\,(8.5 \pm 1.3)\%$
$M_{\chi_{b2(2P)}} = 10268.65 \pm 0.22 \pm 0.50$	$\chi_{b2(2P)} \to \gamma\Upsilon(2S)\,(16.2 \pm 2.4)\%\,,\ \gamma\Upsilon(1S)\,(7.1 \pm 1.0)\%$

While many aspects of the Bottomonium are very similar to the properties of the Charmonium, the bottomonium states, being much heavier than the charmonium states, decay in many more channels, in particular also into charmonium and charmed meson states. Famous is the the $\Upsilon(4S)$ state as it is just above the $B\bar{B}$ threshold of B anti-B meson production. Running an e^+e^-–annihilation machine at the $\Upsilon(4S)$ is a B–factory. B–factories have been relaized at SLAC with the BaBar detector and at KEK Tsukuba/Japan with the Belle detector.

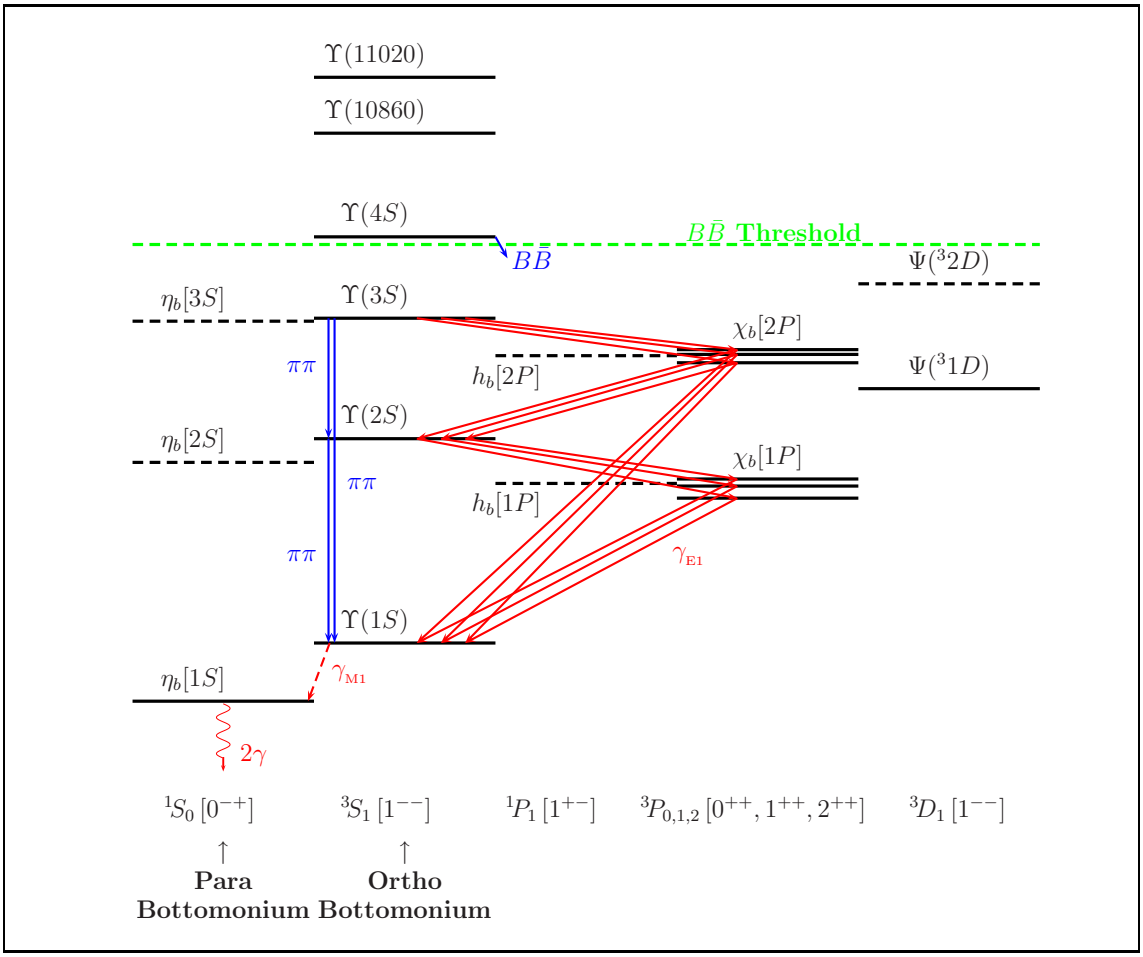


Figure 9.8: Experimentally establisehd states of the bottomonium spectrum and radiative (red) and some hadronic (blue) transitions in full lines. Dashed lines: states and transitons expected from potential model predictions. See also Fig. 9.3 for other possible transitions.

9.4 Bottom Mesons

The b–flavored mesons, usually called B–mesons, are composed of the 3rd family b–quark, the bottom partner of the top quark in the weak iso–doublet, an one of the lighter quarks u, d, s, or c:

quark content	$u\bar{b}$	$d\bar{b}$	$s\bar{b}$	$b\bar{u}$	$b\bar{d}$	$b\bar{s}$
pseudo scalars $J^P = 0^-$	B^+	B^0	B_s^0	B^-	$\overline{B^0}$	$\overline{B_s^0}$
vector mesons $J^P = 1^-$	B^{*+}	B^{*0}	B_s^{*0}	B^{*-}	$\overline{B^{*0}}$	$\overline{B_s^{*0}}$

and the charmed B mesons, we obtain be replacing the u by the c quark in the table.

The bottom pseudoscalar mesons B^0 and $B^\pm$ were established in 1976, the $B_s^\pm$ in 1977. Main production mechanism is $\Upsilon(4S) \to B\bar{B}$ ($\gtrsim 96\%$ [B^+B^- : 51.6 ± 0.6 %, $B^0\overline{B^0}$: 48.4 ± 0.6 %]). All decays are parity violating and hence proceed via weak interactions. The main properties of

the normal B mesons are

$$\begin{aligned} M_{B^\pm} &= 5279.15 \pm 0.31 \text{ MeV} \; ; \; \tau_{B^\pm} = (1.638 \pm 0.011) \cdot 10^{-12} \text{ s} \\ M_{B^0} &= 5279.53 \pm 0.33 \text{ MeV} \; ; \; \tau_{B^0} = (1.530 \pm 0.009) \cdot 10^{-12} \text{ s} \end{aligned}$$

Note that $M_{B^0} - M_{B^\pm} = 0.37 \pm 024$ MeV and $\tau_{B^\pm}/\tau_{B^0} = 1.071 \pm 0.009$.

Like Kaon physics, B physics plays a key role in the flavor structure of the SM. Both show CP violating $X^0 \leftrightarrow \overline{X^0}$ oscillations. In 1973 Kobayashi and Maskawa observed that incorporating the observed CP violation of the K–system in a natural way is possible only in a SM with 3 quark–lepton families. Given the CP pattern in the Kaon sector the three family SM predicted many aspects of B physics and in particular the $O(1)$ CP voilation in the B sector, which in 2001 was fully confirmed at the B–factories by Belle and BaBar.

9.5 Non-Relativistic Potential Models

In spite of the relatively high energy and the related relatively small effective strong coupling, quarkonium resonances are a non–perturbative phenomenon and pQCD is of limitted value in predicting their precise properties. Nevertheless, it is possible to give QCD inspired heuristc arguments which allow us to model quite realistically the physics of quarkonia. The arguments are as follows: at distances $\gtrsim 1 \text{ GeV}^{-1}$ we know quarks must be confined, whereas deep inelastic electron–proton scattering (see later) tells us that for $Q^2 \gtrsim 1 \text{ GeV}^2$ quarks in a proton are "free", as a consequence of asymptotic freedom. For a meson $M = (q\bar{q})$ with a confinement radius R_M the uncertaintity relation tells us that the quark's momentum is $p \sim R_M^{-1}$. For light quarks $m_q \lesssim R_M^{-1}$ we are in a relativistic situation as it applies for the light hadrons. For quarkonia, we are confronted with hadronic bound states of heavy quarks $m_c, m_b \gg R_M^{-1}$. For charmonium typically: $M_{c\bar{c}} \sim 3 - 4$ GeV and binding energy $\Delta E_{c\bar{c}} \sim 100 - 600$ MeV $\ll M_{c\bar{c}}$. Therefore a non–relativistic describtion seen not unrealistic, which suggests a potential model description. How could a $q\bar{q}$–potential look like?
1) At large distances confinement requires a linearly rising potential as known from the Regge behavior of excited hadrons: $V(r) \propto r$. Such behavior is supported by strong coupling expansions as well as by numerical simulations in lattice regulated QCD.
2) At short distances, by virtue of asymptotic freedom, we expect a Coulomb like potential mediated by the massless gluon: $V(r) \propto -1/r$. These two limits suggest a model:

$$V(r) = -\frac{a}{r} + V_0 + \sigma\, r \ . \tag{9.2}$$

One important observation is the approximate congruence of $c\bar{c}$ and $b\bar{b}$ meson spectral as shown in Fig. 9.9. which would be exact for a logarithmic potential $V(r) = C \ln(r/r_0) + C_0$, and which in some sence is an interpolation between the $\propto -1/r$ and the $\propto r$ behavior expected in QCD (Quigg and Rosner 1977). Similarly, a power potential $V(r) = A + B\, r^\alpha$ with $\alpha \simeq 0.1$ leads to a succesful description of the $c\bar{c}$ and the $b\bar{b}$ spectroscopies (Martin 1981).

Note, there does not exist a "derivation" of a potential model! What is used are QCD motivated ansätze. Also, a potential does not account for the full dynamics as it assumes interactions to be instantaneous. Therefore, be aware of the limitations of potential models, which nevertheless usually work rather well. Let us look at this a little bit more specific:

a) The mas spectrum of a heavy quark $q\bar{q}$ bound state is of the form

$$M_n = 2\, m_q + E_n(m_q, V) \ ,$$

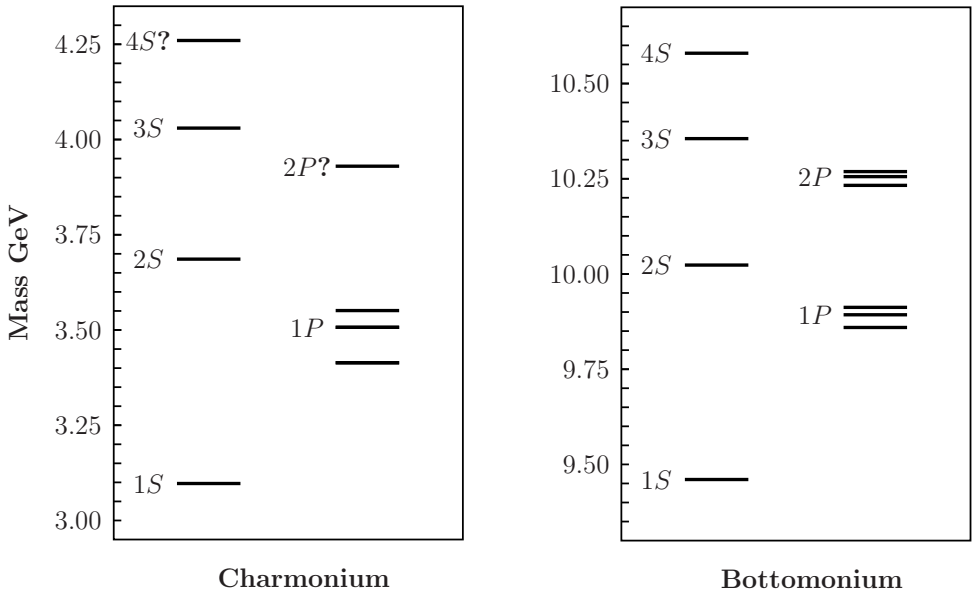


Figure 9.9: Congruence (=equal level spacings) of charmonium and bottomoium spectra.

where m_q is an appropriately defined version of the quark mass, E_n is the energy eigenvalue of a non-relativistic Schrödinger equation and $V(r)$ a suitable Schrödinger potential, which is flavor independent, because the string interaction are flavor independent besides the flavordependent quark mass effects. For the potential we first look at the spin–independent part and discuss spin effects later.

b) The asymptotic behavior of the two–body $q\bar{q}$- potential should satisfy

1) $V(r) \simeq -\frac{4}{3}\frac{\alpha_s(\Lambda\, r)}{r} \; + \;$ corrections ; $r \to 0$,

where the corrections are obtained by the low velocity v/c–expansion of the one–gluon exchange potential ($\Lambda = \Lambda_{\rm QCD}$).

2) $V(r) \simeq \sigma\, r \; + \;$ corrections ; $r \to \infty$,

imposing manifest confinement, whith $\sigma > 0$ a spin and flavor independent *string tension* constant[44].

Point **1)** requiring a Coulomb like potential at short distances suggests the application of the corresponding technique applied for the positronium (or hydrogen–atom) in QED, which yields the ground state splitting of the 0^{-+} (Para-) and 1^{--} (Ortho-) ground states via spin–spin

[44]If a heavy quark antiquark pair is separated the color electric field is concentrated into a string of collimated flux lines. The *shape of the string* can be characterized by the chromo–electric field energy density

$$\mathcal{E}(x) \sim < q\bar{q}|{\rm Tr}\vec{E}^{\,2}(x)|q\bar{q} > - < q\bar{q}|q\bar{q} >< 0|{\rm Tr}\vec{E}^{\,2}(x)|0 > \quad .$$

Quantitatively the string picture manifests itself by the $\ln r$–behavior of the thickness of the "chromoelectric flux tube" (Lüscher, Münster, Weisz 1980 [14]):

$$\sigma^2 \equiv \frac{\int d^2x_\perp x_\perp^2\; \mathcal{E}(x)}{\int d^2x_\perp\; \mathcal{E}} \sim \ln r \quad .$$

This behaviour is confirmed by corresponding lattice QCD simulations (see e.g. [15]).

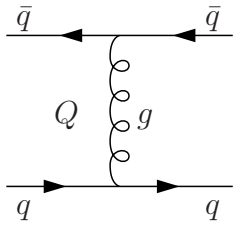


Figure 9.10: Quark–antiquark scattering models the short distance tail of the heavy quark $q\bar{q}$ (two–body) potential.

interaction ($(\uparrow\downarrow)$ vs. $(\uparrow\uparrow)$), as well as the fine– and hyperfine–splitting by other spin dependent effects (see below).

The strategy for "deriving" a "realistic" potential attempts to use all information we have from QCD. The short distance tail shoud be reliably described by pQCD. At lowest order the potential is given by the one–gluon exchange as shown in Fig. 9.10.

$$V(Q^2) = -C_2(R)\,\frac{g_s^2}{Q^2}$$

where Q is the space–like four momentum transfer vector and $C_2(R) = 4/3$ is the $SU(3)_c$ Casimir. This factor is due to the non–Abelian structure of QCD: the two quark–gluon vertices exhibit factors T_i and T_k, respectively, and the gluon propagator is diagonal ($\propto \delta_{ik}$) such that we obtain

$$\left(T_i^R T_i^R\right)_{ab} = \sum_{i=1}^{r}\sum_{c=1}^{d(R)} \left(T_i^R\right)_{ac}\left(T_i^R\right)_{cb} = \delta_{ab}\,C_2(R)\ .$$

The strong coupling constant g_s is to be identified with the effective running QCD coupling governed by the RG (see Sect. 8). We will adopt the convention introduced earlier within the context of effective QCD in Sect. 8.10. Thus $\alpha_s = \frac{g_s^2}{4\pi}$ and $a = \frac{\alpha_s}{\pi}$ and the β–function normalized by $\beta(a) = Q^2\,\frac{\partial}{\partial Q^2}\,a(Q^2)$ (see (8.15)). The β–function should satisfy the following properties:

① By asymptotic freedom we have

$$\beta(a) = -a^2\,\left(\beta_0 + \beta_1\,a + \beta_2\,a^2\right) + O(a^3)\ ;\ \ a \to 0\ .$$

Asymptotically, for large Q^2, exanding in addition with respect to leading logs, we have

$$a(Q^2) = \frac{1}{\beta_0\,\ln Q^2/\Lambda^2} - \frac{\beta_1}{\beta_0^3}\,\frac{\ln\ln Q^2/\Lambda^2}{(\ln Q^2/\Lambda^2)} + O(\frac{1}{(\ln Q^2/\Lambda^2)^3})\ ;\ \ Q \to \infty\ .$$

Note that this in fact means that the potential at short distances is logarithmically modified

$$V(r) \sim \frac{1}{r\,\ln\left(\frac{1}{\Lambda^2 r^2}\right)}\ ;\ \ r \to 0\ . \tag{9.3}$$

② Confinement suggests a β–function which has no IR fixed point (no further zero beyond the trivial one at $a = 0$). In this case the strong coupling constant grows to $a(Q^2) \to \infty$ as $Q^2 \to 0$. Such statements are to be taken with caution, however. At low Q^2 the quark gluon coupling looses its meaning, as large numbers of quarks and gluons get strongly coupled and behave collectively. In other words, using perturbative concepts like α_s in the non–perturbative regime is highly ambiguous, one could say *extremely scheme dependent*. Nevertheless, we use this concept as it

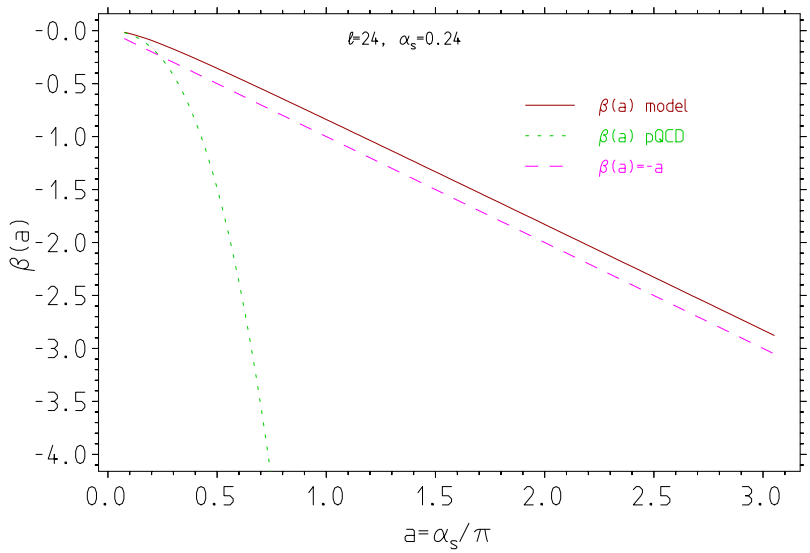


Figure 9.11: Model β–function adopted for modelling the quarkonium potential. Parameters: $\ell = 24$ and $\alpha_s = 0.24$.

has been introduced in the original papers on the subject (Richardson, Buchmüller and Tye and many others [16–21]). In fact it will turn out that the calculations of quarkonium spectra, located in the ranges about $2m_q\pm$ at most a few GeV, do not depend very much on the detailed treament of this long range asymptotics. Inspired by $V(r) \to \sigma\, r$ as $r \to \infty$ and the fact that perturbatvely $V(r) \propto a(\Lambda r)$ the string like behavior of the potential may be obtained by assuming[45]

$$a(Q^2) = \frac{K}{Q^2}\,[1 + o(1)]\ ;\ \ Q^2 \to 0\ ,$$

such that

$$\beta(a) = -a\ (1 + o(1))\ \ ;\ \ a \to \infty\ .$$

Matching the $V(r) \sim \sigma\, r$ string behavior, which in turn is related to the phenomenologically known Regge slope $(\alpha')^{-1} \simeq 1$ GeV^{-2}, one finds

$$(\alpha')^{-1} = 2\pi\,\sigma = 4\pi^2\, C_2(R)\, K\ , \tag{9.4}$$

which fixes $\sigma \sim 0.159$ GeV^{-2} as well as $K \simeq 1.9 \cdot 10^{-2}$ GeV^{-2}. Today one can obtain the string constant reliably from QCD on a lattice, a recent value obtained in [22] is

$$\sigma \simeq 0.2162\ \text{GeV}^{-2}\ , \tag{9.5}$$

see (9.13) below, which amounts to an increase by a factor 1.36, relative to the one obtained from the Regge slope.

We now may choose $a(Q^2)$ such that ① and ② are the boundary conditions. The general ansatz is

$$\beta(a) = -\beta_0\, a^2 - \beta_1\, a^3 + \sigma(a)\ ,$$

[45]The symbols O(x) and o(x) have the usual meaning defined in mathematics of asymptotic expansions: "$f(x) = O(g(x))$ as $x \to \infty$" means, there exist real positive numbers c and x_0 such that $|f(x)| \le c\,|g(x)|$ for all $x > x_0$. In contrast, "$f(x) = o(g(x))$ as $x \to \infty$" means that $g(x)$ grows faster than $f(x)$, i.e., $f(x)/g(x) \to 0$ as $x \to \infty$.

where $\sigma(a)$ must be choosen such that we satisfy ②. With the appropriate choice of $\Lambda_{\rm QCD}$, we have

$$\ln \frac{Q^2}{\Lambda'^2} = \int\limits_{a(Q^2)}^{a(\Lambda'^2)} \frac{\mathrm{d}x}{\beta(x)} \ ,$$

and adopting the strategy introduced in Sect. 8.5, when discussing the "higher order" effects, we may define Λ by ①

$$\ln \frac{Q^2}{\Lambda^2} = \frac{1}{\beta_0\, a} + \frac{\beta_1}{\beta_0^2}\, \ln(\beta_0\, a) + \int\limits_0^a \mathrm{d}x\, \left(\frac{1}{\beta_0 x^2} - \frac{\beta_1}{\beta_0^2}\, \frac{1}{x} + \frac{1}{\beta(x)} \right)$$

for $a(\Lambda^2) > a(Q^2)$ and ②

$$\ln \frac{K}{Q^2} = \ln a + \int\limits_a^\infty \mathrm{d}x\, \left(\frac{1}{x} + \frac{1}{\beta(x)} \right)$$

for $a(K) < a(Q^2)$. The consistency condition of ① and ② requires:

$$\ln \frac{K}{\Lambda^2} \ = \ \frac{1}{\beta_0\, a} + \frac{\beta_1}{\beta_0^2}\, \ln(\beta_0\, a) + \int\limits_0^a \mathrm{d}x\, \left(\frac{1}{\beta_0 x^2} - \frac{\beta_1}{\beta_0^2}\, \frac{1}{x} + \frac{1}{\beta(x)} \right) + \ln a + \int\limits_a^\infty \mathrm{d}x\, \left(\frac{1}{x} + \frac{1}{\beta(x)} \right)$$

independent of Q^2 and therefore of a. A model β–function which satisfies both conditions is given by the explicit function

$$\left\| \ \frac{1}{\beta(a)} = -\frac{1}{\beta_0\, a^2 \left(1 - \exp(-\frac{1}{\beta_0\, a})\right)} + \frac{\beta_1}{\beta_0^2}\, \frac{1}{a}\, \exp(-\ell\, a) \ . \right. \tag{9.6}$$

One easily checks that it satisfies the boundary conditions:

$$\beta(a) = -\beta_0\, a^2 - \beta_1\, a^3 - \frac{\beta_1\, (\beta_1 - \ell\, \beta_0)}{\beta_0}\, a^4 + O(a^5) \ ; \ \ a \to 0$$

$$\beta(a) = -a + \frac{1}{2\beta_0} + O(1/a) \ ; \ \ a \to \infty \ .$$

However, the second condition is satisfied only when $\ell > 0$ such that the second term of the ansatz is damped at large a. On the other hand, the asymptotic form for a small seems to determine ℓ uniquely by β_2 (up to a sign the coefficient of the a^4–term, given in (8.15)). The result we obtain is

$$\ell = \frac{\beta_1^2 - \beta_0 \beta_2}{\beta_0 \beta_1} \simeq -11.8[-48.1] \quad \text{for} \quad N_f = 3[0] \ ,$$

a nonsensical value, because of the wrong sign. Note that the model ansatz only makes sense for positive ℓ. It is not very surprising that this does not work. It is a typical example of abuse of the perturbative concept of the running coupling in the infrared. We conclude that the potential (9.2) is a reasonable ansatz if we treat the three parameters as what they are: uncorrelated fit parameters with the exeption of the first term which we know in QCD has a well predicted form

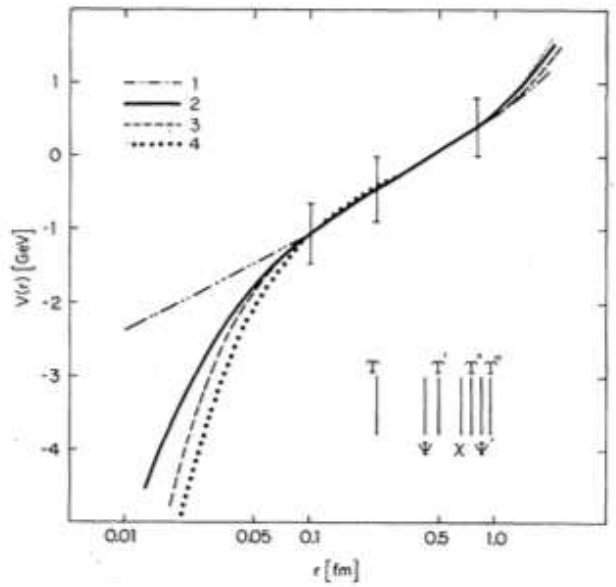


Figure 9.12: A comparison of various quarkonia potentials. Labels 1: Martin [24], 2: Buchmüller, Grunberg and Tye [23], 3: Bhanot and Rudaz [25] 4: Cornell group [16]. Also shown are some mean–square radii for Υ and J/ψ states. (taken from Ref. [18])

(9.3), plus higher order corrections. With a fitted value for ℓ Eq. (9.6), works rather well. In [23] specifically, the QCD correction to two–loops are incorporated:

$$V(r) \sim -\frac{4}{3}\frac{1}{r}\frac{4\pi}{\beta_0 \ln\left(1/r^2\Lambda^2_{\overline{\rm MS}}\right)} \times \left\{1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\left[\ln\left(1/r^2\Lambda^2_{\overline{\rm MS}}\right)\right]}{\ln\left(1/r^2\Lambda^2_{\overline{\rm MS}}\right)} + c\frac{1}{\ln\left(1/r^2\Lambda^2_{\overline{\rm MS}}\right)} + O\left(\frac{1}{\ln^2\left(1/r^2\Lambda^2_{\overline{\rm MS}}\right)}\right)\right\} , \quad (9.7)$$

where $c = \frac{1}{360}\left(31 - \frac{10}{3}N_f\right) + 2\gamma_E$ and $\gamma_E = 0.5772\cdots$ is Euler's constant. The RG coefficient are as given in (8.5).

A set of successful potentials are shown in Fig. 9.12. Note that the radii of the states are intermediate and neither probe the short distance nor the long distance asymptotic regimes. The various potentials have been shifted by a constant such that they coincide at $r_0 = 0.5$ fm, and the error bars indicate the uncertainty in this normalization. Typical predictions are given in Tables 1 and 2. They have been obtained based on the spin independent model: Coulomb–term supplemented with the β–function model (9.6), which at long distances matches a linear potential with appropriate string constant. The parameter $\ell = 24$ has been fixed such that experimetal result for the mass difference $M_{\Upsilon'} - M_{\Upsilon}$ is reproduced correctly. Note that in the spin independent approximation pseudoscalar (0^{-+}) vs. vector states (1^{--}) splitting is neglected. Similarly for the h_c vs. χ_{c2} etc. Remarkably, this crude model reflects the spectrum quite well.

In general one can say that the non-relativistic description for charmonium is quite crude, and as expected, works much better for bottomonium. For much more recent and more detailed reviews we refer to the Refs. [20, 21, 26, 27].

Table 1: $c\bar{c}$ spectrum: masses, ratios of widths, velocities, and mean–square radii. The mass of the ground state is input; it determines the c–quark mass $m_c = 1.48$ GeV [18].

State	Mass (GeV)	$\Gamma_{ee}/\Gamma_{ee}(1S)$	$\langle v^2/c^2 \rangle$	$\langle r^2 \rangle^{1/2}$ (fm)
$J/\psi(1S)$	3.10	1	0.23	0.42
$\chi_{c2}(1P)$	3.52	0.25	0.67	
$\psi'(2S)$	3.70	0.46	0.29	0.85
$\psi(1D)$	3.81	0.29	0.87	
$\chi'_{c2}(2P)$	3.79	0.32	1.05	
$\psi''(3S)$	4.12	0.32	0.36	1.20
$\psi(2D)$	4.19	0.36	1.22	
$\psi'''(4S)$	4.48	0.25	0.44	1.48

Table 2: $b\bar{b}$ spectrum: masses, ratios of widths, velocities, and mean–square radii. The mass of the ground state is input; it determines the b–quark mass $m_b = 4.87$ GeV [18].

State	Mass (GeV)	$\Gamma_{ee}/\Gamma_{ee}(1S)$	$\langle v^2/c^2 \rangle$	$\langle r^2 \rangle^{1/2}$ (fm)
$\Upsilon(1S)$	9.46	1	0.077	0.23
$\chi_{b2}(1P)$	9.89		0.069	0.39
$\Upsilon(2S)$	10.02	0.44	0.075	0.50
$\Upsilon(1D)$	10.14		0.072	0.53
$\chi'_{b2}(2P)$	10.25		0.078	0.65
$\Upsilon(3S)$	10.35	0.32	0.085	0.75
$\Upsilon(2D)$	10.43		0.083	0.77
$\chi''_{b2}(3P)$	10.53		0.090	0.87
$\Upsilon(4S)$	10.62	0.26	0.098	0.95
$\Upsilon(3D)$	10.68		0.098	0.97
$\Upsilon(10860)$	10.86	0.25	0.130	1.10

Table 3: $c\bar{c}$ spectrum: masses in MeV from lattice QCD in the quenched approximation

J^{PC}	State	CP-PACS [28]	Columbia [29]	QCD-TARO [30]	experiment
0^{-+}	η_c	3013(1)	3014(4)	3010(4)	2980(1)
	η_c'	3739(46)	3707(20)		3654(10)
1^{--}	J/ψ	3085(1)	3084(4)	3087(4)	3097
	$\psi(2S)$	3777(40)	3780(43)		3686
1^{+-}	h_c	3474(10)	3774(20)	3528(25)	3525
	h_c'	4053(95)	3886(92)		–
0^{++}	χ_{c0}	3408	3413(10)	3474(15)	3415(1)
1^{++}	χ_{c1}	3472(9)	3462(15)	3524(16)	3511
2^{++}	χ_{c2}	3503(24)	3488(11)		3556

Today it is possible to calculate the charmonium spectra in lattice QCD, although there are still some limitations, like quenched approximation (no sea quark fluctuations), extrapolations to physical parameters, corrections for lattice artefacts etc. Table 3 shows a set of typical results for Charmonium. When simulating heavy quarks of mass m on a lattice of spacing a one inevitably has to cope with ma or $(ma)^2$ corrections, i.e. the condition for $ma \ll 1$ must be satisfied. While this can be reached for charmonium, bottomonium is still at the limit of the possibilities. One can expect that reliable simulations will be possible in not to far future.

9.5.1 More on the heavy quark potential

The leading heavy quark potential (9.2) has a non-perturbative string like IR piece, which actually can be calcuated by lattice QCD, and a perturbative Coulombic UV piece which recieves higher order perturbative QCD corrections [31, 32].

The static potential is defined in a manifestly gauge invariant way in terms of the vacuum expectation value of a Wilson loop [33, 34]

$$V(r) = -\lim_{T\to\infty} \frac{1}{T} \ln \left\langle \mathcal{W}_\Gamma \right\rangle \ , \quad \mathcal{W}_\Gamma = \widetilde{\mathrm{Tr}}\, \mathcal{P} \exp\left(\mathrm{i}\, g \oint_\Gamma \mathrm{d}x_\mu A_\mu \right) \ . \tag{9.8}$$

The path Γ is taken to be rectangular loop with time extension T and spatial extension r (see Fig. 9.13). The prescription $\mathcal{P}$ indicates that the gauge fields A_μ are path-ordered along the loop. The color trace is normalized according to $\widetilde{\mathrm{Tr}}(..) = \mathrm{Tr}(..)/\mathrm{Tr}\mathbb{1}$.

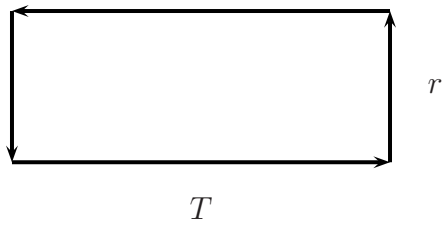


Figure 9.13: Wilson loops (closed path Γ) used in the definition of the static potential.

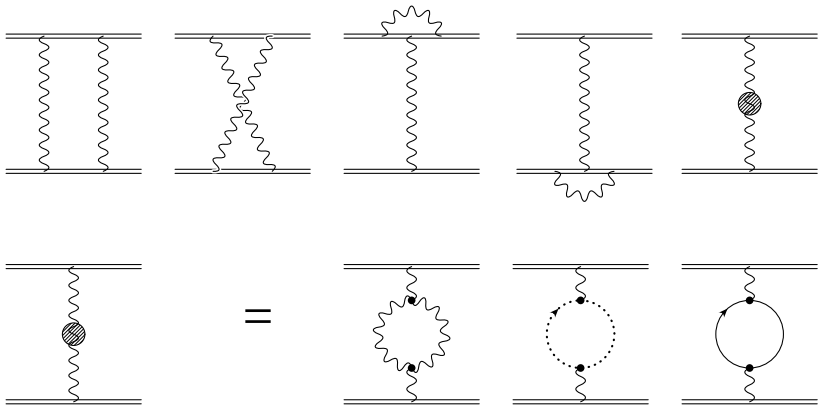


Figure 9.14: The one–loop diagrams contributing to the static potential. Double, wiggly, dotted and solid lines denote source, gluon, ghost and (light) fermion propagators, respectively. A blob on a gluon line stands for one–loop self–energy corrections.

It can be shown that contributions to Eq. (9.8) containing connections to the spatial components of the gauge fields $A_i(\vec{r}, \pm T/2)$ vanish in the limit of large time extension T in perturbation theory (at least up to two loops). Therefore, Eq. (9.8) can be simplified to

$$V_{\rm pert} = -\lim_{T\to\infty} \frac{1}{T} \ln \left\langle \widetilde{\rm Tr}\, T \exp\left(\int_x J_\mu^a A_\mu^a \right) \right\rangle \ , \tag{9.9}$$

where T now means time ordering and the static sources separated by the distance $r = |\vec{r} - \vec{r}\,'|$ are given by

$$J_\mu^a(x) = {\rm i}g\,\delta_{\mu 0}\,T^a \ \left[\, \delta(\vec{x} - \vec{r}) - \delta(\vec{x} - \vec{r}\,') \,\right] \ ,$$

where T^a are the generators in the fundamental representation.

Expanding the expression in Eq. (9.9) perturbatively, one encounters in addition to the usual Feynman rules the source–gluon vertex ${\rm i}\,g\delta_{\mu 0}T^a$, with an additional minus sign for the antisource. Furthermore, the time–ordering prescription generates step functions, which can be viewed as source propagators, analogous to the heavy–quark effective theory (HQET).

The perturbative corrections have been calculated to two loops. The relevant diagrams at one–loop are displayed in Fig. 9.14 and have been calculated in [33, 35]. Shown are the bare diagrams which have to be supplemented as usual by the relevant counterterms. The two–loop corrections stem from diagrams as shown in Fig. 9.15. The renormalization counterterms have to be included as usual. The calculations have been performed in [36–38]. More recently also the three–loop [39] and four–loop [40, 41] leading log correction have been calculated.

Important checks of the calculation include

- gauge independence of appropriate classes of diagrams,
- confirmation of cancelation of infrared divergences,
- correct renormalization properties.

The result for the static potential in coordinate space including the complete two–loop as well as the logarithically enhanced three–loop and four–loop terms is given by

$$V(r) = -C_F\,\frac{\alpha_V(1/r)}{r} \ , \tag{9.10}$$

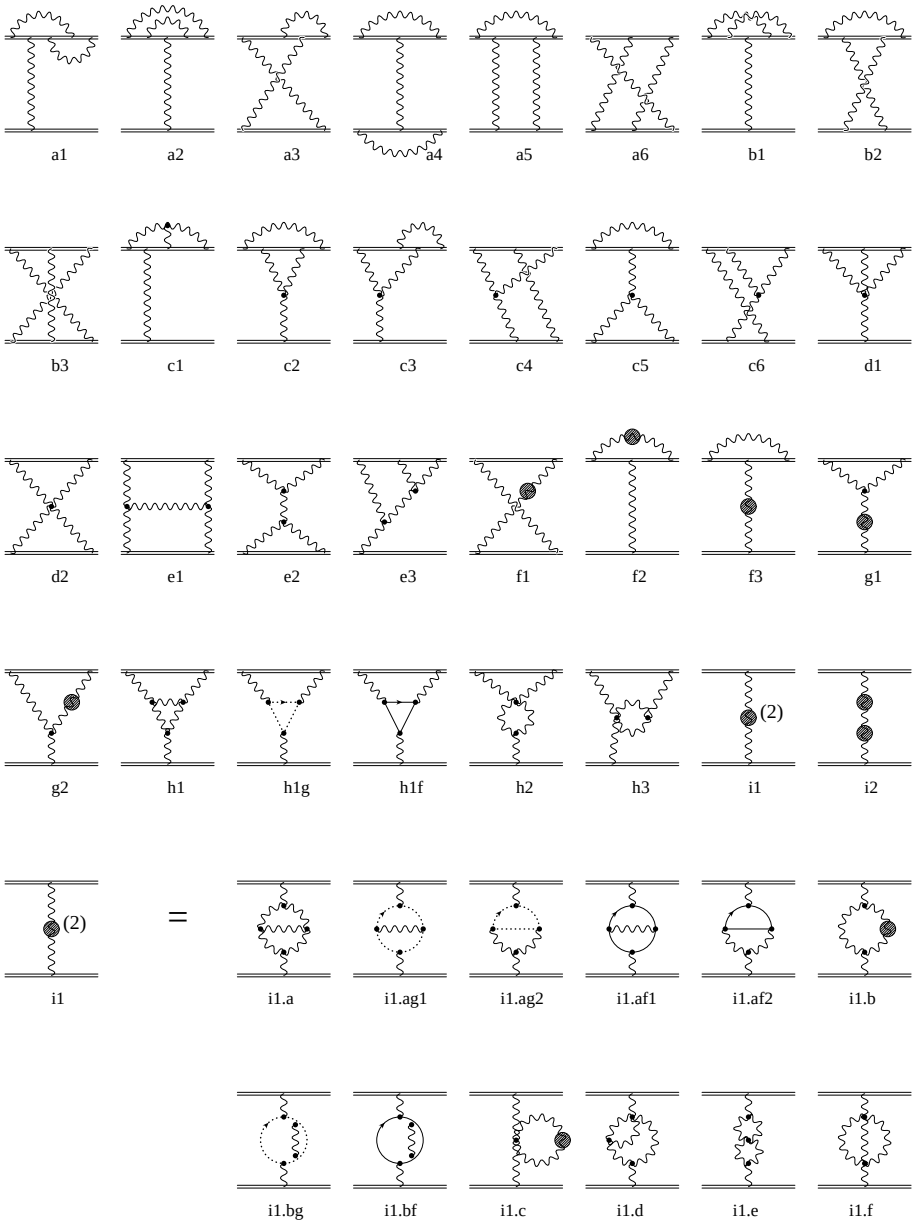


Figure 9.15: Classes of two–loop diagrams contributing to the static potential. Double, wiggly, dotted and solid lines denote source, gluon, ghost and (light) fermion propagators, respectively. A blob on a gluon line stands for one–loop self–energy corrections (from [36]).

with

$$\begin{aligned}\alpha_V(1/r) &= \alpha_s(1/r)\Bigg\{1+\left(\frac{\alpha_s(1/r)}{4\pi}\right)\,[a_1+2\gamma_E\beta_0] \\ &+\left(\frac{\alpha_s(1/r)}{4\pi}\right)^2\left[a_2+\left(\frac{\pi^2}{3}+4\gamma_E^2\right)\,\beta_0^2+2\gamma_E\,(\beta_1+2\beta_0 a_1)\right]+\left(\frac{\alpha_s(1/r)}{4\pi}\right)^3\left[a_3^L\,\ln\frac{C_A\alpha_s(1/r)}{2}+\tilde{a}_3\right] \\ &+\left(\frac{\alpha_s(1/r)}{4\pi}\right)^4\left[a_4^{L2}\,\ln^2\frac{C_A\alpha_s(1/r)}{2}+a_4^L\,\ln\frac{C_A\alpha_s(1/r)}{2}+\tilde{a}_4\right]+\cdots\Bigg\}\,. \qquad (9.11)\end{aligned}$$

Here, $\alpha_s(1/r)=\alpha_{\overline{\rm MS}}(\mu=1/r)$ and γ_E is Euler's constant. The first two terms of the beta function needed here are given by $\beta_0=\frac{11}{3}\,C_A-\frac{4}{3}\,T_FN_f$ and $\beta_1=\frac{34}{3}\,C_A^2-4C_FT_FN_f-\frac{20}{3}\,C_AT_FN_f$. The expansion coefficients are given by

$$a_1 \;=\; \frac{31}{9}\,C_A-\frac{20}{9}\,T_FN_f\;,$$

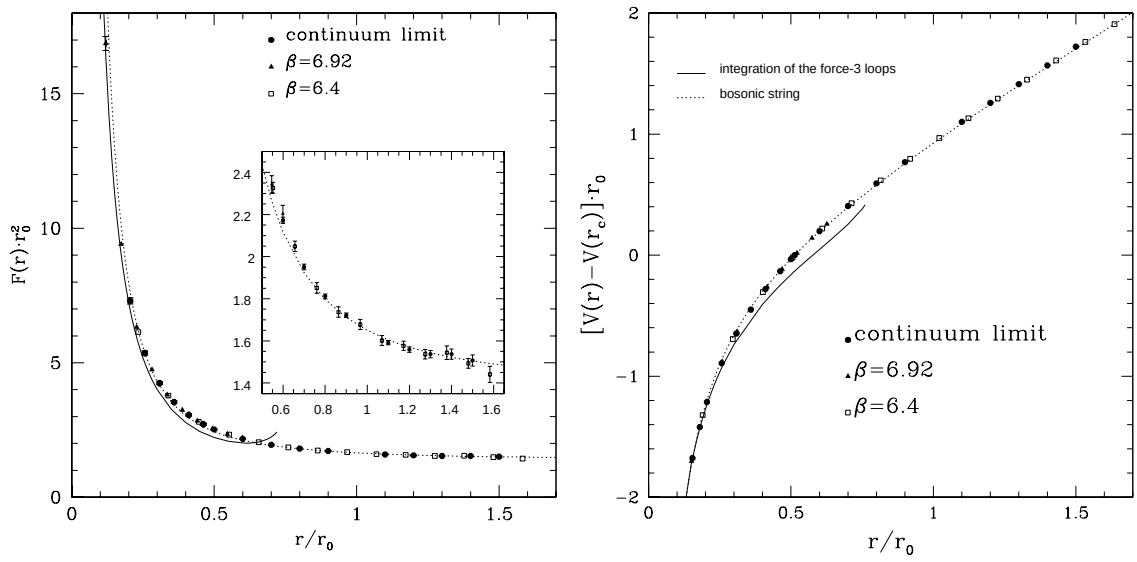


Figure 9.16: The force and the static potential. The dashed line represents the bosonic string model and the solid line the prediction of perturbation theory as detailed in the text. From [22].

$$
\begin{aligned}
a_2 &= \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right) C_A^2 - \left(\frac{55}{3} - 16\zeta(3)\right) C_F T_F N_f \\
&\quad + \frac{400}{81}\, T_F^2 N_f^2 - \left(\frac{1798}{81} + \frac{56}{3}\,\zeta(3)\right) C_A T_F N_f \ , \\
a_3^L &= \frac{16\pi^2}{3}\, C_A^3 \qquad (9.12) \\
a_4^{L2} &= \frac{16\pi^2}{3}\, C_A^3 \left(-\frac{11}{3}\, C_A + \frac{4}{3}\, T_F\, N_f\right) \\
a_4^L &= 16\pi^2 C_A^3 \left[a_1 + 2\gamma_e\beta_0 + T_F N_f \left(-\frac{40}{27} + \frac{8}{9}\,\ln 2\right) + C_A \left(\frac{149}{27} - \frac{22}{9}\,\ln 2 + \frac{4}{9}\,\pi^2\right)\right] \ .
\end{aligned}
$$

This establishes the "constant" a of Eq. (9.2) with remarkable accuracy. Mass effects to two–loops have been calculated in [38, 42]. The three and four loop results are incomplete: the constant pieces $\tilde{a}_3$ and $\tilde{a}_4$ are not yet known. We have not included the results for the "ultrasoft" contributions to a_3 and a_4 which have been given in [43–46]. Since corrections are potentially large and very much scheme dependent, a complete calculation of a_3 is highly needed.

As already mentioned, lattice QCD allows one to compute the string constant σ of Eq. (9.2). A computation which explored the static potential in the infrared and down to perturbative short distance scales has been performed in [22]. For large values of r, the force $F(r)$ is expected to be given by a constant, the string tension σ. The correction term is proportional to $1/r^2$ and has a universal coefficient $\pi/12$ (Lüscher term) [47]. This yields the parameter free bosonic string model,

$$
F(r) = \frac{\mathrm{d}V(r)}{\mathrm{d}r} = \sigma + \frac{\pi}{12\, r^2} \ , \quad \sigma r_0^2 = 1.65 - \pi/12 \, , \ r_0 = 0.5 \text{ fm} \ , \qquad (9.13)
$$

which is in excellent agreement with the $N_f = 0$ lattice result Fig. 9.16 for a surprisingly wide range down to $r \geq 0.8\, r_0$. The agreement at lower r at first sight looks accidental. In any case one would expect corrections to this formula to be negligible only for much larger r. However, in a high precision computation [48] the coefficient of the subeading term proportional to r^{-2}

was shown to agree with the one in the string model with high accuracy. Also the next term in the expansion has been determined recently [49]. In any case eq. (9.13) is a very good effective description of $F(r)$ for $0.8\,r_0 \le r \le 1.6\,r_0$.

The reference length scale r_0 (Sommer scale) [50] used in this analysis is defined in terms of the force $F(r)$ between static quarks by the implicit equation

$$r^2 F(r)|_{r=r(c)} = c\,, \quad r_0 = r(1.65)\,.$$

In QCD, r_0 has a value of about 0.5 fm. At short distances the force may be obtained by an integration of the perturbative renormalization group [for $N_f = 0$],

$$F(r) = C_{\rm F}\,\bar{g}^2_{\rm qq}(r)/(4\pi r^2)\,, \quad C_{\rm F} = 4/3\,, \tag{9.14}$$

$$-r\,\frac{\rm d}{{\rm d}r}\,\bar{g}_{\rm qq} = \beta(\bar{g}_{\rm qq}) = -\sum_{\nu=0}^{2} b_\nu\,\bar{g}^{2\nu+3}_{\rm qq}\,, \tag{9.15}$$

$$b_0 = \frac{11}{16\pi^2}\,,\ b_1 = \frac{102}{(16\pi^2)^2}\,,\ b_2 = \frac{1}{(4\pi)^6}\left(-3470 + 2519\frac{\pi^2}{3} - 99\frac{\pi^4}{4} + 726\zeta(3)\right)$$

Here the 3-loop coefficient, b_2, could be extracted from [36, 37, 42]. Inserting the result

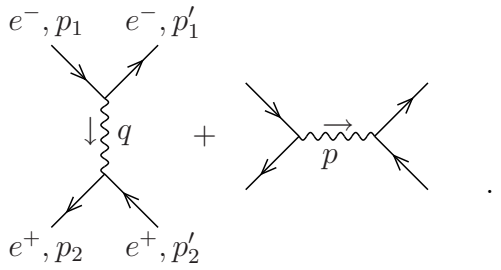
$$\Lambda_{\overline{\rm MS}}\,r_0 = 0.602(48) \tag{9.16}$$

from [51], as well as the known relation between $\Lambda_{\overline{\rm MS}}$ and $\Lambda_{\rm qq}$ [33, 35] to fix the integration constant, one has a parameter free perturbative prediction. For short distances, $r < 0.3\,r_0$, the perturbative prediction $V(r) = V(0.3\,r_c) + \int_{0.3\,r_c}^{r} {\rm d}y F(y)$ agrees rather well with the non-perturbative results. However, the convergence of the pQCD result very much depends on the renormalization scheme utilized, this is discussed in detail in [52]. A corresponding lattice QCD calculation including dynamical light quarks ($N_f = 3$) at present does not exist, but would be very important. For more recent investgations of the heavy quark potential see also [53] and [43–46].

9.6 A digression: positronium in QED

Positronium is an e^+e^- bound state, a state very much like a hydrogen atom, except that the two partners are of equal mass and the positron is the antiparticle of the electron. The theory of the positronium is a classical topic in QED textbooks and there are a number of excellent expositions (see e.g. *Landau and Lifshitz Course of Theoretical Physics Vol 4*).

Starting point is the $e^+e^- \to e^+e^-$ scattering amplitude at tree level:



The amplitude is given by

$$\begin{aligned} T_{fi}^{e^+e^-} &= e^2 \left\{ \bar{u}(p'_1, r'_1)\, \gamma^\mu\, u(p_1, r_1)\, \bar{v}(p_2, r_2)\, \gamma^\nu\, v(p'_2, r'_2)\, D_{\mu\nu}(q) \right. \\ &\quad \left. -\bar{v}(p_2, r_2)\, \gamma^\mu\, u(p_1, r_1)\, \bar{u}(p'_1, r'_1)\, \gamma^\nu\, v(p'_2, r'_2)\, D_{\mu\nu}(p) \right\} \ . \end{aligned}$$

We will justify below that the non-relativistic limit is well justified as a leading approximation. A systematic expansion in small velocities v/c may be worked out in a straight forward manner if one writes quantities in natural units, with the appropriate factors in c, in particular. The spinors then read

$$\begin{aligned} u(p,r) &= \frac{1}{\sqrt{c}} \sqrt{p^0 + mc} \begin{pmatrix} U(r) \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+mc}\, U(r) \end{pmatrix} \\ v(p,r) &= \frac{1}{\sqrt{c}} \sqrt{p^0 + mc} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{p^0+mc}\, V(r) \\ V(r) \end{pmatrix} \end{aligned}$$

and the photon propagator

$$\begin{aligned} D_{\mu\nu} &= -\left(g_{\mu\nu} - (1-\xi)\, \frac{q_\mu q_\nu}{q^2} \right) \frac{1}{q^2 + \mathrm{i}\varepsilon} \\ &= D_{\mu\nu}^{\text{Coulomb}}(q) + D_{c\,\mu\nu}(q)\,, \quad \text{with} \\ D_{\mu\nu}^{\text{Coulomb}}(q) &: \quad D_{00} = \frac{1}{\vec{q}^{\,2}}\ ; \quad D_{0i} = 0\ ; \quad D_{ik} = \frac{1}{q^2 + \mathrm{i}\varepsilon} \left(\delta_{ik} - \frac{q_i q_k}{\vec{q}^{\,2}} \right) \ , \\ D_{c\,\mu\nu}(q) &= O(1/c) \ . \end{aligned}$$

The Schrödinger wave functions for the scattering states then are given by

$$\begin{aligned} U(r) &= U_-(r) \qquad \text{for the electron} \ , \\ V(r) &= \mathrm{i}\,\sigma_2\, U_+(r) \qquad \text{for the positron} \ . \end{aligned}$$

We then obtain

$$T_{fi}^{e^+e^-} = -2\, m_1\, 2\, m_2\, U_-^{+\prime}\, U_+^{+\prime}\, U(\vec{p}_1, \vec{p}_2, \vec{q})\, U_+\, U_- \ ,$$

with the following total interaction operator (Breit potential) for e^+e^-–scattering:

$$\begin{aligned} U(\vec{p}_1,\vec{p}_2,\vec{q}) &= -e^2\Big\{\frac{1}{\vec{q}^{\,2}} - \frac{1}{8m_1^2c^2} - \frac{1}{8m_2^2c^2} + \mathrm{i}\,\frac{\vec{\sigma}_-\,(\vec{q}\times\vec{p}_1)}{4m_1^2c^2\vec{q}^{\,2}} - \mathrm{i}\,\frac{\vec{\sigma}_+\,(\vec{q}\times\vec{p}_2)}{4m_2^2c^2\vec{q}^{\,2}} \\ &+\frac{(\vec{p}_1\cdot\vec{q})(\vec{p}_2\cdot\vec{q})}{m_1m_2c^2\vec{q}^{\,4}} - \frac{(\vec{p}_1\cdot\vec{p}_2)}{m_1m_2c^2\vec{q}^{\,2}} + \mathrm{i}\,\frac{(\vec{q}\times\vec{p}_1)\cdot\vec{\sigma}_+}{2m_1m_2c^2\vec{q}^{\,2}} - \mathrm{i}\,\frac{(\vec{q}\times\vec{p}_2)\cdot\vec{\sigma}_-}{2m_1m_2c^2\vec{q}^{\,2}} \\ &+\frac{(\vec{\sigma}_-\cdot\vec{q})\,(\vec{\sigma}_+\cdot\vec{q})}{4m_1m_2c^2\vec{q}^{\,2}} - \frac{\vec{\sigma}_+\vec{\sigma}_-}{4m_1m_2c^2} - \frac{3+\vec{\sigma}_+\vec{\sigma}_-}{8m_2m_2c^2}\Big\}\ . \end{aligned}$$

We have given the result of the expansion up to $O((1/c)^3)$for different masses, for our purpose we set $m_1 = m_2 = m$. It is useful to write the corresponding Hamiltonian in configuration space in the c.m. system:

$$\begin{aligned} \mathbf{H} &= \frac{\vec{p}^{\,2}}{m} - \frac{\alpha}{r} + V_1 + V_2 + V_3\ , \\ V_1 &= -\frac{\vec{p}^{\,4}}{4m^3c^2} + 4\pi\,\mu_0^2\,\delta(\vec{r}) - \frac{\alpha}{2m^2c^2}\,\frac{1}{r}\,\Big[\vec{p}^{\,2} + \frac{\vec{r}\,(\vec{r}\,\vec{p})\,\vec{p}}{r^2}\Big]\ , \\ V_2 &= 6\,\mu_0^2\,\frac{1}{r^3}\,\vec{\ell}\cdot\vec{S}\ , \\ V_3 &= 6\,\mu_0^2\,\frac{1}{r^3}\,\Big[\frac{(\vec{S}\cdot\vec{r})(\vec{S}\cdot\vec{r})}{r^2} - \frac{1}{3}\,\vec{S}^{\,2}\Big] + 4\pi\mu_0^2\,(\frac{7}{3}\vec{S}^{\,2} - 2)\,\delta(\vec{r})\ , \end{aligned} \tag{9.17}$$

where

$$\begin{aligned} &\mu_0 = \tfrac{e\hbar}{2mc} && \text{Bohr magneton,} \\ &\hbar\vec{\ell} = \vec{r}\times\vec{p} && \text{orbital angular momentum,} \\ &\vec{S} = \tfrac{\vec{\sigma}_+ + \vec{\sigma}_-}{2} && \text{total spin } (\vec{S}^{\,2} = \tfrac{1}{2}\,(3 + \vec{\sigma}_+\vec{\sigma}_-)). \end{aligned}$$

Using this Hamiltonian we are able to treat the positronium bound state problem perturbatively. The undisturbed Hamiltonian is given by

$$\mathbf{H} = \frac{\vec{p}^{\,2}}{m} - \frac{\alpha}{r}\ ,$$

which can be solved, and yields the hydrogen spectrum with the replacement $m \to m/2$:

$$E_n = -\frac{m\alpha^2}{4\hbar^2 n^2}\ ; \quad n \text{ the principal quantum number.}$$

The perturbations V_i $(i = 1,2,3)$ yield the fine structure splitting, with the classification of the levels according to the angular momentum. Since only $\vec{S} = \frac{\vec{\sigma}_+ + \vec{\sigma}_-}{2}$, not however $\vec{\mu}_S = \mu_0\,(\vec{\sigma}_+ - \vec{\sigma}_-)$, the magnetic spin moment, are present in the Hamiltonian, we have

$$[\mathbf{H}, \vec{S}^{\,2}] = 0\ ,$$

which means that the total spin is a good quantum number(holds exactly):

$$\begin{aligned} S = 0 \quad &: \quad \text{para–positronium,} \\ S = 1 \quad &: \quad \text{ortho–positronium.} \end{aligned}$$

Furthermore, parity P, time reversal T and charge conjugation C provide constraints. For a particle-antiparticle pair the phases of P, T and C are given by

$$\eta_P \bar{\eta}_P = (-1)^{2j} \; ; \;\; \eta_T \bar{\eta}_T = \eta_C \bar{\eta}_C = 1 \;\; ,$$

with j the spin. The total parity of the e^+e^-–system therefore is

$$P = (-1)^{\ell+1} \;\; .$$

The wave function of the e^+e^-–system is therefore odd under exchange of the positions, the spin and charge conjugation:

$$(-1)^{\ell} \cdot (-1)^{S+1} \cdot C = -1 \;\; ,$$

such that

$$C = (-1)^{S+\ell} \; ; \;\; PC = (-1)^{S+1} \;\; ,$$

which are symmetries of the electro-magnetic interactions. For the positronium system this means that the spin is conserved. If j is given, we thus have the two possibilities:

$$\begin{aligned} {}^1S_0 \;\; &: \;\; S = 0 \;\; ; \;\; \ell = j \;\; , \\ {}^3S_1 \;\; &: \;\; S = 1 \;\; ; \;\; \ell = j, j \pm 1 \;\; . \end{aligned}$$

Rather than going into a discussion of level splitting by the relativistic corrections here, we are interested in the possible positronium decay channels.

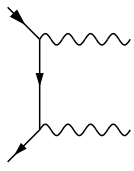
9.6.1 e^+e^-–annihilation into photons: $e^+e^- \to 2\gamma,\; 3\gamma$

The differential rate is given by the standard decay rate formula

$$\mathrm{d}\bar{\sigma}_{n\gamma} = \frac{(2\pi)^4 \, \delta^{(4)}(P_f - P_i)}{2\sqrt{\lambda(s, m_e, m_e}} \, \frac{1}{4} \sum_{\mathrm{pol}} |T_{fi}|^2 \mathrm{d}\mu(k_1) \cdots \mathrm{d}\mu(k_n) \;\; .$$

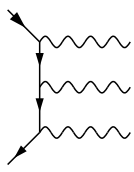
The leading processes are

T_{fi} : 2γ :



+ 1 permutation

3γ :



+ 5 permutation .

The total cross–sections in the non–relativistic approximation are given by

$$\begin{aligned} \bar{\sigma}_{2\gamma} &= \pi \left(\frac{e^2}{mc^2} \right)^2 \frac{c}{v_{\mathrm{rel}}} \;\; , \\ \bar{\sigma}_{3\gamma} &= \frac{4\,(\pi^2 - 9)}{3} \, \alpha \left(\frac{e^2}{mc^2} \right)^2 \frac{c}{v_{\mathrm{rel}}} \;\; , \end{aligned} \tag{9.18}$$

where $v_{\rm rel}$ is the relative velocity between electron and positron.

The transition probability enters the cross–section definition as

$$\mathrm{d}\sigma = \frac{\mathrm{d}w}{j} \ ,$$

where $\mathrm{d}w$ is the process probability and j the incoming particle current, in the c.m. system given by

$$j = \frac{v_1 + v_2}{V} \ .$$

Thus

$$\mathrm{d}w = j\,\mathrm{d}\sigma \ ,$$

and for the total rates averaged over spin. Thus

$$\bar{w}_{n\gamma} = j\,\bar{\sigma}_{n\gamma}$$

are the probability rates for $e^+e^- \to n\gamma$. In the c.m. system we have

$$j = \frac{v_{\rm rel}}{V}$$

for a free e^+e^--pair. For an e^+e^--pair bound in the state $\psi(r)$

$$j = v_{\rm rel}\,|\psi(0)|^2 \,,$$

in the non–relativistic approximation. For positronium in the ground state, we have

$$\psi(r) = \frac{1}{\sqrt{\pi a^3}}\,\mathrm{e}^{-r/a} \ ; \ \ a = \frac{2\hbar^2}{me^2} = 2\,r_0$$

where r_0 is the Bohr radius of the H-atom. Positronium may be considered as a H-atom with reduced mass $m \to m/2$.

We are now able to consider the decay of positronium.

Positronium annihilation: we consider the decay of the ground state. Energy–momentum conservation requires at least 2 γ's. As $P = (-1)^{\ell+1}$ and $CP = (-1)^{S+1}$ charge conjugation is $C = (-1)^{\ell+S}$. Therefore we have the correspondence

$$\begin{aligned} {}^1S_0 \quad &: \quad S = 0 \quad \text{para–positronium:} \quad C = \ \ 1 \to 2n \ \ \gamma' s \,, \\ {}^3S_1 \quad &: \quad S = 1 \quad \text{ortho–positronium:} \quad C = -1 \to 2n+1 \ \ \gamma' s \,, \end{aligned}$$

where $n = 1, 2, \cdots$, and the number of possible spin states is 1 for para and 3 for ortho, i.e., in total four. Note that our ground state wave function $\psi(r)$ is normalized to describe four degenerate states. Switching on the spin interaction responsible for the ground state splitting, para describes 1, ortho 3 spin states out of the 4. The corresponding properly normalized wave functions are then $\psi_{{}^1S_0}(r) = \frac{4}{1}\,\psi(r)$ and $\psi_{{}^3S_1}(r) = \frac{4}{3}\,\psi(r)$, respectively.

The probability for para–positronium decaying into 2 γ's is

$$w_0 = 4\bar{w}_{2\gamma} \quad = \quad 4\,|\psi(0)|^2 \ (v\,\bar{\sigma}_{2\gamma})\big|_{v\to 0} = \frac{mc^2\alpha^5}{2\hbar} \ , \tag{9.19}$$

with $\alpha = e^2/\hbar c$.

The life time is then given by

$$\tau_0 = w_0^{-1} = \frac{2\hbar}{mc^2\alpha^5} \simeq 1.23 \times 10^{-10}\text{sec} \ . \tag{9.20}$$

The corresponding width is

$$\Gamma_0 = \hbar/\tau_0 = \frac{mc^2\alpha^5}{2} \ll |E_0| = \frac{mc^2\alpha^2}{4} \ ,$$

and we can conclude that positronium is a quasi stationary system, as required for a non–relativistic treatment. The same exercise we can do for ortho–positronium.

Similarly, the probability for ortho–positronium decaying into 3 γ's is

$$w_1 = \frac{4}{3}\bar{w}_{3\gamma} \quad = \quad \frac{4}{3}\,|\psi(0)|^2 \ (v\,\bar{\sigma}_{3\gamma})\Big|_{v\to 0} = \frac{2\,(\pi^2-9)}{9\pi}\,\frac{mc^2\alpha^6}{\hbar} \ . \tag{9.21}$$

The life time is then

$$\tau_1 = w_1^{-1} = \frac{9\pi}{2\,(\pi^2-9)}\,\frac{\hbar}{mc^2\alpha^6} \simeq 1.4 \times 10^{-7}\text{sec} \ . \tag{9.22}$$

We note that $\tau_1 \gg \tau_0$ and we have the hierarchy: $\Gamma_1 \ll \Gamma_0 \ll |E_0|$. Ortho-positronium is much more stable than para-positronium, in spite of the fact that ortho-positronium lies higher in energy:

$$\Delta E_0 = E(^3S_1) - E(^1S_0) = \frac{7}{2}\,\alpha^2\,\frac{me^4}{\hbar^2} \simeq 8.2 \times 10^{-4} \quad \text{eV} \ .$$

The reason for the enhanced stability is that it cannot decay into 2 γ's, and the decay into 3 γ's requires one additional power in α:

$$w_1 = w_0\,\frac{4\,(\pi^2-9)}{9\pi}\,\alpha \ .$$

In addition, we observe a cancellation of two large terms, an irrational π^2 minus the rational 9, such that the factor $\pi^2 - 9 \simeq 0.87$ is an order smaller than the individual terms.

□ *End of the Digression*

For the application to Quarkonia systems we summarize a few relevant results from positronium (we go back to the unit system $\hbar = c = 1:\ e^2 \to \alpha$ here) in the following. The decay rates are

$$\begin{aligned} \Gamma(^1S_0 \to 2\gamma) \quad &= \quad 4\pi\,\frac{\alpha^2}{m^2}\,\left|\psi_{^1S_0}(0)\right|^2 \\ \Gamma(^3S_1 \to 3\gamma) \quad &= \quad \frac{4}{3}\,\frac{4}{3}\,(\pi^2-9)\,\frac{\alpha^3}{m^2}\,\left|\psi_{^3S_1}(0)\right|^2 \ , \end{aligned} \tag{9.23}$$

where $\psi_{^1S_0}(r)$ and $\psi_{^3S_1}(r)$ are the ground state wave functions for the $^1S_0$ spin singlet and the $^3S_1$ spin triplet,respectively. In the digression above $\psi(r)$ denoted the ground state wave function for the mixed state of all 4 spins. Therefore, the decay of the normalized state $\psi(r)$ proceeds with 1/4 propabiity to 2 γ and with 3/4 probability to 3 γ, and $\psi_{^1S_0}(r) = \frac{4}{1}\,\psi(r)$ and $\psi_{^3S_1}(r) = \frac{4}{3}\,\psi(r)$ in terms of the wave function normalized for the degenerate ground state.

The energy splitting between $^1S_0$ and $^3S_1$ states is explained by the spin dependent part of the potential. Since in the ground state the orbital angular momentum is zero, $\Delta\mathbf{H} = V_3$ only, contributes. Since the ground state is spherically symmetric, we have

$$\langle \frac{1}{r^3} \left(\frac{(\vec{S}\cdot\vec{r})(\vec{S}\cdot\vec{r})}{r^2} - \frac{1}{3}\vec{S}^{\,2} \right) \rangle = 0 \ .$$

As

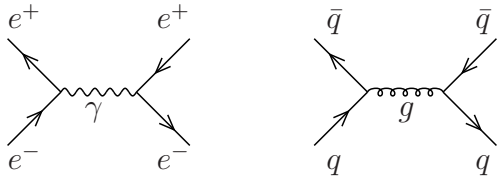
$$\vec{S}^{\,2} = \begin{cases} 0 & ; \quad S=0 \ , \\ S\,(S+1) = 2 & ; \quad S=1 \ , \end{cases}$$

we obtain

$$\Delta E_0 \ = \ \langle \Delta\mathbf{H}_{SS} \rangle = \int \mathrm{d}^3 r\, \psi^*(r)\, V_3(\vec{r})\, \psi(r) = 4\,\pi\,\mu_0^2 \cdot 2\,|\psi(0)|^2 = \frac{14}{3}\,\pi\,\frac{\alpha}{m^2}|\psi(0)|^2 \ ,$$

with $\mu_0 = \frac{e\hbar}{2mc}$.

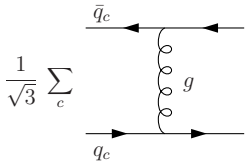
An important conceptual difference concerns the role of the annihilation diagramms



in QED on the one hand and in QCD on the other hand. In QED at the order discussed above the annihilation diagram only contributes a term

$$\delta\mathbf{H} = \frac{\pi\alpha}{m^2}\,\vec{S}^{\,2}\,\delta(\vec{r})$$

to the Hamiltonian. In QCD in place of (e^+e^-) we have a color singlet $\frac{1}{\sqrt{3}}\sum_c(\bar{q}_c q_c)$ and since the gluon carries color the annihilation diagram contribution vanishes (because of $\sum_c (T_i)_{cc} = \mathrm{Tr}\, T_i = 0$) for any colorless state. This means that in QCD only the scattering diagram



contributes to the $q\bar{q}$–potential. This leads to a modified spin-spin interaction term

$$\delta\mathbf{H} = \frac{4}{3}\,\frac{\pi\alpha}{m^2}\,\vec{S}^{\,2}\,\delta(\vec{r}) \ ,$$

with coefficient 4/3 replacing 7/3. The contribution of the QED scattering diagram to the ground state splitting is

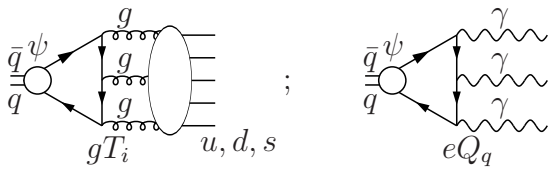
$$\Delta E_0^{(exchange)} = \frac{8}{3}\,\frac{\pi\alpha}{m^2}\,|\psi(0)|^2 \ . \tag{9.24}$$

9.7 Quarkonia Decays

The first question asked after the discovery of the J/ψ and later the Υ was: why are these heavy states so stable?

The answer is given by the Okubo-Zweig-Iizuka (OZI) rule (1964): decays of heavy resonances, which require disconnected quark diagrams are suppressed. In QCD language: processes which proceed via exchange of intermediate gluons only are suppressed. Thereby it is important to note that physical processes cannot take place via single gluon exchange, since this violates the color singlet condition of physical states. In fact the lowest quarkonia states have negative binding energy and lie below the $q\bar{q}$ thershold. Since QCD preserves quark flavors a decay into lighter quark flavors is possible only via intermediate gluons.

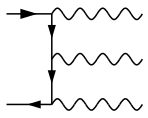
We first consider the J/ψ and Υ, which are the lowest lying $J^{PC} = 1^{--}$ vector mesons, and which correspond to ortho-positronium, i.e., they can decay only via $2n+1$ gluons ($n \geq 1$). It is interesting to compare directly the hadronic with the radiative decays:



The respective rates read

$$\Gamma(1^{--} \to 3g) = \frac{10}{81}\,\alpha_s^3\,4\,(\pi^2-9)\,\frac{|\psi(0)|^2}{m_q^2}\;;\;\; \Gamma(1^{--} \to 3\gamma) = \frac{4}{3}\,\alpha^3\,Q_q^6\,4\,(\pi^2-9)\,\frac{|\psi(0)|^2}{m_q^2} \tag{9.25}$$

The result may be obtained by using the parton cross–section



$$\bar{\sigma}_{3\gamma} = \frac{4\,(\pi^2-9)}{3}\,\frac{\alpha^3\,Q_q^6}{m_q^2 v}\;,$$

which determines the width of the bound $q\bar{q}$ system according to

$$\Gamma = \tau^{-1} = \lim_{v\to 0} v\,|\psi(0)|^2\,\bar{\sigma}(v)\;.$$

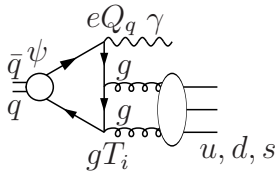
We can then calculate the ratio between hadronic and radiative dacays: with $T_i = \frac{\lambda_i}{2}$ the generators of $SU(3)$ in the fundamental representation we find

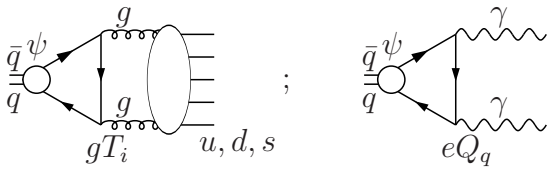
$$\begin{aligned}\frac{\Gamma(1^{--} \to 3g)}{\Gamma(1^{--} \to 3\gamma)} &= \frac{\alpha_s^3}{Q_q^6\,\alpha^3}\sum_{i,j,k}\left(\frac{\mathrm{Tr}\;(\lambda_i\lambda_j\lambda_k)_{\mathrm{symmetrized}}}{8\cdot 3}\right)^2 \\ &= \frac{\alpha_s^3}{Q_q^6\,\alpha^3}\sum_{i,j,k}\left(\frac{d_{ijk}}{12}\right)^2 = \frac{5}{54}\,\frac{\alpha_s^3}{Q_q^6\,\alpha^3}\;.\end{aligned} \tag{9.26}$$

We have used $\lambda_i\lambda_j + \lambda_j\lambda_i = \frac{4}{3}\,\delta_{ij} + 2\,d_{ijk}\,\lambda_k$ and $\sum_{ijk}(d_{ijk})^2 = \frac{40}{3}$. There is also a mixed single photon radiative decay

with a rate given by

$$\Gamma(1^{--} \to \gamma 3g) = \frac{8}{9}\,\alpha\,Q_q^2\,\alpha_s^2\,4\,(\pi^2-9)\,\frac{|\psi(0)|^2}{m_q^2}\;. \tag{9.27}$$



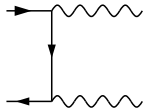


The pseudoscalar η_c as a $J^{PC} = O^{-+}$ state decays to an even number of gluons or photons, preferably to $2g$ or 2γ,

with rates given by

$$\Gamma(0^{-+} \to 2g) = \frac{8}{3}\,\alpha_s\,\pi\,\frac{|\psi(0)|^2}{m_q^2}\;;\;\; \Gamma(0^{-+} \to 2\gamma) = 12\,\alpha^2\,Q_q^4\,\pi\,\frac{|\psi(0)|^2}{m_q^2}\;\;. \tag{9.28}$$

Again, the result may be obtained starting from the parton annihilation cross–section



$$\bar{\sigma}_{2\gamma} = \pi\,\frac{\alpha^2\,Q_q^4}{m_q^2 v}\;\;,$$

from which we get width of the bound $q\bar{q}$ system via

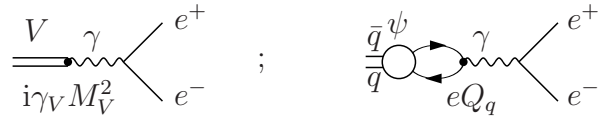
$$\Gamma = \tau^{-1} = \lim_{v\to 0} v\,|\psi(0)|^2\,\bar{\sigma}(v)\;\;.$$

Here the ratio between hadronic and radiative decay widths is given by

$$\begin{aligned}\frac{\Gamma(0^{-+} \to 2g)}{\Gamma(0^{-+} \to 2\gamma)} &= \frac{\alpha_s^2}{Q_q^4\,\alpha^2}\sum_{i,j}\left(\frac{\mathrm{Tr}\;(\lambda_i\lambda_j)}{4\cdot(\sqrt{3})^2}\right)^2 \\ &= \frac{\alpha_s^2}{Q_q^4\,\alpha^2}\sum_{i,j}\left(\frac{\delta_{ij}}{6}\right)^2 = \frac{2}{9}\,\frac{\alpha_s^2}{Q_q^4\,\alpha^2}\;\;.\end{aligned} \tag{9.29}$$

Note the color factors: color singlet states have a wave function $\propto \frac{1}{\sqrt{3}}\,\delta_{cc'}$ and $\sum_{\mathrm{color}} \frac{1}{\sqrt{3}}\,\delta_{cc'} = \sqrt{3}$, which in rates yields a factor $(\sqrt{3})^2 = 3$, as we learned earlier.

For the 1^{--} states leptonic decays play a central role as they proceede via the direct coupling to the photon. Thereby the VMD picture confronts the $q\bar{q}$ bound state picture:



The latter, characterized by the second diagram, is encoded by the van Royen-Weisskopf formula (see below)

$$\Gamma(V \to e^+e^-) = 16\,\pi\,\alpha^2\,Q_q^2\,\frac{|\psi(0)|^2}{M_V^2}\;\;, \tag{9.30}$$

where in our case $V = J/\psi, \cdots, \Upsilon, \cdots$.

For completeness let us calculate the given decay rate: the basic matrixelement for the transition $q\bar{q} \to \gamma^* \to e^+e^-$ is obtained with the transition operator

$$T_{\rm op} = -\frac{4\pi\,\alpha\,Q_q}{q^2}\,j^e_\mu\,j^{\mu q}\ ,$$

where $j^e_\mu = -\bar{e}\,\gamma_\mu\,e$ and $j^q_\mu = \bar{q}\,\gamma_\mu\,q$. We have used that from the photon propagator $D_{\mu\nu} = -g_{\mu\nu}\frac{1}{q^2}+$ gauge terms, only the first term contributes by gauge invariance. In vector meson decay $q^2 = M_V^2$. Then

$$\langle e^+e^-|T_{\rm op}|V\rangle = -\frac{4\pi\,\alpha\,Q_q}{M_V^2}\,\langle e^+e^-|j^e_\mu(0)|0\rangle_{\rm lep}\langle 0|j^{\mu q}(0)|V\rangle_{\rm had}\ ,$$

where $\langle e^+e^-|j^e_\mu(0)|0\rangle_{\rm lep} = -\bar{u}(p_-,r_-)\,\gamma_\mu\,v(p_+,r_+)$ and $\langle 0|j^{\mu q}(0)|V\rangle_{\rm had} = \bar{\gamma}_V\,M_V^2\varepsilon^\mu(\lambda)$. The width then follows from the standard 2–body decay formula

$$\Gamma = \frac{1}{2M}\,\frac{\overline{|T|^2}}{8\pi}\,\sqrt{1-\frac{4m_e^2}{M^2}}\ ,$$

with $\overline{|T|^2}$ is the spin averaged amplitude $\frac{1}{3}\,\sum_\lambda\,|T|^2$. The factor $\frac{1}{8\pi}\sqrt{1-4m_e^2/M_V^2} \simeq \frac{1}{8\pi}$ comes from the phase space intergal $(2\pi)^4\,\int {\rm d}\mu(p_+)\,{\rm d}\mu(p_-)\,\delta^{(4)}(p_V-p_+-p_-)$. In the terminology of Sect. **??**, the matrix elements squared read

$$\sum_{\rm spins}\,\bar{v}(p_+,r_+)\gamma_\mu u(p_-,r_-)\bar{u}(p_-,r_-)\gamma_\nu v(p_+,r_+) = 4\ell_{\mu\nu}$$

and

$$\sum_{\rm spins}\,\langle 0|j_\mu(0)|V(p_V,\lambda)\rangle\langle V(p_V,\lambda)|j_\nu(0)|0\rangle = \sum_\lambda\,\varepsilon_\mu(p_V,\lambda)\varepsilon^*_\nu(p_V,\lambda)\,M_V^4\bar{\gamma}_V^2 = h_{\mu\nu}$$

with

$$X_{\mu\nu} = \sum_\lambda\,\varepsilon_\mu(p_V,\lambda)\varepsilon^*(p_V,\lambda) = -g_{\mu\nu} + \frac{p_{V\mu}p_{V\nu}}{M_V^2}\ ;\ \ p_V \equiv q\ .$$

In the rest frame of the vector meson V we have $p_{V\mu} = (M_V,\vec{0}\,)$ and hence

$$X_{\mu\nu} = \left\{\begin{array}{ccc} 0 & ; & (\mu\nu) = (0,0) \\ \delta_{ik} & ; & (\mu\nu) = (i,k)\ . \end{array}\right.$$

This means that we need the spacial part of $\ell_{\mu\nu}$ only:

$$4\ell_{ik} = 2\,\left\{M_V^2\,\delta_{ik} - 4\,p_ip_k\right\}\ .$$

The momenta are $p_V = p_+ + p_- = (M_V,\vec{0}\,)$, thus $\vec{p}_+ = -\vec{p}_- = \vec{p}$ and $E_+ = E_- = E = M_V/2$. With $\vec{p}_+ - \vec{p}_- = 2\,\vec{p}\,;\ \ E^2 - m_e^2 = \vec{p}^{\,2}$ we arrive at

$$\left(M_V^4\bar{\gamma}_V^2\right)^{-1}\,4\ell_{\mu\nu}\,h^{\mu\nu} = 4\,\ell_{ik}\,\delta_{ik} = 4\,\sum_i\,\ell_{ii} = 2\,\left(3\,M_V^2 - 4\,\vec{p}^{\,2}\right) = 4\,M_V^2\,\left(1+\frac{2m_e^2}{M_V^2}\right)\ .$$

Finally, with

$$\sum|T|^2 = \frac{(4\pi\alpha Q_q)^2}{M_V^2}\cdot 4\,M_V^2\,\left(1+\frac{2m_e^2}{M_V^2}\right)\,\bar{\gamma}_V^2\,M_V^4\ ,$$

we arrive at the decay rate

$$\Gamma = \frac{1}{3}\,\frac{16\pi^2}{2M_V}\,\frac{\alpha^2Q_q^2}{M_V^2}\,\frac{4}{8\pi}\,\left(1+\frac{2m_e^2}{M_V^2}\right)\,\sqrt{1-\frac{4m_e^2}{M_V^2}}\,\bar{\gamma}_V^2\,M_V^4\ ,$$

and using the van Royen–Weisskopf formula (Nuovo Cimento **40** (1967))

$$\gamma_V^2 \doteq Q_q^2\,\bar{\gamma}_V^2 = 4\,N_c\,\langle Q_q\rangle^2\,|\psi(0)|^2/M_V^3$$

and the approximation $m_e \ll M_V$ we find

$$\Gamma = 16\,\pi\,\alpha^2\,\langle Q_q\rangle^2\,\frac{|\psi(0)|^2}{M_V^2} = \frac{4}{3}\,\pi\,\alpha^2\,\gamma_V^2\,M_V\ .$$

Note that $\dim\psi = 3/2$ and like in positronium, where $|\psi(0)|^2 = \frac{1}{\pi a^3} = \frac{m^3\alpha^3}{8\pi}$.

9.8 Matrix elements of mesonic bound states

Here we reproduce the derivation of the Van Royen - Weisskopf formula in detail. This exercise at the same time may be considered as a repetitorium of basic QFT concepts.
We first consider a quark field, single quark states and their properties:

$$\psi_{\alpha a}(x) \;=\; \sum_r \int \mathrm{d}\mu(p)\, \left\{ u_\alpha(p,r)\, a_a(\vec{p},r)\, \mathrm{e}^{-\mathrm{i}px} + v_\alpha(p,r)\, b_a^+(\vec{p},r)\, \mathrm{e}^{\mathrm{i}px} \right\}$$

with creation and annihilation operators satisfying the normalization

$$\left[a_a(\vec{p},r), a_{a'}^+(\vec{p}\,',r') \right]_+ = \delta_{aa'}\, \delta_{rr'}\, (2\pi)^3\, 2\omega_{\vec{p}\,'}\, \delta^{(3)}(\vec{p}-\vec{p}\,') \qquad \text{etc.}$$

The single quark states are then represented by

$$|\vec{p},r,a\rangle = a_a^+(\vec{p},r)\,|0\rangle \;\;,$$

and normalized according to

$$\langle \vec{p}\,',r',a'|\vec{p},r,a\rangle = \delta_{aa'}\, \delta_{rr'}\, (2\pi)^3\, 2\omega_{\vec{p}\,'}\, \delta^{(3)}(\vec{p}-\vec{p}\,') \;\;.$$

The integral measure

$$\mathrm{d}\mu(p) = \frac{\mathrm{d}^3p}{\hbar^3\,(2\pi)^3}\,\frac{1}{2\omega_p} \;;\;\; \omega_p = \sqrt{\vec{p}^{\,2}+m^2} \;\;,$$

denotes the relativistically invariant on–mass shell element equivalent to $\Theta(p^0)\,\delta(p^2-m^2)\,\mathrm{d}^4p$, which is normalized to one particle per unit volume element:

$$\sum_{a',r'} \int \mathrm{d}\mu(p)\, \langle \vec{p}\,',r',a'|\vec{p},r,a\rangle = 1 \;\;.$$

The particle density operator is thus given by

$$\vec{n}_a(\vec{p},r) = \frac{1}{(2\pi)^3\,2\omega_p}\, a_a^+(\vec{p},r)\, a_a(\vec{p},r) \;\;,$$

and the particle number operator

$$N_a = \sum_r \int \mathrm{d}^3p\, \vec{n}_a(\vec{p},r) = \sum_r \int \mathrm{d}\mu(p)\, a_a^+(\vec{p},r)\, a_a(\vec{p},r)$$

satisfies the comutation relation

$$\left[N_a, a^\pm(\vec{p},r) \right] = \pm\, \delta_{ab}\, a^\pm(\vec{p},r) \;\;.$$

The one–particle wave functions are represented by the Dirac spinors $u(p,r)$ for quarks and $v(p,r)$ for antiquarks, which are solutions of the momentum space Dirac equation and in our notation satisfy

$$\sum_r u_\alpha(p,r)\bar{u}_\beta(p,r) = (\not{p}+m)_{\alpha\beta} \quad \text{etc.}$$

The free wave pakets may be replaced by more general solutions of the Klein-Gordon equation

$$\mathrm{e}^{-\mathrm{i}px} \to f_{\vec{p}}(x) \;\;.$$

They are independent of any quantum numbers. We then have the single particle matrix elements

$$\begin{aligned}\langle 0|\psi_{\alpha b}(x)|\vec{p},r,a\rangle &= \sum_{r'}\int \mathrm{d}\mu(p')\,\left[a_b(\vec{p}\,',r'),a_a^+(\vec{p},r)\right]_+ \times u_\alpha(p',r')\,f_{\vec{p}\,'}(x)\\ &= u_\alpha(p,r)\,f_{\vec{p}\,'}(x)\ ,\end{aligned}$$

which represents the complete one particle wave function.

In the stationary case we may write

$$\begin{aligned}f_{\vec{p}}(x) &= \mathrm{e}^{-\mathrm{i}\omega_p x^0}\,g_{\vec{p}}(\vec{x})\ ;\ \ \omega_p=\omega(|\,\vec{p}\,|)\\ g^*_{\vec{p}}(\vec{x}) &= g_{-\vec{p}}(\vec{x})\ .\end{aligned}$$

We are interested in mesonic bound states of quarks. Interpolating fields for these states with the correct tranformation properties and the appropriate quantum numbers can be constructed with the help of the free quark fields as "composite" operators:

$$M_{\Gamma ik}(x)\ =\ \frac{1}{\sqrt{3}}\sum_c\ \bar{\psi}_{\alpha ci}(x)\ (\Gamma)_{\alpha\beta}\ \psi_{\beta ck}(x)\ , \tag{9.31}$$

where c is the color index and i,k are flavor indices. The field $M(x)$ in any case is a color singlet and depending on Γ we have

Γ =	$\mathbf{1}$	$\mathrm{i}\,\gamma_5$	γ^μ	$\gamma^\mu\gamma_5$	$\sigma^{\mu\nu}$
	scalar	pseudoscalar	vector	axialvector	tensor

defining the transformation property under Lorentz–transformations.

Consider a meson, a quark–antiquark color neutral bound state. Depending on the spin and parity combination, the meson may have spin 0, 1 or 2 with parities

Γ =	$\mathbf{1}$	$\mathrm{i}\,\gamma_5$	γ^μ	$\gamma^\mu\gamma_5$	$\sigma^{\mu\nu}$
spin	0	0	1	1	2
parity	+	−	−	+	+

The Fourier–transform of the composite field is

$$\tilde{M}_{\Gamma ik}(P)=\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}Px}\,M_{\Gamma ik}\ , \tag{9.32}$$

where P is the four momentum in the center of mass system of the meson.

We consider the "meson" in the c.m. frame at rest ($\vec{P}=0$) at time $t=x^0=0$:

$$|M(\vec{P}=0)\rangle=\int \mathrm{d}^3x\,\frac{1}{\sqrt{3}}\sum_c\ :\bar{\psi}_{\alpha ci}(x)\ (\Gamma)_{\alpha\beta}\ \psi_{\beta ck}(x):\Big|_{x^0=0}|0\rangle\ .$$

By the Wick ordering we achieve that when experessed in terms of creation and annihilation operators only the term with two creation operators in non-vanishing. This term creates the

desired quark-antiquark color singlet:

$$\begin{aligned}
|M(\vec{P}=0)\rangle &= \frac{1}{\sqrt{3}}\sum_c\sum_{r,r'}\int \mathrm{d}\mu(p)\mathrm{d}\mu(p')\,\bar{u}_\alpha(p,r)\;(\Gamma)_{\alpha\beta}\;v_\beta(p',r')\times \\
&\quad\times\underbrace{\int \mathrm{d}^3x\;\mathrm{e}^{\mathrm{i}\,(p'+p)\,x}\Big|_{x^0=0}}_{(2\pi)^3\,\delta^{(3)}(\vec{p}\,'+\vec{p})}\;a^+_{ci}(\vec{p},r)\,b^+_{ck}(\vec{p}\,',r')\,|0\rangle \\
&= \sum_{r,r'}\int \mathrm{d}\mu(p)\frac{1}{2\omega_p}\,\bar{u}_\alpha(\vec{p},r)\,(\Gamma)_{\alpha\beta}\;v_\beta(-\vec{p},r')\;\cdot\;\frac{1}{\sqrt{3}}\sum_c\;a^+_{ci}(\vec{p},r)\,b^+_{ck}(-\vec{p},r')\,|0\rangle\;\;,
\end{aligned}$$

where the $\vec{x}$ integration yielded a δ–function, which allowed us to perform the $\vec{p}\,'$ integration trivially. As expected, quark and antiquark have opposite momenta, such that $\vec{P}=0$.

Of course the quarks in a bound state are not free, such that the free wave functions $u(p,r)\,\mathrm{e}^{-\mathrm{i}\,px}$ etc. have to be replaced by corresponding, generally unknown, wave function of the bound state. The wave function of the bound state also determines the spin and flavor properties of the bound system. A creation operator for a mesonic bound state in the rest system, formally has the form:

$$C^+_M(\vec{P}=0,\lambda)=N\cdot\sum_{cc',ii'}\sum_{r,r'}\int \mathrm{d}\mu(p)\;f(\vec{p})\,\varphi_{cc';ii'}(\lambda;r,r')\;\cdot\;a^+_{ci}(\vec{p},r)\,b^+_{c'i'}(-\vec{p},r')\;\;,$$

where $f(\vec{p})$ is the momentum distribution of the valence quarks and the coefficients $\varphi_{cc';ii'}(\lambda;r,r')$ have to be determined in such a way that the meson creation operator C^+_M has the desired quantum numbers. The creation operators a^+ and b^+ of course do not create free quarks, but the so called valence quarks, which characterize the macroscopic properties of the sysyem. Besides the valence quarks there are the virtual quarks, called sea–quarks, and the gluons. N is a normalization factor. The state

$$|M(\vec{P}=0,\lambda)\rangle=C^+_M(\vec{P}=0,\lambda)|0\rangle \tag{9.33}$$

describes a meson at rest. In order to fix the normalization we need a propagating state. Our standard normalization is then given by

$$\langle M(\vec{P},\lambda')|M(\vec{0},\lambda)\rangle=\delta_{\lambda\lambda'}\,2P^0\,(2\pi)^3\,\delta^{(3)}(\vec{P})\;\;,$$

where $P^0=M$ is the mass of the meosn. Note that with this normalization only, our standard formulae for T–matrix elements, cross sections and decay widths remain valid.

Formally, a moving quark–antiquark state at time $t=0$ is given by

$$\begin{aligned}
|M(\vec{P},\lambda)\rangle &= \int \mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}\,\vec{P}\,\vec{x}}\,\frac{1}{\sqrt{3}}\sum_c\;:\bar{\psi}_{\alpha ci}(x)\;(\Gamma)_{\alpha\beta}\;\psi_{\beta ck}(x):|0\rangle \\
&= \frac{1}{\sqrt{3}}\sum_c\sum_{r,r'}\int \mathrm{d}\mu(p)\mathrm{d}\mu(p')\,\bar{u}_\alpha(p,r)\;(\Gamma)_{\alpha\beta}\;v_\beta(p',r')\times \\
&\quad\times\underbrace{\int \mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}\,(\vec{P}-\vec{p}\,'-\vec{p})\,\vec{x}}}_{(2\pi)^3\,\delta^{(3)}(\vec{P}-\vec{p}\,'-\vec{p})}\;a^+_{ci}(\vec{p},r)\,b^+_{ck}(\vec{p}\,',r')\,|0\rangle \\
&= \sum_{r,r'}\int \mathrm{d}\mu(p)\frac{1}{2\omega_{P-p}}\,\bar{u}_\alpha(\vec{p},r)\,(\Gamma)_{\alpha\beta}\;v_\beta(\vec{P}-\vec{p},r')\;\cdot\;\frac{1}{\sqrt{3}}\sum_c\;a^+_{ci}(\vec{p},r)\,b^+_{ck}(\vec{P}-\vec{p},r')\,|0\rangle\;\;.
\end{aligned}$$

We thus can calculate the normalization

$$\begin{aligned}
&\langle 0|C_M(\vec{P},\lambda')\,C_M^+(\vec{0},\lambda)|0\rangle \\
= &\sum_{s,s'}\int \mathrm{d}\mu(q)\,\frac{1}{2\omega_{\vec{P}-\vec{q}}}\,\bar{v}(\vec{P}-\vec{q},s')\,\Gamma_{\lambda'}\,u(\vec{q},s)\times\sum_{r,r'}\int \mathrm{d}\mu(p)\,\frac{1}{2\omega_{\vec{p}}}\,\bar{u}(\vec{p},r)\,\Gamma_{\lambda}\,v(-\vec{p},r') \\
&\times\sum_{c,c'}\left(\frac{1}{\sqrt{3}}\right)^2\,\delta_{cc'}\,\delta_{ii'}\,\delta_{cc'}\,\delta_{kk'}\,\delta_{rs}\,\delta_{r's'}\,(2\pi)^6\,2\omega_{\vec{p}}\,2\omega_{\vec{P}-\vec{q}}\times\delta^{(3)}(\vec{q}-\vec{p})\,\delta^{(3)}(\vec{P}-\vec{q}+\vec{p}) \\
= &\;3\,\delta_{ii'}\delta_{kk'}\sum_{r,r'}\int \mathrm{d}\mu(p)\,\frac{1}{2\omega_{\vec{p}}}\,\bar{v}(-\vec{p},r')\,\Gamma_{\lambda'}\,u(\vec{p},r)\,\bar{u}(\vec{p},r)\,\Gamma_{\lambda}\,v(-\vec{p},r')\times(2\pi)^3\,\delta^{(3)}(\vec{P})\ .
\end{aligned}$$

We now have to replace the free wave functions by the ones of the bound state. This is achieved by the substitution

$$\sum_{ii'}\sum_{cc'}\sum_{rr'}\frac{1}{\sqrt{3}}\,\frac{1}{2\omega_{\vec{p}}}\,\bar{u}(p,r)\,\Gamma_\lambda\,v(-\vec{p},r')\,\delta_{cc'}\,\cdots\Rightarrow\sum_{ii'}\sum_{cc'}\sum_{rr'}N\,f(\vec{p})\,\varphi_{cc';ii'}(\lambda,r,r')\,\cdots$$

and results in

$$\begin{aligned}
&\langle M(\vec{P},\lambda')|M(\vec{0},\lambda)\rangle=\sum_{\bar{c}\bar{c}'}\sum_{\bar{i}\bar{i}'}\sum_{cc'}\sum_{ii'}\sum_{ss'}\sum_{rr'}\varphi^*_{\bar{c}\bar{c}';\bar{i}\bar{i}'}(\lambda',s,s')\,\varphi_{cc';ii'}(\lambda,r,r') \\
\times\;&\delta_{\bar{c}c}\,\delta_{\bar{c}'c'}\,\delta_{\bar{i}i}\,\delta_{\bar{i}'i'}\,\delta_{rs}\,\delta_{r's'}\;\times\int \mathrm{d}\mu(p)\,|f(\vec{p})|^2\,2\,\omega_{\vec{p}}\,(2\pi)^3\,\delta^{(3)}(\vec{P})\cdot N^2\ .
\end{aligned}$$

Our normalization thus requires the bound state wave function factors to have the normalizations:

$$\begin{aligned}
&1.\qquad \sum_{cc'}\sum_{ii'}\sum_{r,r'}\varphi^*_{cc';ii'}(\lambda,r,r')\,\varphi_{cc';ii'}(\lambda,r,r')=\delta_{\lambda'\lambda}\ , \\
&2.\qquad \int\frac{\mathrm{d}^3p}{(2\pi)^3}\,|f(\vec{p})|^2=1\ , \\
&3.\qquad N=\sqrt{2M}\ .
\end{aligned}\tag{9.34}$$

Herewith we are equipped to calculate simple matrix elements. We are interested here in the matrix element

$$\langle 0|J_\mu(0)|M(\vec{P}=0.\lambda)\rangle\tag{9.35}$$

of the electromagnetic current

$$J_\mu(x)=\sum_{c,i}Q_i\ :\bar{\psi}_{ci}(x)\,\gamma_\mu\,\psi_{ci}(x):\tag{9.36}$$

Only the term of J_μ exhibiting two annihilation operators yields a contribution,

$$\langle 0|:J_\mu(0):=\sum_{\bar{c},f}Q_f\sum_{ss'}\int \mathrm{d}\mu(q)\,\mathrm{d}\mu(q')\,\bar{v}_\alpha(q',s')\;(\gamma)_{\alpha\beta}\,u_\beta(q,s)\;\cdot\;\langle 0|b_{\bar{c},f}(\vec{q}',s')\,a_{\bar{c},f}(\vec{q},s)\ ,$$

and we obtain

$$\begin{aligned}
&\langle 0|J^\mu(0)\,C_M^+(\vec{P}=0,\lambda)|0\rangle \\
= &\;\sqrt{M}\sum_{\bar{c},f}Q_f\sum_{s,s'}\sum_{c,c'}\sum_{i,i'}\sum_{r,r'}\int \mathrm{d}\mu(q)\,\mathrm{d}\mu(q')\int \mathrm{d}\mu(p)\,f(\vec{p})\times\bar{v}_\alpha(q',s')\,\gamma^\mu_{\alpha\beta}\,u_\beta(q,s)
\end{aligned}$$

$$
\begin{aligned}
&\times \varphi_{cc';ii'}(\lambda;r,r')\,\delta_{\bar{c}c}\,\delta_{fi}\,\delta_{sr}\,(2\pi)^3\,2\omega_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q})\,\delta_{\bar{c}c'}\,\delta_{fi'}\,\delta_{s'r'}\,(2\pi)^3\,\delta^{(3)}(-\vec{p}-\vec{q}\,') \\
= \;& \sqrt{M}\sum_{\bar{c},f} Q_f \sum_{s,s'} \int \mathrm{d}\mu(p)\, f(\vec{p})\,\bar{v}_\alpha(-\vec{p},s')\,\gamma^\mu_{\alpha\beta}\,u_\beta(\vec{p},s)\,\varphi_{\bar{c}\bar{c};ff}(\lambda;s,s') \\
= \;& \sqrt{M}\sum_{\bar{c},f} Q_f \sum_{s,s'} \bar{v}(0,s')\,\gamma^\mu\,u(0,s)\int \mathrm{d}\mu(p)\, f(\vec{p})\,\frac{2m}{\omega_{\vec{p}}+m}\,\varphi_{\bar{c}\bar{c};ff}(\lambda;s,s') \\
\simeq \;& \sqrt{M}\sum_{\bar{c},f} Q_f \sum_{s,s'} \bar{v}(0,s')\,\gamma^\mu\,u(0,s)\,\varphi_{\bar{c}\bar{c};ff}(\lambda;s,s')\,\psi(0)\ ,
\end{aligned}
$$

where in the last step we assumed the non-relativistic limit to be appropriate. Let us work out the non-relativistic limit for the electromagnetic current explicitly:

$$
\begin{aligned}
& \bar{v}(-\vec{p},s')\,\gamma^\mu\,u(\vec{p},s) = v^+(-\vec{p},s')\,\gamma^0\,\gamma^\mu\,u(\vec{p},s) \\
= \;& (p^0+m)\left(V^+(s')\,\frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}, V^+(s')\right)\gamma^0\,\gamma^\mu\begin{pmatrix} U(r) \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\,U(r)\end{pmatrix} \\
= \;& \begin{cases} 0 & ;\ \mu=0 \\ \frac{2m}{p^0+m}\,V^+(s')\,\sigma_i\,U(s) & ;\ \mu=i \end{cases} \\
= \;& \frac{2m}{p^0+m}\,\bar{v}(0,s')\,\gamma^\mu\,u(0,s)\ .
\end{aligned}
$$

We have been using $\vec{\sigma}\cdot\vec{p}\,\sigma_i\,\vec{\sigma}\cdot\vec{p} = \vec{p}^{\,2}\,\sigma_i$ and $p^0 = \omega_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}$. Furthermore, taking the non-reletivistic approximation we have

$$
\int \mathrm{d}\mu(p)\,\frac{2m}{\omega_{\vec{p}}+m}\,f(\vec{p}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\frac{2m}{\omega_{\vec{p}}\,(\omega_{\vec{p}}+m)}\,f(\vec{p}) \simeq \frac{1}{m}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,f(\vec{p})\ .
$$

In fact, the non-relativistic wave function $\psi(\vec{r})$ of the bound state is given by

$$
\psi(\vec{r}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{e}^{\,\mathrm{i}\,\vec{p}\vec{r}} f(\vec{p})
$$

and hence

$$
\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,f(\vec{p}) = \psi(0)\ .
$$

This last relation we have been using above, already.

Color, flavor and spin:
Color ,flavor and spin are independent and hence factorize

$$
\varphi_{\bar{c}\bar{c};ff}(\lambda;s,s') = \Phi^{(c)}_{cc'}\cdot\Phi^{(F)}_{ff'}\cdot\chi^{\mathrm{spin}}(\lambda;s,s')\ .
$$

1. Color: we consider color first: the mesons are color singlet

$$
\Phi^{(c)}_{cc'} = \frac{1}{\sqrt{3}}\,\delta_{cc'}\ .
$$

Note that

$$
\sum_c \Phi^{(c)}_{cc'} = \sqrt{3}
$$

yields the color factors in matrix elements.

2. Flavor: $Q_f = \frac{2}{3}$; $f = u,c,t$ and $Q_f = -\frac{1}{3}$; $f = d,s,b$. Our main interest here are the vector mesons $J^{PC} = 1^{--}$:

$$
\begin{array}{rcccll}
\rho^0 & \hat{=} & \frac{1}{\sqrt{2}}(u\bar{u}-d\bar{d}) & : & \Phi^{\rho^0}_{uu}=\frac{1}{\sqrt{2}}, \quad \Phi^{\rho^0}_{dd}=-\frac{1}{\sqrt{2}}, & \sum_f Q_f\,\Phi^{\rho^0}_{ff}=\frac{1}{\sqrt{2}}(\frac{2}{3}+\frac{1}{3})=\frac{1}{\sqrt{2}} \\
\omega & \hat{=} & \frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d}) & : & \Phi^{\omega}_{uu}=\frac{1}{\sqrt{2}}, \quad \Phi^{\omega}_{dd}=\frac{1}{\sqrt{2}}, & \sum_f Q_f\,\Phi^{\omega}_{ff}=\frac{1}{\sqrt{2}}(\frac{2}{3}-\frac{1}{3})=\frac{1}{3\sqrt{2}} \\
\phi & \hat{=} & s\bar{s} & : & \Phi^{\phi}_{ff\neq s}=0, \quad \Phi^{\phi}_{ss}=1, & \sum_f Q_f\,\Phi^{\phi}_{ff}=-\frac{1}{3} \\
J/\psi & \hat{=} & c\bar{c} & : & \Phi^{J/\psi}_{ff\neq c}=0, \quad \Phi^{J/\psi}_{cc}=1, & \sum_f Q_f\,\Phi^{J/\psi}_{ff}=\frac{2}{3} \\
\Upsilon & \hat{=} & b\bar{b} & : & \Phi^{\Upsilon}_{ff\neq b}=0, \quad \Phi^{\Upsilon}_{bb}=1, & \sum_f Q_f\,\Phi^{\Upsilon}_{ff}=-\frac{1}{3}\ .
\end{array}
$$

In general color– and flavor–factors take to form

$$
\sum_c \sum_f \Phi^{(c)}_{cc}\,\Phi^{(F)}_{ff} \;=\; \sqrt{3}\,\langle Q\rangle = \sqrt{3}\,\left\{\frac{1}{\sqrt{2}},\ \frac{1}{3\sqrt{2}},\ -\frac{1}{3},\quad \frac{2}{3},\ -\frac{1}{3},\ \cdots\right\} \qquad (9.37)
$$
$$
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \rho^0 \quad \omega \quad \phi \quad J/\psi \quad \Upsilon
$$

3. Spin: $\chi^{\rm spin}(\lambda;s,s')$

We again consider a massive vector boson (spin 1). The possible spin states are

$$
\begin{array}{rcccc}
 & & |\uparrow\uparrow\rangle\ , & \frac{1}{\sqrt{2}}\,|\uparrow\downarrow+\downarrow\uparrow\rangle\ , & |\downarrow\downarrow\rangle \\
J_3 & : & +1\ , & 0\ , & -1
\end{array}
$$

The spin dependence of

$$
\varphi_{cc';ii'}(\lambda;s,s') = \Phi^{(c)}_{cc}\,\Phi^{(F)}_{ff}\,\chi(\lambda;s,s')
$$

is characterized by the quark–antiquark wave function

$$
\bar{u}(0,s)\,\gamma_\mu\,v(0,s') = \left\{\begin{array}{ll} 0 & ;\,\mu=0 \\ -U^+(s)\,\sigma_i\,V(s') & ;\,\mu=i=1,2,3 \end{array}\right. ,
$$

which couples in an obvious manner to the electromagnetic current:

$$
J^\mu(0) = \cdots \bar{v}(0,s')\,\gamma^\mu\,u(0,s)\ ,
$$

which means

$$
\begin{array}{rcl}
J^\mu(0)\,\bar{u}(0,s)\,\gamma_\mu\,v(0,s') & = & J^0(0)\,\bar{u}(0,s)\,\gamma^0\,v(0,s') - J^i(0)\,\bar{u}(0,s)\,\gamma^i\,v(0,s') \\
 & = & J^i(0)\,\left(-U^+(s)\,\sigma_i\,V(s')\right)\ ,
\end{array}
$$

where the minus sign comes from the metric $\gamma^i=-\gamma_i$. We now look for states with fixed J_3: we have

$$
U(\uparrow)=\begin{pmatrix}1\\0\end{pmatrix},\ U(\downarrow)=\begin{pmatrix}0\\1\end{pmatrix};\ \ V(\uparrow)=\begin{pmatrix}0\\1\end{pmatrix},\ V(\downarrow)=-\begin{pmatrix}1\\0\end{pmatrix}\ ,
$$

and with $\sigma_1=\sigma_+ + \sigma_-$; $\sigma_2=-\mathrm{i}\,(\sigma_+-\sigma_-)$, i.e.,

$$
\sigma_+ = \frac{\sigma_1+\mathrm{i}\sigma_2}{2} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\ ;\ \ \sigma_- = \frac{\sigma_1-\mathrm{i}\sigma_2}{2} = \begin{pmatrix}0&0\\1&0\end{pmatrix}\ ,
$$

the relationship between the normalized J_3 eigenstates and $-U^+(s)\,\sigma_i\,V(s')$ is fixed:

$$
\begin{array}{rcrcl}
\delta_{s\uparrow}\,\delta_{s'\uparrow} & = & U^+(s)\,\sigma_+\,V(s') & : & \lambda=+ \\
\delta_{s\downarrow}\,\delta_{s'\downarrow} & = & -U^+(s)\,\sigma_-\,V(s') & : & \lambda=- \\
\frac{1}{\sqrt{2}}\left(\delta_{s\uparrow}\,\delta_{s'\downarrow}+\delta_{s\downarrow}\,\delta_{s'\uparrow}\right) & = & -\frac{1}{\sqrt{2}}\,U^+(s)\,\sigma_3\,V(s') & : & \lambda=0\ .
\end{array}
$$

We thus have

$$\begin{aligned}
& -U^+(s)\,\sigma_i\,V(s') = -\delta_{ik}\,U^+(s)\,\sigma_k\,V(s') \\
= & \quad \delta_{i1}\,U^+(s)\,(-\sigma_+ - \sigma_-)\,V(s') + \delta_{i2}\,U^+(s)\,(\mathrm{i}\sigma_+ - \mathrm{i}\sigma_-)\,V(s') + \delta_{i3}\,U^+(s)\,(-\sigma_3)\,V(s') \\
= & \quad (-\delta_{i1} + \mathrm{i}\,\delta_{i2})\;\delta_{s\uparrow}\,\delta_{s'\uparrow} + (\delta_{i1} + \mathrm{i}\,\delta_{i2})\;\delta_{s\downarrow}\,\delta_{s'\downarrow} + \sqrt{2}\,\delta_{i3}\,\frac{1}{\sqrt{2}}\left(\delta_{s\uparrow}\,\delta_{s'\downarrow} + \delta_{s\downarrow}\,\delta_{s'\uparrow}\right) \\
= & \quad \sqrt{2}\,\delta_{i\lambda}\,\chi(\lambda;s,s') \;\; .
\end{aligned}$$

The coefficients $\chi(\lambda;s,s')$ are just the Clebsch-Gordon coefficients of the reduction $1/2 \otimes 1/2 = 0 \oplus 1$. Non-vanishing components are

$$\begin{aligned}
\chi(+;\uparrow\uparrow) = 1 \quad &; \quad \chi(-;\downarrow\downarrow) = 1 \\
\chi(0;\uparrow\downarrow) = \frac{1}{\sqrt{2}} \quad &; \quad \chi(0;\downarrow\uparrow) = \frac{1}{\sqrt{2}} \;\; ,
\end{aligned}$$

and we have defined

$$\delta_{i\lambda}: \quad \delta_{i+} \doteq \frac{-\delta_{i1} + \mathrm{i}\delta_{i2}}{\sqrt{2}} \;\; , \quad \delta_{i-} \doteq \frac{\delta_{i1} + \mathrm{i}\delta_{i2}}{\sqrt{2}} \;\; , \quad \delta_{i0} \doteq \delta i3 \;\; .$$

The wave function of the massive spin 1 boson are given by the polarization vector $\varepsilon(P,\lambda)$, with the properties:

$$\begin{aligned}
& \varepsilon^0(P,\lambda) = 0 \;; \;\; \vec{P}\cdot\vec{\varepsilon}(P,\lambda) = 0 \;\; , \\
& \varepsilon^\mu(P,\lambda)\,\varepsilon^*_\mu(P,\lambda') = -\delta_{\lambda\lambda'} \\
& \sum_{\lambda=0,\pm} \varepsilon_\mu(P,\lambda)\,\varepsilon^*_\nu(P,\lambda) = -g_{\mu\nu} + \frac{P_\mu P_\nu}{M^2} \stackrel{\vec{P}=0}{=} \left\{ \begin{array}{lcl} 0 & ; & (\mu,\nu) \neq (i,k) \\ \delta_{ik} & ; & (\mu,\nu) = (i,k) \end{array} \right. \;\; .
\end{aligned}$$

In the rest system of the meson, in the kartesian basis, we thus have $e_i(\lambda) = \delta_{i\lambda}$ for $i = 1,2,3$. This translates into $\varepsilon_i(\pm) = \frac{1}{\sqrt{2}}\,(\mp e_i(1) - \mathrm{i}\,e_i(2))$; $\varepsilon_i(0) = e_i(3)$, or

$$\varepsilon_i(+) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -\mathrm{i} \\ 0 \end{pmatrix} \;\; ; \;\; \varepsilon_i(-) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathrm{i} \\ 0 \end{pmatrix} \;\; ; \;\; \varepsilon_i(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\; .$$

Back to the matrix element

$$\langle 0|J^\mu(0)|M(\vec{P}=0,\lambda)\rangle \;\; ,$$

we know it must be proportional to the above polarization vectors:

$$\begin{aligned}
\sum_{s,s'} \; & \underbrace{\bar{v}(0,s')\,\gamma^\mu\,u(0,s)} \;\; \chi(\lambda;s,s') \propto \varepsilon^\mu(\lambda) \;\; , \\
& = \left\{ \begin{array}{l} 0 \\ V^+(s')\sigma_i U(s) \end{array} \right.
\end{aligned}$$

whereby

$$\begin{aligned}
\lambda = + \;\; : \;\; & V^+(s')\sigma_i U(s) = \;\; (0,1)\sigma_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1,\mathrm{i},0) = -\sqrt{2}\,\varepsilon_i(+) \\
\lambda = - \;\; : \;\; & V^+(s')\sigma_i U(s) = -(1,0)\sigma_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (-1,\mathrm{i},0) = -\sqrt{2}\,\varepsilon_i(-) \\
\lambda = 0 \;\; : \;\; & V^+(s')\sigma_i U(s) = -(1,0)\sigma_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0,1)\sigma_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0,0,-2) = -2\,\varepsilon_i(0) \;\; .
\end{aligned}$$

Table 4: Properties of leading $J^{PC} = 1^{--}$ vector mesons

V	ρ	ω	ϕ	J/ψ	Υ
Mass [MeV]	775.49 ± 0.34	782.65 ± 0.12	1019.455 ± 0.020	3096.916 ± 0.011	9460.30 ± 0.26
Γ [MeV]	149.4 ± 3.0	8.49 ± 0.08	4.26 ± 0.04	0.0932 ± 0.0021	0.05402 ± 0.00125
$\Gamma_{e^+e^-}$ [keV]	7.04 ± 0.06	0.60 ± 0.02	1.26 ± 0.02	5.55 ± 0.14	1.340 ± 0.018
$\langle Q_f \rangle$	1/2	1/18	1/9	4/9	1/9
$\lambda_V = e\,\gamma_V$	0.0611	0.0178	0.0225	0.0271	0.0076
$\lvert\psi(0)\rvert^2$	6.327	44.487	39.627	100.674	3629.101

Taking into account $\varepsilon^i(\lambda) = -\varepsilon_i(\lambda)$ $(i = 1, 2, 3)$ and the Clebsch-Gordon coefficients we obtain indeed:

$$\sum_{s,s'} \bar{v}(0,s')\,\gamma^\mu\,u(0,s)\,\chi^{(\text{spin})}(\lambda;s,s') = \sqrt{2}\,\varepsilon^\mu(\lambda)$$

with $\varepsilon^\mu(\lambda) = \varepsilon^\mu(P,\lambda)|_{\vec{P}=0}$. Alltogether, we have obtained

$$\left\| \quad \langle 0|J^\mu(0)|M(\vec{P}=0,\lambda)\rangle = 2\sqrt{M}\,\sqrt{N_c}\cdot\langle Q_f\rangle_M\,\varepsilon^\mu(\vec{P}=0,\lambda)\cdot\psi(0)\ . \right. \tag{9.38}$$

We finally may compare this result with the matrix element of an elementary vector meson V in the VDM

$$\langle 0|J^\mu(0)|M_V(\vec{P}=0,\lambda)\rangle = \gamma_V\,M_V^2\,\underbrace{\langle 0|V^\mu(0)|M_V(\vec{P}=0,\lambda)\rangle}_{\varepsilon^\mu(\vec{P}=0,\lambda)}\ .$$

The VDM ansatz $J_V^\mu(x) \doteq \gamma_V\,M_V^2\,V^\mu(x)$ implies $J_V^\mu(x)A_\mu(x) = \gamma_V\,M_V^2\,V^\mu(x)\,A_\mu(x)$ as an effective coupling to the photon. Note, J^μ has $\dim J{=}3$ while V^μ has $\dim V{=}1$. We thus have the relationship

$$\boxed{\gamma_V\,M_V^2 = 2\sqrt{3}\,\sqrt{M_V}\,\langle Q_f\rangle_V\,\psi(0)}$$

between the phenomenological VDM ansatz, which defines the phenomenological $V-\gamma$ coupling parameter γ_V, and the bound state wave function at the origin $\psi(0)$, which appears as the only as yet undetermined quantity in the above bound state calculation, in the non–relativistic limit. Other notation frequently used are: $f_V = \gamma_V\,M_V^2$ and $g_V = 1/\gamma_V$ etc. In our notation

$$\gamma_V^2 = 4\,N_c\,\langle Q_f\rangle^2\,\frac{|\psi(0)|^2}{M_V^3} = \frac{3\Gamma_{V\to e^+e^-}}{4\pi\alpha^2 M_V}\ , \tag{9.39}$$

incorporates the compositeness of the vector meson. Like $\langle Q_f\rangle$ which is the effectve charge of the vector meson, which is composed of constituents of different charges Q_f.

In Table 4 we have collected the parameters of the known leading vector mesons. The parameter λ_V we have defined as

$$\lambda_V = e\,\gamma_V = \left(\frac{3\Gamma_{V\to e^+e^-}}{\alpha M_V}\right)^{1/2}\ .$$

Lit.:
R. van Royen and U. Weisskopf, Nuovo Cim. 40 (1967), 617;
H. Ito, Prog. Theor. Phys. 77 (1987), 681

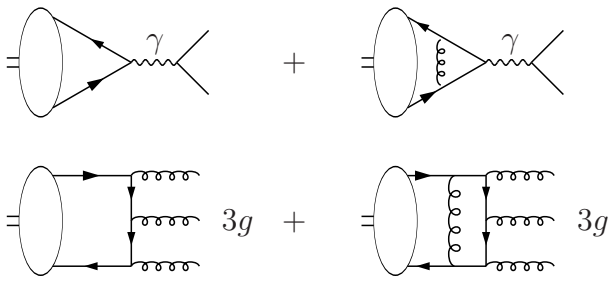
9.9 Applications

9.9.1 Determination of α_s

In our withs predictions typically depend on the wave function $\psi(0)$. The latter non–perturbative quantity is generally not well known, theoretical predictions in general are model dependent. It is therefore interesting to consider ration which are independent of $\psi(0)$. Such a ratio is

$$B^{\Upsilon}_{h/\ell} = \frac{\Gamma(\Upsilon \stackrel{3g}{\to} \text{hadrons})}{\Gamma(\Upsilon \to e^+e^-)} = \frac{10\,(\pi^2-9)}{81\,\pi\alpha^2\,Q_b^2}\,\alpha_s^3(M_\Upsilon)\ . \tag{9.40}$$

This is a very interesting result as it is $\propto \alpha_s^3$ in leading order, and hence potentally an ideal observable the determine α_s. The price to pay is that the both widthes are not known with the same precision because the leptonic one is one order of magnitude smaller. Nevertheless it is an interesting observalbe for the determination of α_s. As usual in pQCD, higher order corrections are important and they have been calculated to $O(\alpha_s^2)$ relative to the leading term. We consider here the leading corrections:



which yields corrections

$$\begin{aligned}\Gamma_{e^+e^-} &= \Gamma_{e^+e^-,0}\left(1-\frac{16}{3}\,\frac{\alpha_s}{\pi}\right)\\ \Gamma_{\text{had}} &= \Gamma_{\text{had},0}\left(1+c\,\frac{\alpha_s}{\pi}\right)\end{aligned} \tag{9.41}$$

with $c = 3.8 \pm 0.5$ (Mackenzie and Lepage 1981), such that

$$B^{\Upsilon}_{h/\ell} = B^{\Upsilon}_{h/\ell,0}\left(1+(1.1\pm 0.5)\,\frac{\alpha_s}{\pi}\right) \tag{9.42}$$

Note that the inclusive decays via $3g$ can be extracted from observable quantities as follows

$$\Gamma(\Upsilon \to 3g \to \text{hadrons}) \simeq \Gamma_{\Upsilon\,\text{tot}} - 3\,\Gamma_{\Upsilon\to e^+e^-} - R(\mu = 10\ \text{GeV})\,\Gamma_{\Upsilon\to e^+e^-} \simeq 34.8 \pm 9.6\ \text{keV}\ .$$

Since $m_e, m_\mu, m_\tau \ll M_\Upsilon$, we have used

$$\Gamma_{\Upsilon\to e^+e^-} + \Gamma_{\Upsilon\to\mu^+\mu^-} + \Gamma_{\Upsilon\to\tau^-\tau^-} \simeq 3\,\Gamma_{\Upsilon\to e^+e^-}\ ;\ \ \Gamma_{\Upsilon\to e^+e^-} = 1.22 \pm 0.05\ \text{keV}\ ,$$

and

$$R\,\Gamma_{\Upsilon\to e^+e^-} = \frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}\cdot\Gamma_{\Upsilon\to\mu^+\mu^-}\ ;\ \ R \simeq 3.7\ ,$$

These are the hadronic channels $\Upsilon \to \gamma \to$ hadrons which do not proceede via the $3g$ intermediate state. Using the experimental branching fraction $B^{\Upsilon} = 28.5 \pm 7.9$ we obtain

$$\alpha_s(M_\Upsilon) = \left(B^{\Upsilon}_{h/\ell} \cdot \frac{81\pi\alpha^2\,\frac{1}{9}}{10\,(\pi^2-9)}\,\frac{1}{1+(1.1\pm 0.5)\frac{\alpha_s}{\pi}} \right)^{1/3} = 0.167 \pm 0.015 \ .$$

With this value we may estimate

$$R(s)^{(udsc)} = 3.5\,\left(1+\frac{\alpha_s}{\pi}\right) \simeq 3.69 \pm 0.02 \ .$$

The approximation works the better the heavier the quarks. However we als expect that we can treat the J/ψ decays in the same way. At $M_{J/\psi}$ we have $R(s)^{(uds)} \simeq 2.1$, $\Gamma(J/\psi \to \text{had}) = 51.86 \pm 1.28$ keV and $\Gamma(J/\psi \to e^+e^-) = 4.7 \pm 0.3$ keV, and subtracting $R \cdot \Gamma_{e^+e^-}$ yields

$$\Gamma(J/\psi \to 3g \to \text{hadrons}) \simeq 42.0 \pm 1.4 \text{ keV} \ ,$$

such that

$$B^{J/\psi}_{h/\ell} = \frac{\Gamma(J/\psi \to \text{hadrons})}{\Gamma(J/\psi \to e^+e^-)} = 8.9 \pm 0.6 \ .$$

From the ratio

$$\frac{B^{J/\psi}_{h/\ell}}{B^{\Upsilon}_{h/\ell}} = \frac{\alpha_s^3(M_{J/\psi})}{\alpha_s^3(M_\Upsilon)} \cdot \frac{Q_b^2}{Q_c^2} = \frac{1}{4}\,\frac{\alpha_s^3(M_{J/\psi})}{\alpha_s^3(M_\Upsilon)}$$

Using $4\,B^{J/\psi}_{h/\ell} \simeq 35.6 > B^{\Upsilon}_{h/\ell} \simeq 28.4$ we may compute $\alpha_s(M_{J/\psi})$:

$$\begin{aligned} \alpha_s(M_{J/\psi}) &= \alpha_s(M_\Upsilon) \cdot \left(\frac{4\,B^{J/\psi}_{h/\ell}}{B^{\Upsilon}_{h/\ell}} \right)^{1/3} = \alpha_s(M_\Upsilon) \times (1.08 \pm 0.042) \\ &= 0.18 \pm 0.07 \ , \end{aligned}$$

which implies that α_s is running! Indeed, the RG with $\alpha_s(M_\Upsilon)$ as an input yields

$$\alpha_s(M_{J/\psi}) = 0.210 \pm 0.028 \ ,$$

in reasonable agreement.

In a similar way one may use the ratio

$$B_{\gamma h} = \frac{\Gamma(V \to \gamma gg \to \gamma + \text{hadrons})}{\Gamma(V \to ggg \to \text{hadrons})} = \frac{36}{5}\,\langle Q_f \rangle_V^2\,\frac{\alpha}{\alpha_s(M_V)}\,\left(1+(2.2\pm 0.6)\,\frac{\alpha_s}{\pi}\right)$$

to determine $\alpha_s(M_V)$.

Here we presented very simple estimates based on NLO corrections. For uptodate determinations including higher order effects I refer to Brambilla et al. 2007 [54] and references therein. This recent determination of α_s from $\Gamma(\Upsilon(1S) \to X\gamma)/\Gamma(\Upsilon(1S) \to X)$ with CLEO data by taking into account color octet contributions and avoiding any model dependence in the extraction, yields $\alpha_s(M_{\Upsilon(1S)}) = 0.184^{+0.015}_{-0.014}$, which corresponds to $\alpha_s(M_Z) = 0.119^{+0.006}_{-0.005}$.

Pre-Coulombic behavior: hyperfine splitting In the region where the one gluon exchange diagram determines the $q\bar{q}$–potential and the the extent that a potential si suitable to describe the leading physics effects, the energy level splitting of the ground state is given by

$$\Delta E_0 = M(1^3S_1) - M(1^1S_0) = \frac{32\pi}{9}\,\frac{\alpha - s(M)}{m^2}\,|\psi(0)|^2 \ ,$$

as in positronium with the replacement $\alpha \to \frac{4}{3}\,\alpha_s$. The states in question here are

$$
\begin{aligned}
1^3S_1 \quad &: \quad \Upsilon \quad \text{or} \quad J/\psi \quad : \quad J^{PC} = 1^{--} \\
1^1S_0 \quad &: \quad \eta_b \quad \text{or} \quad \eta_c \quad : \quad J^{PC} = 0^{-+}
\end{aligned}
$$

and $|\psi(0)|^2$ can be determined experimentally from $\Gamma(1^3S_1 \to e^+e^-)$ or calculated within quarkonia potential models. Using van Royen–Weisskopf relationship

$$
|\psi(0)|^2 = \frac{M_V^2\,\Gamma_{V\to e^+e^-}}{16\,\pi\,\alpha^2\,Q_q^2}
$$

we obtain

$$
\Delta E_0 = \frac{2}{9\,Q_q^2\,\alpha^2}\,\frac{M_V^2}{m_q^2}\,\alpha_s(M_V)\,\Gamma(V \to e^+e^-)\ . \tag{9.43}
$$

For the Υ–system then

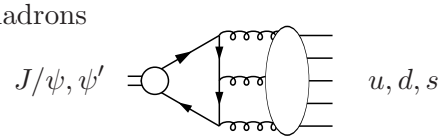
$$
M_\Upsilon - M_{\eta_b} \simeq \frac{2}{\alpha^2}\,\frac{M_\Upsilon^2}{m_b^2}\,\alpha_s(M_\Upsilon)\cdot\Gamma(\Upsilon \to e^+e^-) \simeq 31\ \text{MeV}
$$

where we have used $m_b(m_b) = 4.7 \pm 0.1$ GeV, $M_\Upsilon \simeq 9.46$ GeV and $\alpha_s \simeq 0.167$. For the J/ψ–system similarly

$$
M_{J/\psi} - M_{\eta_c} \simeq \frac{2}{\alpha^2}\,\frac{M_{J/\psi}^2}{m_c^2}\,\alpha_s(M_{J/\psi})\cdot\Gamma(J/\psi \to e^+e^-) \simeq 66\ \text{MeV}
$$

where we have used $m_c(m_c) = 1.27 \pm 0.05$ GeV, $M_{J/\psi} \simeq 3.01$ GeV and $\alpha_s \simeq 0.25$. The experimental value is $M_{J/\psi} - M_{\eta_c} = 116 \pm 9$ MeV. Charmonium models yield $\simeq 99$ MeV, which shows that nonperturbative effects are substantial.

As a last example we illistrate the valididy and meanung of the OZI-rule: the stability of the states J/ψ and ψ' is the result of the fact that they are OZI forbidden! Hadronically, they only decay via gluons into light hadrons



because their masses are below the charm threshold. This is different for ψ'' which predominantly decays into a pair of the charmed D mesons (see Fig. 9.17)

$$
\psi'' \to D\bar{D}\ .
$$

The D mesons into which ψ'' decay at the 90% level are the $J^P = 0^-$ staes D^0, D^+, $\overline{D^0}$ and D^-, with quark content $c\bar{u}$, $c\bar{d}$, $\bar{c}u$ and $\bar{c}d$, respectively. The decays are OZI allowed, such that

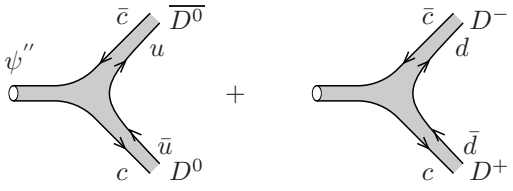


Figure 9.17: Charm conserving ψ'' decay modes (with c and $\bar{c}$ as spectator quarks).

the process proceeds witout purely gluonic intermediate states and the original quarks which

made up the ψ'' are preserved and appear in the final states hadrons. Such quarks are called spectator quarks. The condition for the decay $\psi \to D\bar{D}$ into charmed mesons obviously is

$$M_\psi > 2\, M_D \simeq 3726.6 \text{ MeV}$$

which is fulfilled for ψ'' not however for the very slightly lighter ψ'. For the D meson masses we have $M(D^0) = 1863.3 \pm 0.9$ MeV and $M(D^\pm) = 1868.3 \pm 0.9$ MeV . If we compare the parameters of ψ'' and ψ'

$M(\psi'')$	=	3770.0 ± 3	MeV	$M(\psi')$	=	3686 ± 0.10	MeV
$\Gamma(\psi'')$	=	25 ± 3	MeV	$\Gamma(\psi')$	=	0.215 ± 0.040	MeV
$\Gamma(\psi'' \to e^+e^-)$	=	0.26 ± 0.05	keV	$\Gamma(\psi' \to e^+e^-)$	=	2.05 ± 0.2	keV

we indeed observe that ψ'' is much less stable then the ψ'.

Further reading: much more details may be found in the collection of expert contribution in [19] and the updates you may find in the Particle Data Tables [55]. All about present and future quarkonium the reader may find in the CERN Yellow report [56].

Exercises: Section 9

① Draw the relevant Feynman diagrams (within the SM) for the decays $D^0 \to K^-\pi^+$, $K^-e^+\nu_e$ and $K^-\pi^+\pi^0$, based on the quark content of the hadrons.

② The decay $D^0 \to K^-\pi^+\pi^0$ is observed with a branching fraction 13.9 %. What is the branching fraction of the charge conjugate decay $D^0 \to K^+\pi^-\pi^0$?

③ Explain why $D_s^+ \to \mu^+\nu_\mu$ has a branching fraction of $6.2{\cdot}10^{-3}$ % only, while $D_s^+ \to \tau^+\nu_\tau$ has one of 6.6 %.

④ The top quark decays via the weak process $t \to Wb$, with subsequent decay of the W and the b (e.g. $W \to e\nu$ and $b \to b$–jet of hadrons). Above threshold for real W production (with $m_t \simeq 172$ GeV and $M_W \simeq 80.1$ GeV well satisfied), in the approximation $m_b \ll m_t$, the tree level QPM top width is given by

$$\Gamma_t = \frac{\sqrt{2} G_F\, m_t^3}{32\pi}\, 2\, (1-y)^2\, (1+2\,y)$$

where $y = (M_W/m_t)^2$. Compare the lifetime of the top quark obtained from this formula with the actual lifetime of the top and with the lifetimes of Υ and J/ψ resonances. Try to explain why no toponium resonances are formed.

References

[1] J. J. Aubert *et al.* [E598 Collaboration], Phys. Rev. Lett. **33** (1974) 1404. See also: http://nobelprize.org/nobel_prizes/physics/laureates/1976/ting-lecture.html

[2] J. E, Augustin et al., Phys. Rev. Lett. 33. 1406 (1974).
G. S. Abram et al., Phys. Rev. Lett. 33, 1453 (1974). See also: http://nobelprize.org/nobel_prizes/physics/laureates/1976/richter-lecture.html

[3] T. Appelquist, H.D. Politzer, Phys. Rev. Lett. **34** (1975) 43.

[4] V. A. Novikov, L. B. Okun, M. A. Shifman, A. I. Vainshtein, M. B. Voloshin and V. I. Zakharov, Phys. Rept. **41** (1978) 1.

[5] J. Gaiser *et al.* [Crystal Ball Collaboration], Phys. Rev. D **34** (1986) 711.

[6] W. Braunschweig and et al. [DASP Collaboration], Phys. Lett. B **57** (1975) 407.

[7] W. M. Tanenbaum *et al.*, Phys. Rev. Lett. **35** (1975) 1323.

[8] J. S. Whitaker *et al.*, Phys. Rev. Lett. **37** (1976) 1596.

[9] T. Skwarnicki, Int. J. Mod. Phys. A **19** (2004) 1030 [arXiv:hep-ph/0311243].

[10] S. W. Herb *et al.*, Phys. Rev. Lett. **39** (1977) 252; W. R. Innes et al., Phys. Rev. Lett. **39** (1977) 1240.

[11] C. Berger *et al.* [Pluto Collaboration], Phys. Lett. B **76** (1978) 243.

[12] C. W. Darden *et al.* [DASP Collaboration], Phys. Lett. B **76** (1978) 246.

[13] J. K. Bienlein *et al.* [DESY-Heidelberg Collaboration], Phys. Lett. B **78** (1978) 360.

[14] M. Lüscher, G. Münster and P. Weisz, Nucl. Phys. B **180** (1981) 1.

[15] G. S. Bali, H. Neff, T. Duessel, T. Lippert and K. Schilling [SESAM Collaboration], Phys. Rev. D **71** (2005) 114513 [arXiv:hep-lat/0505012].

[16] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T. M. Yan, Phys. Rev. D **17** (1978) 3090 [Erratum-ibid. D **21** (1980) 313].

[17] J. L. Richardson, Phys. Lett. B **82** (1979) 272.

[18] W. Buchmüller and S. H. H. Tye, Phys. Rev. D **24** (1981) 132.

[19] A. Ali, P. Söding, *HIGH-ENERGY ELECTRON POSITRON PHYSICS*, (World Scientific, Singapore, 1988) 873 p.

[20] J. L. Rosner, J. Phys. G **34** (2007) S127 [arXiv:hep-ph/0609195].

[21] E. Eichten, S. Godfrey, H. Mahlke and J. L. Rosner, Rev. Mod. Phys. **80** (2008) 1161 [arXiv:hep-ph/0701208].

[22] S. Necco and R. Sommer, Nucl. Phys. B **622** (2002) 328 [arXiv:hep-lat/0108008].

[23] W. Buchmüller, G. Grunberg and S. H. H. Tye, Phys. Rev. Lett. **45** (1980) 103 [Erratum-ibid. **45** (1980) 587].

[24] A. Martin, Phys. Lett. **100B** (1981) 511.

[25] G. Bhanot, S. Rudaz, Phys. Lett. **78B** (1978) 119.

[26] M. B. Voloshin, Prog. Part. Nucl. Phys. **61** (2008) 455 [arXiv:0711.4556 [hep-ph]].

[27] A. Martin, arXiv:0705.2353 [hep-ph].

[28] M. Okamoto et al. [CP-PACS Collaboration], Phys. Rev. D **65** (2002) 094508, [arXiv:heplat/0112020].

[29] P. Chen, Phys. Rev. D **64** (2001) 034509, [arXiv:hep-lat/0006019].

[30] S. Choe et al. [QCD-TARO Collaboration], JHEP **0308** (2003) 022, [arXiv:hep-lat/0307004].

[31] L. Susskind , Coarse grained quantum chromodynamics. In: lectures given at the Les Houches Summer School on High energy physics (1976).

[32] T. Appelquist, M. Dine and I. J. Muzinich, Phys. Lett. B **69** (1977) 231; Phys. Rev. D **17** (1978) 2074.

[33] W. Fischler, Nucl. Phys. B **129** (1977) 157.

[34] E. Eichten and F.L. Feinberg, Phys. Rev. Lett. **43**791205; Phys. Rev. **23**812724.

[35] A. Billoire, Phys. Lett. B **92** (1980) 343.

[36] Y. Schröder, Phys. Lett. B **447** (1999) 321 [arXiv:hep-ph/9812205].

[37] M. Peter, Phys. Rev. Lett. **78** (1997) 602 [arXiv:hep-ph/9610209]; Nucl. Phys. B **501** (1997) 471 [arXiv:hep-ph/9702245].

[38] B. A. Kniehl, A. A. Penin, M. Steinhauser and V. A. Smirnov, Phys. Rev. D **65** (2002) 091503 [arXiv:hep-ph/0106135].

[39] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D **60** (1999) 091502 [arXiv:hep-ph/9903355].

[40] A. Pineda and J. Soto, Phys. Lett. B **495** (2000) 323 [arXiv:hep-ph/0007197].

[41] N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo, Phys. Lett. B **647** (2007) 185 [arXiv:hep-ph/0610143].

[42] M. Melles, Phys. Rev. D **62** (2000) 074019 [arXiv:hep-ph/0001295].

[43] A. A. Penin, A. Pineda, V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **699** (2004) 183 [arXiv:hep-ph/0406175].

[44] A. A. Penin, V. A. Smirnov and M. Steinhauser, Nucl. Phys. B **716** (2005) 303 [arXiv:hep-ph/0501042].

[45] M. Beneke, Y. Kiyo and K. Schuller, Nucl. Phys. B **714** (2005) 67 [arXiv:hep-ph/0501289].

[46] M. Beneke, Y. Kiyo and A. A. Penin, Phys. Lett. B **653** (2007) 53 [arXiv:0706.2733 [hep-ph]].

[47] M. Lüscher, Nucl. Phys. B **180** (1981) 317.

[48] M. Lüscher and P. Weisz, JHEP **0207** (2002) 049 [arXiv:hep-lat/0207003].

[49] M. Lüscher and P. Weisz, JHEP **0407** (2004) 014 [arXiv:hep-th/0406205].

[50] R. Sommer, Nucl. Phys. B **411** (1994) 839, hep-lat/9310022.

[51] S. Capitani, M. Lüscher, R. Sommer and H. Wittig [ALPHA Collab.], Nucl. Phys. B544 (1999) 669, hep-lat/9810063.

[52] S. Necco and R. Sommer, Phys. Lett. B **523** (2001) 135 [arXiv:hep-ph/0109093].

[53] N. Brambilla, X. G. i. Tormo, J. Soto and A. Vairo, arXiv:0906.1390 [hep-ph].

[54] N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo, Phys. Rev. D **75** (2007) 074014 [arXiv:hep-ph/0702079].

[55] C. Amsler *et al.* [Particle Data Group], Phys. Lett. B **667** (2008) 1.

[56] N. Brambilla *et al.* [Quarkonium Working Group], arXiv:hep-ph/0412158.

10 Jets

In this lecture we introduce in a more quantitative manner the concept of hadronic jets. We start with a brief summary of the outline given in Sect. **??**, already.

For highly energetic processes ("hard kicks") asymptotic freedom suggests that perturbative QCD should work, with the provisos discussed earlier. This manifests itself in the correct prediction of the total hadronic cross section in electron–positron annihilation $\sigma_{\rm tot}(e^+e^- \to \gamma^* \to \text{hadrons})$ in non–resonant energy ranges. The consequences of the applicability of pQCD go far beyond that. According to pQCD, the production of hadrons in e^+e^-–annihilation proceeds via the primary creation of a quark–antiquark pair (up to effects suppressed by one power in α) whereby the quarks hadronize into jets (see Fig. 8.11).

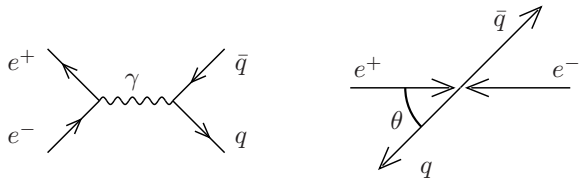
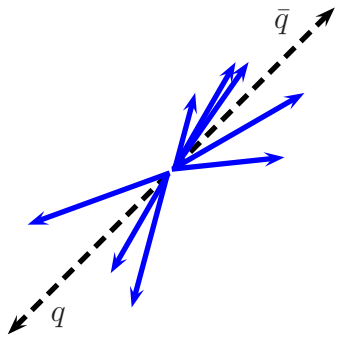


Figure 10.1: Quark–pair production as a leading primary process in hadron production in e^+e^-–annihilation. At high energies the two highly energetic quarks are produced at fixed production angle θ.

The elementary process $e^+e^- \to q\bar{q}$ implies that in a highly energetic collision of e^+ and e^- in the c.m. system q and $\bar{q}$ are produced with high energy in opposite directions as suggested by Fig. 10.1. In pQCD the leading contribution is given in this case by the *quark–parton model* (strong interaction switched off), and the corresponding lowest order differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e^+e^- \to q\bar{q}) = \frac{4}{3}\,\frac{\alpha^2}{s}\,\sum_f Q_f^2\,\left(1+\cos^2\theta\right) \quad , \tag{10.1}$$

with $s = E_{\rm cm}^2$ and θ the production angle. This is an angular distribution typical for the production of a pair of spin 1/2 particles, up to a color factor identical with the cross section for muon pair production. Indeed one observes that the two quarks hadronize each by itself by forming jets of hadrons. At high enough energies, these bundles of hadrons show up in relatively narrow angular cones:



The quarks are not observable, however their production angle can be identified with the *jet–axis*, which, in principle, is reconstructible from the hadrons in the jets [46]. The angular distribution is

[46] For first attempts to find and analyze jet events at the SPEAR ring at SLAC see [1].

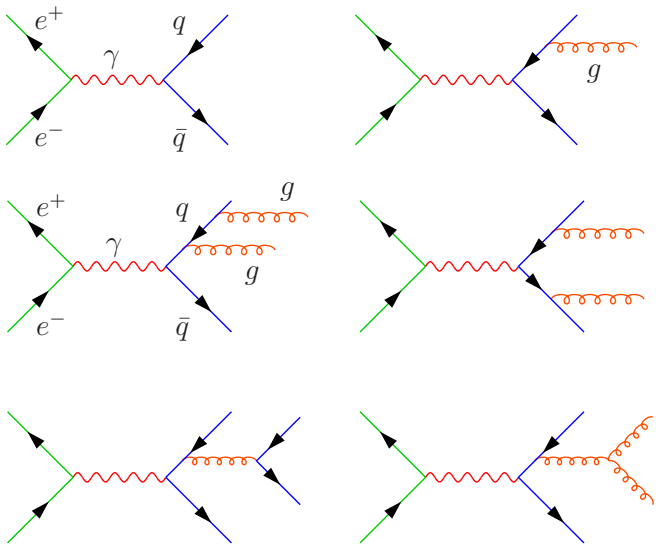


Figure 10.2: Feynman diagrams for the production of up to 4 hard partons resulting into 2– 3– and 4–jets. With 4 partons in the final state for the first time the triple gluon self-interaction (non-Abelian gauge structure) comes into play (in the last diagram).

thus observable as a *distribution of the jet axis*. While color and fractional charge are balanced via infrared effects (sea–quarks and gluons) between the jets, in such a way that only color singlet hadrons of integer charge show up in the final state, the momentum has not time enough to distribute evenly. The point is that the primary quarks supply themselves with sea–quarks from the vacuum to form hadrons, which allows to balance color and charge without the need to distribute the momentum isotropically. As a rule: in highly energetic kicks perturbation theory may be utilized if one identifies "hard partons (quarks, antiquarks and gluons)" with "hadron jets".

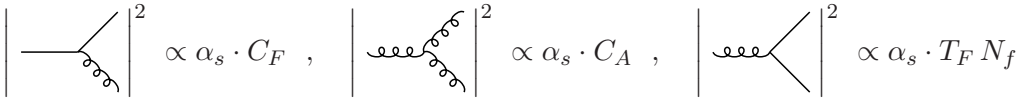
The key points are that, in the e^+e^- c.m. system (laboratory system of the electron positron storage ring), 2–jet events are back-to-back, while 3–jets are coplanar (in a common plane) as it follows from the primary parton kinematics.

The discovery of the gluon at the PETRA e^+e^- storage ring at DESY marked the beginning of a new epoch in QCD and strong interaction physics. First clear evidence was reported by the TASSO collaboration in 1979 [6] immediately followed by JADE, MARK J and PLUTO [7]. It made possible to determine the spin of the gluon, which proved to be a vector particle as required by QCD. Also the "string–effect", the hadronization of quarks and gluons via the formation of color strings, could be established. Last but not least jet physics is an important tool for the precise measurement the strong coupling constant as the ratio of rates 3–jet/2–jet is directly proportional to α_s modulo higher order effects.

In 1989 the Large Electron Positron (LEP) collider went into operation at CERN and due to the very large event samples QCD tests, α_s determinations and jet analyses did become possible at much higher precision. For example, it was possible now to investigate 4–jet signals in an unambiguous way, which allowed one to determine the group structure of QCD. Types of Feynman diagrams for perturbative jet cross-section calculations in QCD are shown in Fig. 10.2. Only typical cases shown, gluons are to be attached to quarks in all possible ways. Final states are $q\bar{q}\,(2\;jets)$, $q\bar{q}g\,(3\;jets)$ and $q\bar{q}, q'\bar{q}'\,(4\;jets)$, $q\bar{q}g\,g\,(4\;jets)$. A nice 4–jet event is shown in Fig. 10.4.

Ratio of gluon jets/quark jets $\propto C_A/C_F$. The Casimir coefficients are obtained in squaring the

sum of amplitudes represented by the Feynman diagrams (10.2), where typically



$\propto \alpha_s \cdot C_F$, $\propto \alpha_s \cdot C_A$, $\propto \alpha_s \cdot T_F\, N_f$

$$\sigma_{4-jet} \sim C_F \cdot \sigma_A \left(+ \cdots \right)$$

$$+ \; C_A \cdot \sigma_B \left(+ \cdots \right)$$

$$+ \; T_F \cdot \sigma_C \left(+ \cdots \right)$$

$$+ \quad \text{interference terms}$$

Including all terms yields a gauge invariant result for any unbroken local gauge group, which implies that at the given perturbative order the terms proportional to the different Casimirs must be gauge invariant separately. Therefore the Casimirs are quasi observales, which may be extracted from experimental jet cross–sections with the help of theory (pCD). In NLO QCD the prediction for an observable like R is

$$R = A\alpha_s + (B_{CF}C_F + B_{CA}C_A + B_{TF}T_F\, N_f)\, C_F\, \alpha_s^2 \ .$$

Usually, experiments fit α_s, C_A/C_F and $(T_F\, N_f)/C_F$, from which the Casimirs may be calculated. Theory values

Theory	C_F	C_A	T_F
QCD	4/3	3	1/2
Abelian	1	0	3

vary widely depending on the gauge group.

Results from 4–jet analysis at LEP typically are:

Experiment	$\alpha_s(M_Z)$	C_A	C_F
ALEPH [8]	0.119 ± 0.027	2.93 ± 0.60	1.35 ± 0.27
OPAL [9]	0.120 ± 0.023	3.02 ± 0.55	1.34 ± 0.26
Ref. [13]	0.119 ± 0.010	2.84 ± 0.24	1.29 ± 0.18

where the last entry was obtained from an event shape analysis. Other results for the gauge structure determinations are

Experiment	C_A/C_F	T_F/C_F
DELPHI [11]	$2.25 \pm 0.06 \pm 0.12$	0.34 ± 0.14
ALEPH	2.24 ± 0.40	0.58 ± 0.29
OPAL [10]	$2.23 \pm 0.01 \pm 0.14$	$0.40 \pm 0.11 \pm 0.14$
DELPHI [12]	2.26 ± 0.16	

The last result was extracted form observed scaling violations in quark and gluon jets [12].

For the average values one finds [15]

$$\begin{aligned} \alpha_s &= 0.1211 \pm 0.0010(\text{exp.}) \pm 0.0018(\text{theo.}) \ , \\ C_A &= 2.89 \pm 0.01(\text{stat.}) \pm 0.21 \ (\text{syst.}) \ , \\ C_F &= 1.30 \pm 0.01(\text{stat.}) \pm 0.09 \ (\text{syst.}) \ , \end{aligned}$$

in excellent agreement with structure constants of QCD, ruling out an Abelian vector gluon model as well as many other possible gauge field structures. Results are depicted in Fig. 10.3, shows the relative n–jet contributions, and which shows how much $SU(3)_c$ is favored by experimental jet data from the LEP experiments.

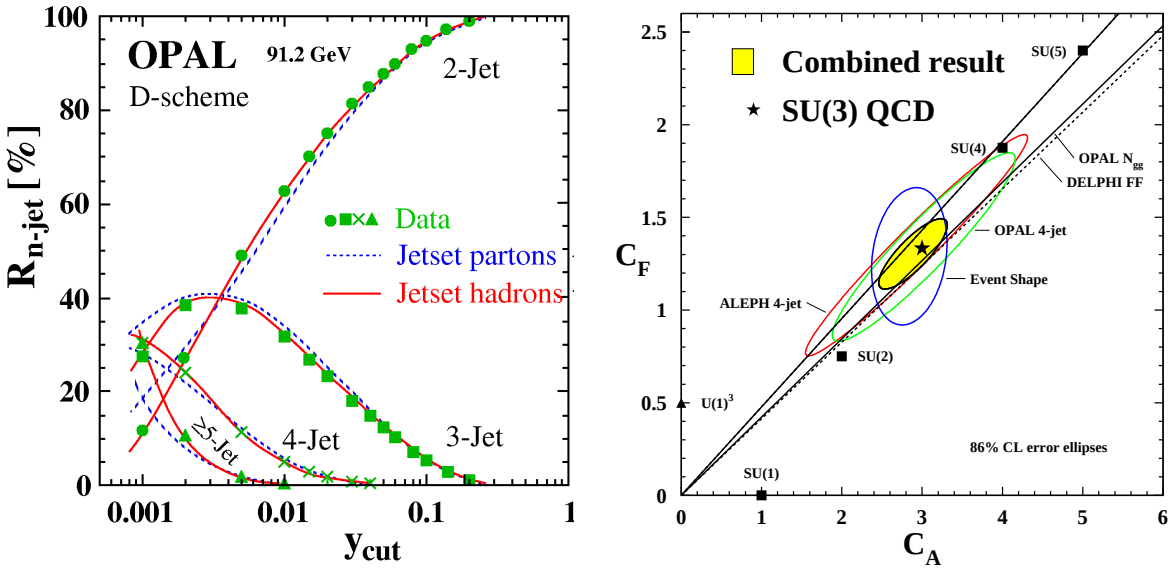


Figure 10.3: Experimental results showing n–jet cross-sections and the status of measurements of the gauge group Casimirs.

Typical LEP jet events are shown in Fig. 10.4. Figure **??** presents another plot of the experimental determination of the gauge structure of QCD.

What precisely is a jet? The original approach is that of Sterman and Weinberg [2] and allows to define this concept to all orders of perturbation theory. Introducing jet resolution parameters takes care of non–perturbative effects by averaging them out. We will discus this, as an example, below in detail. In fact it turned out soon that other concepts of characterizing hadronic final states are more practical and also are calculable in pQCD and after appropriate smearing of soft and collinear effects can be compared directly with the data. The key requirement is that the observables must be infrared and collinear safe. What this means we will investigate by simple analytic calculations below. Infrared safe means that the quantity, in order to be an

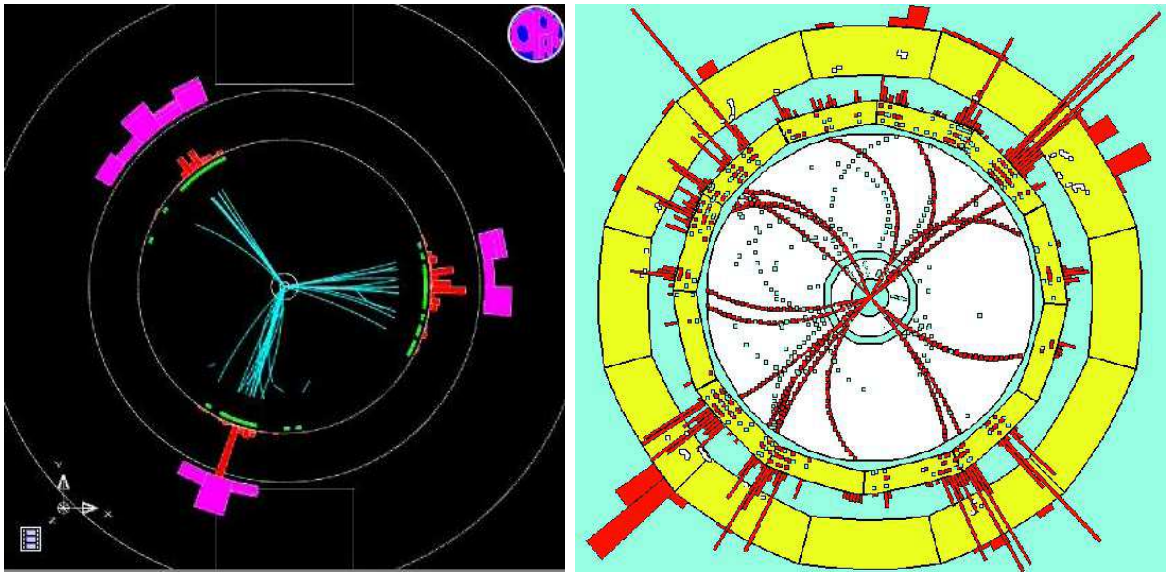


Figure 10.4: A 3–jet event from OPAL and a 4–jet event from the ALEPH detectors at LEP.

true observable, should not change discontinuously if one adds a soft particle to the final state. Similarly, collinear safe means that the quantity should not change abruptly if one splits one final state particle into two particles with equal momentum.

In general, the properties of hadronic events may be analyzed using a set of event shape observables. These may be used to classify the distribution of particles and the event topology (planar, spherical etc.). The characteristic features can be computed either using the measured charged particle tracks and calorimeter clusters, or using the true hadrons or partons in simulated events.

Historically, one of the first tools for the analysis of jet events was the sphericity tensor. Unfortunately the latter is not collinear safe and would not be used in practice any more. Nevertheless, we discuss it here as a pedagogical example, as it was used in some of the pioneering experiments. Conceptually similar save observables we will considered as a next step below.

If in a $e^+e^- \to q\bar{q}$ event r hadrons with momenta $\vec{p}_a$ $(a = 1, \cdots, r)$ are produced, one can construct the sphericity tensor

$$M_{ij} = \frac{\sum_a p_{ia} p_{ja}}{\sum_a p_{ia} p_{ia}}$$

and performs a transformation to principal axes. The corresponding eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ which we assume ordered according to $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$ and normalized to $\lambda_1 + \lambda_2 + \lambda_3 = 1$. They determine the sphericity $S = \frac{3}{2}(\lambda_1 + \lambda_2)$ and the acoplanarity $A = \frac{3}{2}\lambda_1$. Extremal cases are
1) isotropic distribution: $S = 1,\ A = \frac{1}{2}$
2) "ideal jet": $S = 0,\ A = 0$
Denoting $\hat{n}_1$. $\hat{n}_2$ and $\hat{n}_3$ the directions of the corresponding main axes, the eigenvalues have to following interpretation:

$$\begin{aligned}
\lambda_1 &= \min_{\hat{n}_1} \frac{\sum_a (\vec{p}_a \cdot \hat{n}_1)}{\sum_a \vec{p}_a{}^2} && \text{"flatness"} && \text{of event} \\
\lambda_2 &= \min_{\hat{n}_2 \perp \hat{n}_1} \frac{\sum_a (\vec{p}_a \cdot \hat{n}_2)}{\sum_a \vec{p}_a{}^2} && \text{"width"} && \text{of event} \\
\lambda_3 &= \max_{\hat{n}_3} \frac{\sum_a (\vec{p}_a \cdot \hat{n}_3)}{\sum_a \vec{p}_a{}^2} && \text{"length"} && \text{of event}
\end{aligned}$$

Events with $\lambda_1 \ll \lambda_2$ are called "coplanar", events with $\lambda_2 \ll \lambda_3$ are called "collinear".

One may use a triangular plot $Y = \frac{\sqrt{3}}{2}(\lambda_2 - \lambda_1)$ as a function of $S = \frac{3}{2}(\lambda_2 + \lambda_1)$ to separate 2–jet, 3–jet and non-coplanar events, as shown in Fig. 10.5.

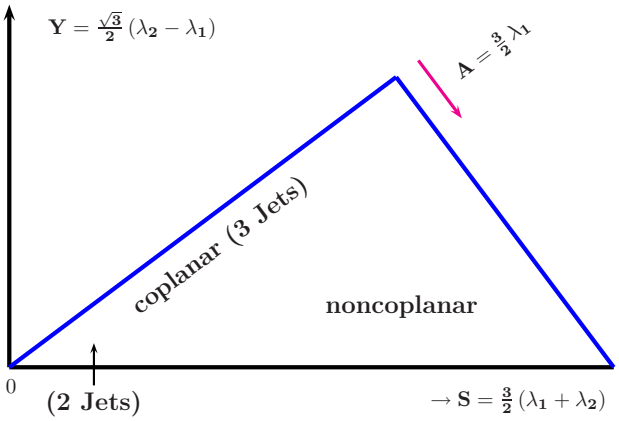


Figure 10.5: Triangular plot which allows to separate 2–jets from 3–jets and noncoplanar events.

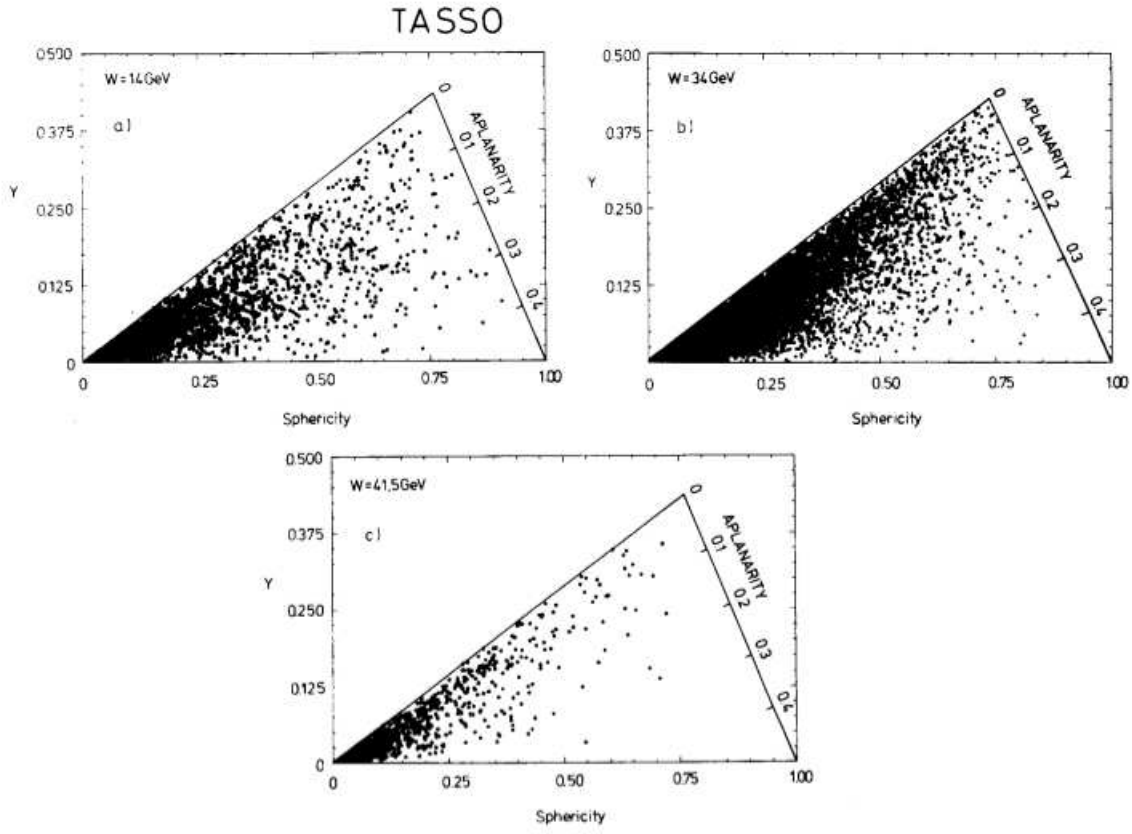


Figure 10.6: TASSO distributions of sphericity versus aplanarity at $W = 14,\ 34$ and 41.5 GeV.

As expected, data show that preferably 2–jet events are produced. If one investigates the observed frequency of 2–jet events, which are produced at an angle θ of the jet-axis $\hat{n}_3$, one indeed observes clearly the perturbative $1 + \cos^2\theta$ dependence typical for the production of a pair of spin 1/2 particles. For scalar quarks one would get a $\sin^2\theta$ dependence instead. The spin 1/2 nature of the hadronic constituents is thus experimentally established.

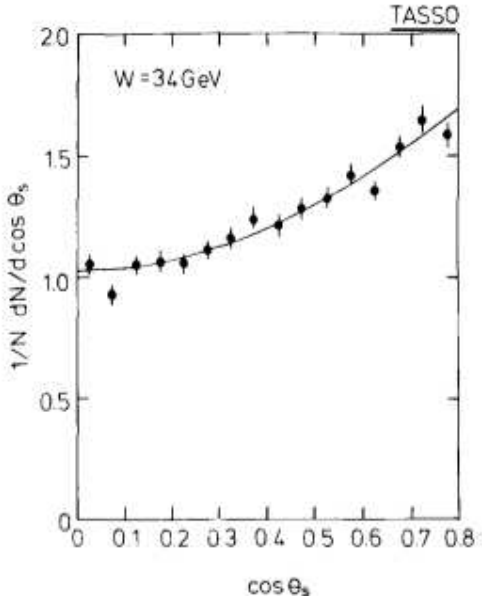


Figure 10.7: The angular distribution of the jet axis determined by sphericity. The curve shows the $1 + \cos^2\theta$ shape.

As mentioned, sphericity

$$S = \frac{3}{2} \frac{\sum_a |\vec{p}_{aT}|^2}{\sum_a |\vec{p}_a|^2} \ ,$$

with $\vec{p}_a$ the four momentum of particle a and $\vec{p}_{aT}$ is taken relative to the axis which minimizes $\sum_a |\vec{p}_{aT}|^2$, is not a good physical observable, because its not collinear safe. It can be replaced by

- Spherocity $\tilde{\boldsymbol{S}}$: defined as

$$\tilde{S} = \frac{4}{\pi} \min \left(\frac{\sum_a |\vec{p}_{aT}|}{\sum_a |\vec{p}_a|} \right)^2 \ , \tag{10.2}$$

which is both infrared and collinear safe. Other frequently utilized event shapes are the following (see e.g. [15] and references therein):

- Thrust $\boldsymbol{T}$: is defined by the expression

$$T = \max_{\vec{n}} \left(\frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |p_i|} \right) \ . \tag{10.3}$$

The thrust axis $\vec{n}_T$ is the direction $\vec{n}$ which maximizes the expression in parentheses. A plane through the origin and perpendicular to $\vec{n}_T$ divides the event into two hemispheres H_1 and H_2.

- $\boldsymbol{C}$-parameter: The linearized momentum tensor $\Theta^{\alpha\beta}$ is defined by

$$\Theta^{\alpha\beta} = \frac{\sum_i (p_i^\alpha p_i^\beta)/|\vec{p}_i|}{\sum_i |\vec{p}_i|} \quad , \quad \alpha, \beta = 1, 2, 3 \quad .$$

The three eigenvalues λ_j of this tensor define C through

$$C = 3(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) \ .$$

- Heavy Jet Mass $\boldsymbol{M_H}$:

The hemisphere invariant masses are calculated using the particles in the two hemispheres H_1 and H_2. Then $M_{\rm H}$ is defined as the heavier mass, divided by $\sqrt{s}$.

- Jet Broadening observables $\boldsymbol{B_T}$ and $\boldsymbol{B_W}$:

These are defined by computing the quantity

$$B_k = \left(\frac{\sum_{i \in H_k} |\vec{p}_i \times \vec{n}_T|}{2 \sum_i |\vec{p}_i|} \right)$$

for each of the two event hemispheres, H_k, defined above. The two observables are defined by

$$B_{\rm T} = B_1 + B_2 \ , \quad \text{and} \quad B_{\rm W} = \max(B_1, B_2)$$

where $B_{\rm T}$ is the total and $B_{\rm W}$ is the wide jet broadening.

- Transition value between 2 and 3 jets $\boldsymbol{y_{23}}$:

Jet algorithms are applied to cluster the large number of particles of an hadronic event into a small number of jets, reflecting the parton structure of the event. Defining each particle initially to be a jet, a resolution variable y_{ij} is calculated for each pair of jets i and j:

$$y_{ij} = \frac{2\min(E_i^2, E_j^2)}{E_{\rm vis}^2}(1 - \cos\theta_{ij}),$$

where E_i and E_j are the energies, $\cos\theta_{ij}$ is the angle between the two jets and $E_{\rm vis}$ is the sum of the energies of all visible particles in the event (or the partons in a theoretical calculation). If the smallest value of y_{ij} is less than a predefined value $y_{\rm cut}$, the pair is replaced by a jet with four momentum $p_{ij}^\mu = p_i^\mu + p_j^\mu$, and the clustering starts again with p_{ij}^μ instead of the momenta p_i^μ and p_j^μ. Clustering ends when the smallest value of y_{ij} is larger than $y_{\rm cut}$. The remaining jets are then counted. The value of $y_{\rm cut}$ at which for an event the transition between a 2-jet and a 3-jet assignment occurs is called y_{23}.

For applications of these concepts we refer to [15].

Exercise:
1) Show that sphericity is infrared safe but not collinear safe.
2) Show that spherocity is both infrared and ultraviolet save.

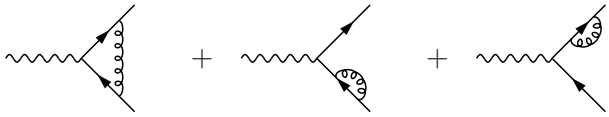
Safe observables have a very useful property: they can be defined in calorimetric terms. In the spherodicity definition a can be used to label the calorimetric cells and $\vec{p}_a$ is the energy detected in cell a times the unit vector pointing to it. One can show that for sufficiently small cells and sufficiently high energy the two definitions coincide, provided hadronization effects are power suppressed.

10.1 QCD corrections $O(\alpha_s)$ to the 2–jet cross–section

The lowest order 2–jet cross–section is given by (10.1). The corrections $O(\alpha_s)$ in pQCD may be calculated exactly like corresponding photon emission corrections in QED. Not surprisingly, in pQCD the same type of problems show up, like infrared and collinear singularities. Going from QED $[U(1)_{\rm em}]$ to QCD $[SU(3)_c]$ the difference is the Casimir operator $C_2(R) = \frac{4}{3}$, which amount to replace α by $\frac{4}{3}\,\alpha_s$. Strong interaction corrections affect only the final state (quarks). We therefore need only consider the sub–process $\gamma^* \to q\bar{q}$ in the following.

10.1.1 Virtual corrections

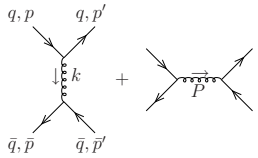
The virtual corrections



for massless gluons and on-shell quarks (i.e., after on-shell renormalization, which because of its gauge invariance is advocated here) is infrared divergent.

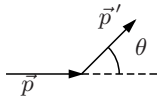
***

Remember: the origin of the troubles are the Coulomb like forces between colored quarks, which make perturbatively defined quark scattering states illdefined. Looking at $q(p)\bar{q}(\bar{p}) \to q'(p')\bar{q}'(\bar{p}')$ in leading order,

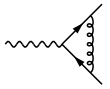


the scattering diagram has a singularity in the physical region, the forward peak. The scattering amplitude is

$$\propto \frac{1}{k^2} = \frac{1}{(p-p')^2} = -\frac{1}{2\vec{p}^{\,2}\,(1-\cos\theta)}$$



and therefore $\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}$ is singular in forward direction. As a consequence the total cross section $\sigma_{\rm tot} = \int_{-1}^{+1} \mathrm{d}\cos\theta\, \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}$ is inexistent, while $\sigma_{\rm cut}(\delta) = \int_{-1}^{1-\delta} \mathrm{d}\cos\theta\, \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}$ increases unboundedly as $\delta \to 0$. This carries over to loops, like



which is infrared singular iff quarks are on–shell. On shell quarks are unphysical due to steady emission and reabsorption of soft gluons, which causes "physical" quarks to be never realy on–shell. In pQCD of quarks we thus encounter exactly the same problems as in QED with the charged leptons.

***

The required calculations we have performed in Sect. 7. As advocated there, in the case of one–loop corrections, we may use and infinitesimal gluon mass m_g to regularize the IR singularities at intermediate steps. In observable quantities the limit $m_g \to 0$ exists and will be taken. In addition we consider the energy regions where the active quarks, in the sense of N_f–flavor effective QCD, have negligible mass: $m_q \ll \sqrt{s}$. Only then the perturbative treatment is adequate. The result then may be worked out easily from the results presented in Sects. 7.1 and 7.4. The result is given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\rm virtual} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\rm parton} \cdot C_{\rm virtual} \tag{10.4}$$

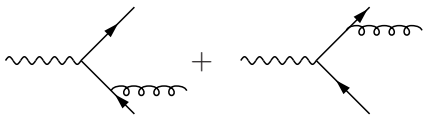
with

$$C_{\rm virtual} = \frac{4}{3}\,\frac{\alpha_s}{2\pi}\left\{4\left(\ln\frac{s}{m_q^2} - 1\right)\,\ln\frac{m_g}{\sqrt{s}} + \ln^2\frac{s}{m_q^2} + \ln\frac{s}{m_q^2} - 4 - 2\,\frac{\pi^2}{3}\right\}\ .$$

Like in QED, an IR finite result can only be obtained by looking for a truly observable quantity. In fact in quantum field theories with Coulomb type long range interactions, with the steady emission and absorption of soft massless quanta, any measurement beyond the finite energy resolution ω of any detector, cannot distinguish between virtual and soft quanta of energy $E_g \leq \omega$. Thus one has to include soft real emission on the same footing to obtain a realistic measurable result. In our case, we thus have to take into account the soft real gluon emission (Bloch–Nordsieck prescription!) in order to get a sensible result.

10.1.2 Real soft gluons

Taking into account soft real gluons of energy $E_g \leq \omega = \varepsilon\sqrt{s}$, we have to calculate the contributions from the diagrams



Since the singular part comes from the very low energetic gluons we may consider the approximation where the gluon four momentum k can be neglected in the energy balance. The corresponding correction is the proportional to the Born term, such that the real soft gluon corrections factorize. This factorization in fact is a necessary condition for the singularities to cancel between virtual and real soft contributions:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{soft bremsstrahlung}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{parton}} \cdot C_{\text{soft bremsstrahlung}} \tag{10.5}$$

with

$$\begin{aligned} C_{\text{softbremsstrahlung}} &= \frac{4}{3}\,\frac{\alpha_s}{2\pi^2} \int\limits_{|\vec{k}|\leq\omega} \frac{\mathrm{d}^3k}{2\omega_k} \left\{ \frac{2\,(p_1p_2)}{(kp_1)(kp_2)} - \frac{m_q^2}{(kp_1)^2} - \frac{m_q^2}{(kp_2)^2} \right\} \\ &= \frac{4}{3}\,\frac{\alpha_s}{2\pi} \left\{ 4\left(\ln\frac{s}{m_q^2} - 1\right) \ln\frac{2\omega}{m_g} - \ln^2\frac{s}{m_q^2} + 2\,\ln\frac{s}{m_q^2} - 2\,\frac{\pi^2}{3} \right\} \ . \end{aligned}$$

This result again considered as a separate contribution is IR singular and it depend on the energy resolution ω. Again we have taken an infinitesimally small gluon mass m_g. Actually, in the $m_g \to 0$ limit we need only the keep the gluon mass in the phase-space integral measure $\frac{\mathrm{d}^3k}{2\omega_k}$ with $\omega_k = \sqrt{\vec{k}^{\,2} + m_g^2}$.

The origin of singular behavior can easily be traced back to the behavior of the virtual quark propagators in the bremsstrahlung diagrams: with

$$\frac{1}{\not{k} + \not{p}_i - m_q + \mathrm{i}\varepsilon} = \frac{\not{k} + \not{p}_i - m_q}{(k+p_i)^2 - m_q^2 + \mathrm{i}\varepsilon}$$

singular behavior comes from the zeros in the denominator of the last expression when gluons are massless $k^2 = 0$ and quarks are on–shell $p_i^2 = m_q^2$. Then,

$$\frac{1}{(k+p_i)^2 - m_q^2 + \mathrm{i}\varepsilon} = \frac{1}{k^2 + 2\,kp_i + p_i^2 - m_q^2 + \mathrm{i}\varepsilon} \to \frac{1}{2\,kp_i + \mathrm{i}\varepsilon} \ .$$

Now,

$$\begin{aligned} kp_i &= k^0 E_i - \vec{k}\,\vec{p}_i = |\vec{k}|\sqrt{\vec{p}_i^{\,2} + m_q^2} - |\vec{k}||\vec{p}_i|\,\cos\theta = |\vec{k}|\,\left(\sqrt{\vec{p}_i^{\,2} + m_q^2} - |\vec{p}_i|\,\cos\theta\right) \\ &\overset{m_q=0}{=} |\vec{k}|\,|\vec{p}_i|\,(1-\cos\theta) \end{aligned}$$

In the bremsstrahlung cross–section we thus get by performing the phase space integral

$$\mathrm{d}^3k = \mathrm{d}|\vec{k}\,|\,|\vec{k}\,|^2\,\mathrm{d}\Omega\;;\;\; \mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = -\mathrm{d}(\cos\theta)\,\mathrm{d}\varphi\;\;,$$

with $m_g = 0$,

$$\int_{|\vec{k}|\geq|\vec{k}|_{\min}} \frac{\mathrm{d}|\vec{k}\,|\,|\vec{k}\,|^2}{2|\vec{k}|}\,\frac{1}{|\vec{k}|^{\,2}} \sim \log|\vec{k}|_{\min}$$

which is logarithmically singular as $|\vec{k}|_{\min} \to 0$. Alternatively, we may keep an infinitesimally small gluon mass in the phase space integration whereupon, with $\omega_k = \sqrt{m_g^2 + \vec{k}^{\,2}}$,

$$\int_{|\vec{k}|\geq 0} \frac{\mathrm{d}|\vec{k}\,|\,|\vec{k}\,|^2}{2\omega_k}\,\frac{1}{|\vec{k}|^{\,2}} \sim \log m_g\;\;.$$

In the limit $m_q = 0$ also the intergral $\int_{-1}^{+1}\mathrm{d}(\cos\theta)\frac{1}{(1-\cos\theta)}$ is singular (collinear singularity), and either a finit m_q or an appropriate angulat cut $1-\cos\theta \geq \delta$ is required in order to get a finite answer.

Exercise:
a) Show that the hard gluon spectrum, after performing the angular intergrations, may be written in the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}k} = \sigma_0(s)\,C_2(R)\,\frac{\alpha_s}{\pi}\left(\ln\frac{s'}{m_q^2} - 1\right)\left(1 + \left(\frac{s'}{s}\right)^2\right)\,\frac{1}{k}$$

where $k = |\vec{k}\,|$ is the gluon energy and $s' = s\left(1 - \frac{k}{E_b}\right)$ and $E_b = \sqrt{s}/2$. Note that s' is the invariat mass square of the $q\bar{q}$–system after radiation of the gluon, which carried away a fraction k/E_b of the energy.

b) In contrast in the soft gluon limit (factorization) may be represented as

$$\sigma(s) = \sigma_0(s) + C_2(R)\,\frac{\alpha_s}{\pi}\int_{k\leq\omega}\mathrm{d}s'\,\overline{P}(s')\,\sigma_0(s)$$

with $\overline{P}(s')$ radiator function

$$P(s') = \left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2}\right)^2$$

integrated over $\cos\theta$. The azimutal integration over φ yield a factor 2π. Note $\mathrm{d}s' = -2\sqrt{s}\,\mathrm{d}k$. Work out $\overline{P}(s')$.

10.1.3 Virtual plus real soft gluons

Both the virtual and the soft real gluon emission are proportional to the lowest order cross–section and both are $O(\alpha_s)$ corrections such that they can, and actually must, be added

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{q\bar{q}\text{jets}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{parton}} (1 + C_{<\omega}) \tag{10.6}$$

with

$$\begin{aligned} C_{<\omega} &= C_{\text{virtual}} + C_{\text{soft bremsstrahlung}} \\ &= \frac{4}{3}\frac{\alpha_s}{2\pi}\left\{4\left(\ln\frac{s}{m_q^2} - 1\right)\ln\frac{2\omega}{\sqrt{s}} + 3\ln\frac{s}{m_q^2} - 4 - 4\,\frac{\pi^2}{3}\right\} \ . \end{aligned}$$

The following points are remarkable about this result:

1) the infrared singular terms (the ones $\propto \ln m_g$) dropped out; thus the sum of virtual plus real soft corrections is IR finite,

2) the big double logarithms proportional to $\ln^2 \frac{s}{m_q^2}$ (Sudakov Logarithms) also canceled,

3) however, a limit $m_q \to 0$ does not exist!, these are the so called collinear mass singularities due to the degeneracy of a massless quark traveling together with the massless gluon along the same direction.

Because of point **3)** the result for QCD is physically not satisfactory. We can remedy this by taking into account the finite angular resolution of identifying the jets. Hard gluons which travel almost parallel to the quark or antiquark cannot be resolved experimentally. Each detector not only has a finite energy detection threshold, but also a limited angular resolution. Before we discuss the angular cuts, let us first look at the result we get if we integrate over all the real gluon (full phase–space integration)

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)(e^+e^- \to q\bar{q}, q\bar{q}g) = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{parton}} (1 + C_{\text{all}}) \tag{10.7}$$

with

$$C_{\text{all}} = \frac{4}{3}\frac{\alpha_s}{2\pi}\frac{3}{2} \ ,$$

a completely regular perturbative sensible result, all kinds of possible singularities are absent. The result tells us that singularities are caused by unphysical "cuts". Physically sensible cuts are those which admit to separate unambiguously 2–jet events from 3–jet events. At the given order at most 3 jets are possible. At the next order also 4 jets are possible, and actually up to 4–jet events have been investigated with remarkable precision at LEP (see e.g. Fig. 10.4). Here we are going to investigate 3–jet structure in more detail. The experimental identification of 3–jet events, as we mentioned earlier, is a direct for the existence of the gluon and an experimental confirmation of the $q\bar{q}g$–gauge coupling.

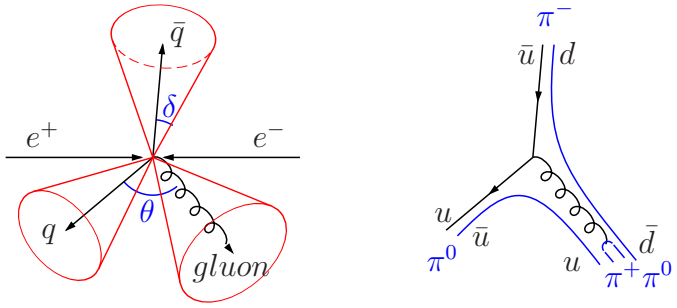


Figure 10.8: Left: three jets separated by cones; δ is the half opening angle of the cone and θ the angle between two cone axes. Right: possible low multiplicity hadronization mode

10.2 3–Jet Events

In the process $\gamma^* \to q\bar{q}g$ one can get 3–jet events, provided q, $\bar{q}$ and g all have large momenta in different directions, as illustrated in the Fig. 10.8

In order to understand the gross features it is sufficient to work in the approximation $m_q \simeq 0$: we use the following notation:

$$\begin{aligned} e_q &= \frac{E_q}{\sqrt{s}} = \frac{1}{2}\left(1 - e_g\left(1 - \cos\bar{\theta}\right)\right) \ , \\ e_{\bar{q}} &= \frac{E_{\bar{q}}}{\sqrt{s}} = \frac{1}{2}\left(1 - e_g\left(1 + \cos\bar{\theta}\right)\right) \ , \\ e_g &= \frac{E_g}{\sqrt{s}} \ ; \ \bar{\theta}: \ \sphericalangle(q,g) \ \text{ in the } \ q\bar{q} \ \text{ cm system} \ . \end{aligned} \tag{10.8}$$

One then has the relations

$$\begin{aligned} e_q &= -e_{\bar{q}} + \left(1 - e_g\right) \ , \\ e_q &= e_{\bar{q}}\,\frac{1-\cos\bar{\theta}}{1+\cos\bar{\theta}}\,\frac{\cos\bar{\theta}}{1+\cos\bar{\theta}} \ . \end{aligned} \tag{10.9}$$

In the laboratory system, which coincides with the e^+e^- c.m. system, the production angle is denoted by $\theta : \sphericalangle(q,g)$ and we have the relations

$$\cos\bar{\theta} = \frac{1 - v\,\sigma}{1 + v\,\sigma} \quad \text{with} \quad v \doteq \frac{1-\cos\theta}{1+\cos\theta} \ ; \ \ \sigma \doteq \frac{(p_1+p_2)^2}{s} \tag{10.10}$$

A 3–jet event is then defied by the cuts

$$e_q, e_{\bar{q}}, e_g > \varepsilon \ \text{ and } \ \theta_{q\bar{q}}, \theta_{qg}, \theta_{\bar{q}g} > \delta \ . \tag{10.11}$$

Of all events a fraction $f(\sqrt{s}, \varepsilon, \delta)$ are 2–jet events:

$$1 - f = \frac{4}{3}\,\frac{\alpha_s}{\pi}\,Q(\sqrt{s}; \varepsilon, \delta) \tag{10.12}$$

has first been calculated by Sterman and Weinberg 1977, in the approximation

$$1 \gg \delta \gg 2\,\frac{m_q}{E} \quad \text{and} \quad \varepsilon \ll 1$$

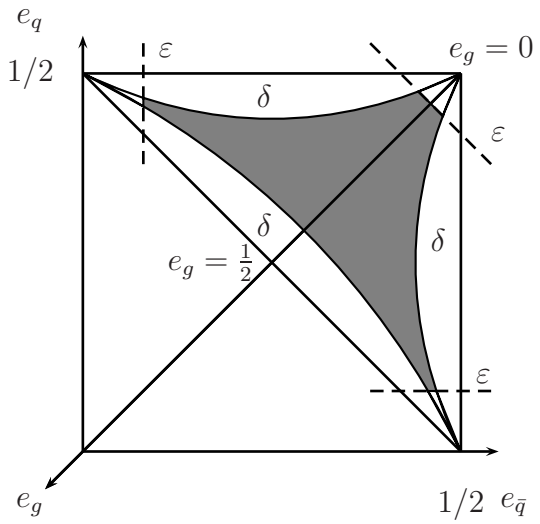


Figure 10.9: Three jet event defined by soft (ε) and collinear (δ) kinematical cuts.

, with the result:

$$Q(\varepsilon,\delta) \simeq (4\,\ln 2\varepsilon + 3)\,\ln\frac{\delta}{2} + \frac{\pi^2}{3} - \frac{7}{4}\ . \tag{10.13}$$

A little thinking tells us that this approximation is not adequate in practice of jet physics, because such sharp cuts are not realizable in hadronic processes. What we need is the result for the above definition of a 3–jet event for arbitrary cuts in δ and ε. Such results were given in [3] for radiative lepton pair production $\gamma^* \to \ell^+\ell^-\gamma$. The necessary integrations are straight forward but quite non-trivial, if one attempts to do them analytically. In the following we just give the results adapted for hadronic jets.

The result for the fraction f of 2–jet final states is given by

$$1 - f = \frac{4}{3}\,\frac{\alpha_s}{\pi}\,Q(\varepsilon,\delta) + O(\alpha_s^2) + O(\alpha_s\,\frac{m_q^2}{s}\ , \tag{10.14}$$

where Q consists of different pieces which we now present one by one.

1.) Configurations of hard non–collinear gluons defined by cuts

$$E_g/E > \varepsilon\ ,\quad \theta qg,\ \theta\bar{q}g > \delta$$

Here

$$Q_I(\varepsilon,\delta) = \frac{C_{>\omega,>\delta_0}}{\alpha/\pi}$$

where $C_{>\omega,>\delta_0}$ is given in Eq. (11) of Ref. [3] with $\lambda = 2\varepsilon$. Thus

$$\begin{aligned} Q_I^{(m_q=0)}(\varepsilon,\delta) \;=\; & \frac{1}{2}\Big\{(4\,\ln 2\varepsilon + (1-2\varepsilon)(3-2\varepsilon))\,\ln\frac{1-\rho}{1+\rho} - 4\,\frac{1+v}{v^2}\,\ln(1+v(1-3\varepsilon) + \frac{4}{v}\,(1-2\varepsilon) \\ & (1-2\varepsilon)(3-2\varepsilon)\,\ln(1-2\varepsilon)\,\frac{2}{3}\,\pi^2 - 4\mathrm{Sp}\,(2\varepsilon) - \frac{3}{2}\,(1-2\varepsilon)^2\Big\} \end{aligned} \tag{10.15}$$

where $\rho = \cos\delta$ and $v = \frac{1-\rho}{1+\rho}$. One easily checks that in the limit $(1 \gg \delta \gg 2\frac{m_q}{E}\ ,\ \ \varepsilon \ll 1)$ we obtain Sterman and Weinbergs result.

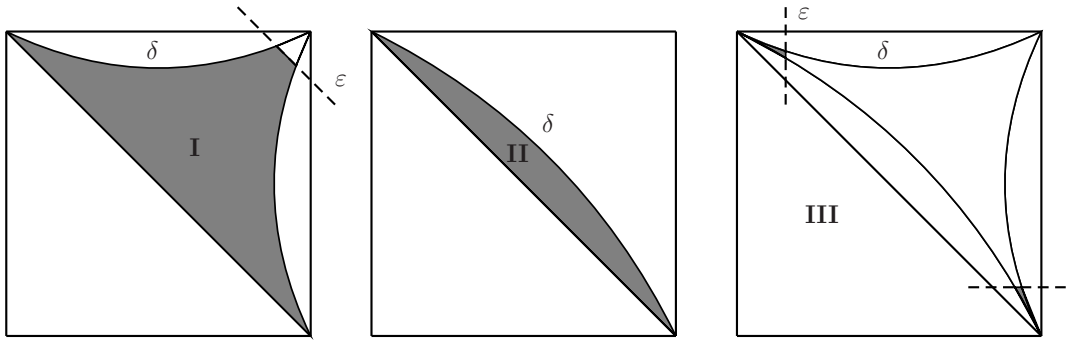


Figure 10.10: The three integration regions I, II and III, referred to in the text.

2.) Configurations with quark and antiquark in the same jet

$$\theta q\bar{q} < \delta$$

$$\Delta Q_{II}(\delta) = \mathrm{Sp}\,(\Delta) + \frac{1}{2}\,\ln^2(1-\Delta) + \frac{1}{2}\,\frac{1+2\Delta}{\Delta^2}\,\ln(1-\Delta) + \frac{1}{2\delta} + \frac{5}{4} \tag{10.16}$$

with $\Delta = \frac{1-\rho_{q\bar{q}}}{2}$, and $\rho_{q\bar{q}} = \cos\delta_{q\bar{q}}$.

3.) Configurations with soft q and/or $\bar{q}$ at angle

$$\sphericalangle(q,g) > \delta\,,\ (\bar{q},g) > \delta \quad \text{and} \quad (q,\bar{q}) > \delta$$

Here one obtains

$$\begin{aligned}\Delta Q_{III}(\varepsilon,\delta) \;&=\; 2\,(\ln(1-2\varepsilon) + \varepsilon\,(1+\varepsilon))\,\ln v + \ln^2(1-2\varepsilon) + 2\,\varepsilon\,(1+\varepsilon)\,\ln(1-2\varepsilon) - 8/,\varepsilon \\ &\quad + \frac{1-4\,\delta}{\delta^2}\,\ln(1-2\varepsilon\Delta) + \frac{2\varepsilon}{\Delta}\;,\end{aligned} \tag{10.17}$$

where $\delta = \delta_{q\bar{q}} = \delta qg = \delta_{\bar{q}g}$, $v = \frac{1-\rho}{1+\rho}$, $\Delta = \frac{1-\rho}{2}$ and $\rho = \cos\delta$. In any case $\delta < \frac{\pi}{2}$ and $\varepsilon \leq \frac{1}{4}$ is assumed. Altogether, we find

$$Q(\varepsilon,\delta) = Q_I(\varepsilon,\delta) - Q_{II}(\delta) + Q_{III}(\varepsilon,\delta) \tag{10.18}$$

Results presented in Tab. 1 illustrate the strong dependence of the rates on the cuts. Plots may be found in Ref. [3]. Especially for narrow cuts one probes the soft ($\varepsilon \to 0$) and the collinear ($\delta \to 0$) singularities. In reality the latter are smoothed by the quark masses, however at high energies the collinear logarithms get huge and may cause the breakdown of the perturbative expansion. The soft logarithms exponentiate, upon which the soft limit gets regularized (screening).

10.3 Summary and Discussion

We have presented results at $O(\alpha_s)$ with all terms $O(\alpha_s\, m_q^2/s)$ neglected, which is a good approximation sufficiently far above thresholds. The results may be summarized as follows:

1.) The differential quark pair production cross section

Table 1: Values for the function $Q(\varepsilon, \delta)$ for various values of the parameters ε and δ.

$\delta^\circ \setminus \varepsilon$	.05	.10	.15	.20	.25
5.0	21.36	12.92	7.92	4.15	.83
10.0	16.92	10.32	6.40	3.43	.83
15.0	14.31	8.79	5.50	3.01	.82
20.0	12.44	7.69	4.86	2.71	.81
25.0	10.98	6.83	4.35	2.47	.80
30.0	9.77	6.12	3.93	2.26	.79
35.0	8.73	5.51	3.57	2.09	.77
40.0	7.81	4.97	3.25	1.93	.75

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)(e^+e^- \to q\bar{q}, q\bar{q}g) = \frac{4}{3}\,\frac{\alpha^2}{s} \sum_{q:4m_q^2 \ll s} Q_q^2\,\left(1+\cos^2\theta\right) \times \left\{1+\frac{\alpha_s}{\pi}\right\}$$

in non-resonant regions agrees with the experiments in magnitude as well as in angular distribution (the number of 2–jet events at given angle θ of the jet axes as shown in Fig. 10.7). The virtual γ^* exchange process can be measured directly below about $\sqrt{s} \sim 30$ GeV before $\gamma - Z$ mixing comes into play. Hadron production at the Z–resonance as very precisely studied at LEP/SLC is a topic to be discussed in a separate chapter. The QCD corrections, known to $O(\alpha_s^4)$ clearly improve the agreement between theory and experiment. Mass effect are known at $O(\alpha_s^2)$ and matter as one approaches thresholds from above (see Fig. 8.7). The strong interaction constant α_s has been determined in an up-to-date reanalysis [14, 15] from the total cross section at PETRA to 20% $\alpha_s(42.4\ \mathrm{GeV}) = 0.144 \pm 0.029$ and from jet event shapes at the 6% level $\alpha_s(M_Z) = 0.1286 \pm 0.0072$ (total error).

2.) Neglecting n–jet ($n \geq 4$) events, i.e. $O(\alpha_s^2)$ corrections,

and denoting by $f(\sqrt{s}, \varepsilon, \delta)$ the fraction of 2–jet event at given energy cut ε and given angular cut δ, the fraction of 3–jet events is of leading order $O(\alpha_s)$, given by

$$\frac{4}{3}\,\frac{\alpha_s}{\pi}\,Q(\sqrt{s}; \varepsilon, \delta) = 1 - f(\sqrt{s}, \varepsilon, \delta)\ .$$

Because of asymptotic freedom, for $s \to \infty$ we have $f(\sqrt{s}, \varepsilon, \delta) \to 1$, i.e., 2-jet events dominate more and more the higher the energy. The 3–jet events directly probe the existence of the gluon and of the gauge coupling $q\bar{q}g$, i.e., quarks only interact with each other via gluon exchange.

The decrease of the number of 3–jet events with increasing energy for fixed cut parameters ε and δ means that events more and more populate the boundaries of phase space. This means that jets narrow with increasing energy.

To illustrate jet narrowing we consider the Sterman-Weinberg approximation $\varepsilon \ll 1$ and $2\,\frac{m_q}{E} \ll \delta \ll 1$ and α_s in leading approximation $\alpha_s = \frac{4\pi}{\beta_0\,\ln s/\Lambda^2}$:

$$\begin{aligned} 1 - f &= \frac{4}{3}\,\frac{\alpha_s}{\pi}\,Q(\varepsilon, \delta) \\ Q(\varepsilon, \delta) &= (4\,\ln 2\varepsilon + 3)\,\ln\frac{\delta}{2} + \frac{\pi^2}{3} - \frac{7}{4}\ . \end{aligned}$$

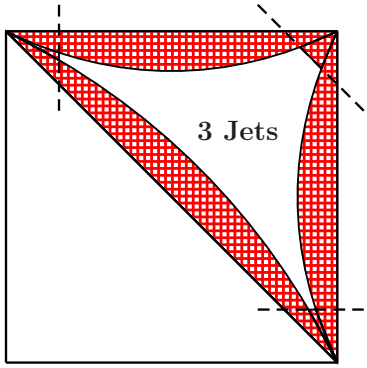


Figure 10.11: 2 jet events (red hatched) at high energies concentrate close to the triangular phase space boundary.

We then ask the question, how do we have to change δ as a function of s such that the fraction of 3–jet events remains the same. To this end we just have to solve the equation $f(\sqrt{s};\varepsilon,\delta)$ =contant for δ, with the result

$$\delta = 2\,\exp\left(\frac{\frac{7}{4}-\frac{\pi^2}{3}}{4\,\ln 2\varepsilon+3}\right)\cdot\underbrace{\exp\left(\frac{3\beta_0}{16}\,\frac{1-f}{4\,\ln 2\varepsilon+3}\,\ln\frac{s}{\Lambda^2}\right)}_{\left(\frac{\sqrt{s}}{\Lambda}\right)^{\frac{3\beta_0}{8}\,\frac{1-f}{4\,\ln 2\varepsilon+3}}}$$

and hence

$$\delta \propto \left(\frac{\sqrt{s}}{\Lambda}\right)^{-d} \quad \text{with} \quad d>0\ .$$

Since $\varepsilon \ll 1$ we have $4\,\ln 2\varepsilon+3<0$, such that QCD jets narrow powerlike with increasing energy. An example: $N_f=3, \varepsilon=0.1, 1-f=0.1$ yields $d=0.098$.

From the observed ratio $\sigma(3\text{ jets})/\sigma(2\text{ jets})$, which at leading order is proportional to α_s one can determine the strong coupling constant. A serious problem is the fact that the cuts ε and δ, which have been imposed for the partons!, do not simply translate into experimental cuts on hadronic events, since the hadronization, also called fragmentation, is modifying the perturbative cuts in a non–perturbative manner. In particular, the angular cut δ is broadening by hadronizatio. This problem is dealt with using phenomenologically tuned fragmentation models. As a typical result at PETRA energies is the one from the JADE Collaboration mentoned under item **1.)** above. As mentioned earlier, LEP experiments at the Z peak allowed one to perform QCD studies with huge event rates with millions of events. There, including the appropriate higher order effects, one could neatly separate also 4 jet events, and the jet analyses were able to experimentally determine the $SU(3)_c$ gauge structure be measuring the characteristic Casimir coefficients as shown once more in Fig. 10.12.

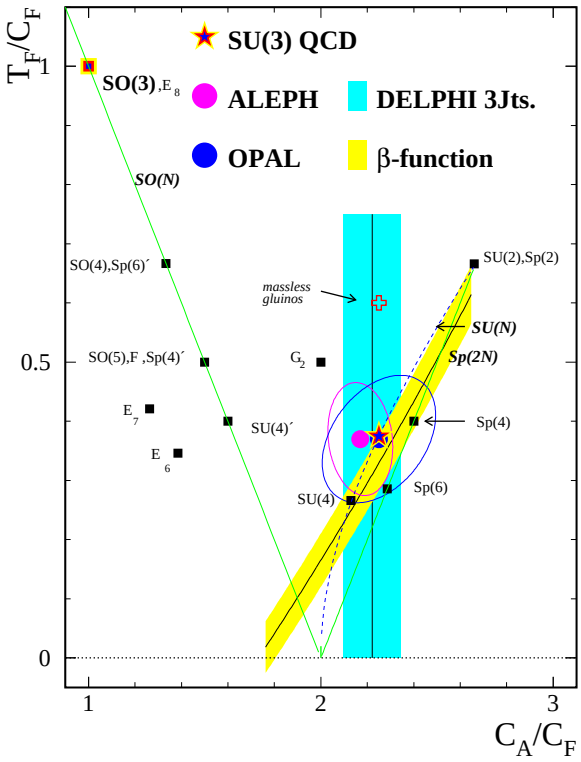


Figure 10.12: Determination of the gauge structure from jet analyses.

Exercises: Section 10

① Show that sphericity is infrared safe but not collinear safe.

② Show that spherocity is both infrared and ultraviolet save.

③ Exercise:
a) Show that the hard gluon spectrum, after performing the angular intergrations, may be written in the form

$$\frac{\mathrm{d}\sigma}{\mathrm{d}k} = \sigma_0(s)\, C_2(R)\, \frac{\alpha_s}{\pi} \left(\ln \frac{s'}{m_q^2} - 1 \right) \left(1 + \left(\frac{s'}{s} \right)^2 \right) \frac{1}{k}$$

where $k = |\vec{k}\,|$ is the gluon energy and $s' = s \left(1 - \frac{k}{E_b} \right)$ and $E_b = \sqrt{s}/2$. Note that s' is the invariat mass square of the $q\bar{q}$–system after radiation of the gluon, which carried away a fraction k/E_b of the energy.

b) In contrast in the soft gluon limit (factorization) may be represented as

$$\sigma(s) = \sigma_0(s) + C_2(R)\, \frac{\alpha_s}{\pi} \int_{k \leq \omega} \mathrm{d}s'\, \overline{P}(s')\, \sigma_0(s)$$

with $\overline{P}(s')$ radiator function

$$P(s') = \left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2} \right)^2$$

integrated over $\cos\theta$. The azimutal integration over φ yield a factor 2π. Note $\mathrm{d}s' = -2\sqrt{s}\,\mathrm{d}k$. Work out $\overline{P}(s')$.

④

References

[1] G. Hanson et al., Phys. Rev. Lett. **35** (1975) 1609

[2] G. Sterman, S. Weinberg, Phys. Rev. Lett. **39** (1977) 1436.

[3] J. Fleischer and F. Jegerlehner, Z. Phys. C **26** (1985) 629.

[4] J. R. Ellis, M. K. Gaillard and G. G. Ross, Nucl. Phys. B **111** (1976) 253 [Erratum-ibid. B **130** (1977) 516].

[5] G. Kramer, G. Schierholz and J. Willrodt, Phys. Lett. B **79** (1978) 249 [Erratum-ibid. B **80** (1979) 433].

[6] R. Brandelik *et al.* [TASSO Collaboration], Phys. Lett. B **86** (1979) 243; Phys. Lett. B **97** (1980) 453.

[7] JADE Collaboration 1980 Phys. Lett.B91 142.
JADE Collaboration 1981 Phys. Lett.B101 129.
MARK J Collaboration 1979 Phys. Rev. Lett.43 830.
MARK J Collaboration 1983 Phys. Rev. Lett.50 2051.
PLUTO Collaboration 1979 Phys. Lett.B86 418.
PLUTO Collaboration 1985 Z. Phys.C28 365.

[8] ALEPH Coll. Heister A et al 2003 Eur. Phys. J. C 27 1 17

[9] OPAL Coll. Abbiendi G et al 2001 Eur. Phys. J. C 20 601 615

[10] OPAL Coll. Abbiendi G et al 2002 Eur. Phys. J. C 23 597 613

[11] DELPHI Coll. Abreu P et al 1999 Phys. Lett. B 449 383

[12] DELPHI Coll. Abreu P et al 2000 Eur. Phys. J. C 13 573

[13] Kluth S et al 2001 Eur. Phys. J. C 21 199 210

[14] C. Pahl, S. Kluth, S. Bethke, P. A. Movilla Fernandez and J. Schieck [JADE Collaboration], arXiv:hep-ex/0408123.

[15] S. Kluth, Rept. Prog. Phys. **69** (2006) 1771 [arXiv:hep-ex/0603011].

A Solved Problems

A.1 Exercises: Section 1

① Show that $8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 10^* \oplus 27$.

✎

$$8 \otimes 8 = \ydiagram{2,2} \times \ydiagram{2,2} = \ydiagram{2,2} \times \begin{smallmatrix} a & a \\ b \end{smallmatrix}$$
$$= \begin{smallmatrix} & a \\ a & b \end{smallmatrix} \oplus \begin{smallmatrix} & a \\ a & b \end{smallmatrix} \oplus \begin{smallmatrix} a & a \\ b \end{smallmatrix} \oplus \begin{smallmatrix} a & a \\ b \end{smallmatrix} \oplus \begin{smallmatrix} a \\ a \\ b \end{smallmatrix} \oplus \begin{smallmatrix} a & a \\ b \end{smallmatrix}$$

In order to append to the first octet the second one in all admissible ways which respect the (anti-) symmetrization, we replace the second one by letters with identical letters in the rows (symmetrized). The different elements of a column if they appear in the same column of the enlarged tableau must appear in the old order (anti-symmetrized). Hence append the elements from the first row in all possible ways to the first tableau, then to the such enlarged ones the elements of the second row etc. For our case one easily reads off the result.

Note that the tableau

$$\begin{smallmatrix} & \\ a & \\ a & \end{smallmatrix}$$

is not allowed. Also

$$\ydiagram{2,1}\,{}^{a\ a\ b}$$

is not allowed, since baa is not an admissible sequence of letters.

✌

② Isospin symmetry was introduced by Heisenberg in 1932 right after the discovery of the neutron (Chadwick 1932). It is obvious that isospin symmetry is violated by electromagnetism: the iso-doublet members (p,n) have different charge $(1, 0)$. The nucleon system is described by a iso-spinor Dirac field

$$\Psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} .$$

Write down the charges Q, B and T_i $(i = 1, 2, 3)$ in terms of the nucleon field operator and show that B commutes with isospin, but Q does not. Write down the $T = 1$ and $T = 0$ two nucleon states.

✎

The generators of the different symmetry operations are given by the operators

$$Q = \int \mathrm{d}^3x\, \bar{\psi}_p \gamma^0 \psi_p = \int \mathrm{d}^3x\, \bar{\Psi} \gamma^0 \frac{1 + \tau_3}{2} \Psi$$

$$
\begin{aligned}
B &= \int \mathrm{d}^3x \, \left(\bar{\psi}_p\gamma^0\psi_p + \bar{\psi}_n\gamma^0\psi_n\right) = \int \mathrm{d}^3x \, \bar{\Psi}\gamma^0\Psi \\
T_3 &= \frac{1}{2}\int \mathrm{d}^3x \, \left(\bar{\psi}_p\gamma^0\psi_p - \bar{\psi}_n\gamma^0\psi_n\right) = \bar{\Psi}\gamma^0\frac{\tau_3}{2}\Psi \\
T_i &= \int \mathrm{d}^3x \, \bar{\Psi}\gamma^0\frac{\tau_i}{2}\Psi
\end{aligned}
$$

The operator version of the Lie algebra follows directly from the $SU(2)$ generator matrices $\tau_i/2$ (τ_i the Pauli matrices):

$$[T_i, T_j] = \mathrm{i}\,\varepsilon_{ijk}T_k \ .$$

We furthermore have $Q = T_3 + \frac{B}{2}$ and since B corresponds the the unity matrix in isospin space such that $[B, T_i] = 0$ we formally obtain

$$[Q, T_i] = [T_3, T_i] \neq 0 \ ; \ \ i = 1, 2$$

the manifestation of isospin breaking by electromagnetism. The two-nucleon states of fixed isospin follow the usual $SU(2)$ spin rules:

$$
\begin{aligned}
T = 1: &\qquad \psi_p^{(1)}\psi_p^{(2)} \ , \frac{1}{\sqrt{2}}\left(\psi_p^{(1)}\psi_n^{(2)} + \psi_n^{(1)}\psi_p^{(2)}\right) \ , \psi_n^{(1)}\psi_n^{(2)} \ , \\
T = 0: &\qquad \frac{1}{\sqrt{2}}\left(\psi_p^{(1)}\psi_n^{(2)} - \psi_n^{(1)}\psi_p^{(2)}\right) \ .
\end{aligned}
$$

✌

③ Discuss the iso-spin properties of the triplet of pions (π^+, π^0, π^-) .

The iso-spin symmetry of the scattering operator S not only leads to relations between matrix elements but also to selection rules: Suppose

(a) T is a generator of a symmetry transformation such that $[T, S] = 0$,
(b) $\mid \alpha >$ and $\mid \beta >$ are eigenstates of T i.e. $T \mid \alpha >= t_\alpha \mid \alpha >, T \mid \beta >= t_\beta \mid \beta >$

What does this imply for the S-matrix elements

$$S_{\beta\alpha} =< \beta \mid S \mid \alpha > \ ?$$

Find a few examples.

✎

The pions form an isospin triplet, for $SU(2)$ the representation is equivalent to the adjoint representation (which has dimension 3) : $(t_a)_{ij} = -\mathrm{i}\,\varepsilon_{ija}$ thus

$$
t_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \ ; \ t_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix} \ ; \ t_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ,
$$

By a change of basis we may arrange t_3 to be diagonal

$$
t_3' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ ; \ t_1' = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ ; \ t_2' = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \ ,
$$

and we may use the Lie algebra relations in the form $T_\pm = T_1 \pm \mathrm{i}\, T_2$, $[T_3, T_\pm] = \pm T_\pm$ *and* $[T_+, T_-] = 2\, T_3$ *to find* t_1' *and* t_2' *for given* t_3'. *The states describing* π^+, π^0, π^- *are*

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; ; \; \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \; ,$$

with $T_3 = +1, 0, -1$.

Two-pion states decompose into $T = 0, 1$ *and* 2 *irreducible parts:*

$$\begin{aligned} 3 \otimes 3 &= \square\square \times \square\square = \begin{array}{|c|c|}\hline \\ \hline \\ \hline \end{array} \oplus \begin{array}{|c|c|c|}\hline \\ \hline \end{array} \oplus \square\square\square\square \\ &= \square \oplus \square\square \oplus \square\square\square\square = 1 \oplus 3 \oplus 5 \; . \end{aligned}$$

The tensor product space is 9 dimensional $(\pi^+\pi^+, \pi^+\pi^0, \pi^+\pi^-, \pi^0\pi^+, \pi^0\pi^0, \pi^0\pi^-, \pi^-\pi^+, \pi^-\pi^0, \pi^-\pi^-)$ *and decomposes into a singlet, a triplet and a quintet. In strong interactions a two pion state has definite total isospin (analogous to angular momentum)* $\vec{T} = \vec{t}^{\,(1)} + \vec{t}^{\,(2)}$ *with*

$$\vec{T}^2 = T\,(T+1) = (\vec{t}^{\,(1)} + \vec{t}^{\,(2)})^2 = (\vec{t}^{\,(1)})^2 + 2\,\vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)} + (\vec{t}^{\,(2)})^2$$

such that with $(\vec{t}^{\,(i)})^2 = t\,(t+1) = 2$ *we find*

$$\vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)} = \frac{1}{2}\, T\,(T+1) - 2 \; ,$$

which may be used to construct a basis of two–pion states of isospin $T = 2, 1$ *and* 0. *Obviously*

$$\vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)} = \begin{cases} -2 & , \quad T{=}0 \\ -1 & , \quad T{=}1 \\ +1 & , \quad T{=}2 \end{cases}$$

such that the combinations

$$\begin{aligned} 2 + \vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)} &= 0, 1, 3 \quad \text{for} \quad T = 0, 1, 2 \\ 1 + \vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)} &= -1, 0, 2 \quad \text{for} \quad T = 0, 1, 2 \\ 1 - \vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)} &= 3, 2, 0 \quad \text{for} \quad T = 0, 1, 2 \end{aligned}$$

project to zero the $T = 0, 1, 2$ *part, respectively. Normalized isospin projection operators then read*

$$\begin{aligned} \Pi_2 &= \frac{1}{6}\,(2 + \vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)})\,(1 + \vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)}) \; , \\ \Pi_1 &= \frac{1}{2}\,(2 + \vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)})\,(1 - \vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)}) \; , \\ \Pi_0 &= \frac{1}{3}\,(\vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)} - 1)\,(1 + \vec{t}^{\,(1)} \cdot \vec{t}^{\,(2)}) \; . \end{aligned}$$

A particular property of the pion triplet is **G-parity** *conservation. G–parity is defined by*

$$G = C\, \mathrm{e}^{\mathrm{i}\pi T_3}$$

Let ϕ *denote the triplet of pion fields. Under charge conjugation*

$$C \phi C^{-1} = \phi^*$$

We may write $\phi^0 = \phi_3$ and $\phi^\mp = \frac{1}{\sqrt{2}}\,(\phi_1 \pm \mathrm{i}\,\phi_2)$, in terms of 3 real Cartesian fields. Then

$$C \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} C^{-1} = \begin{pmatrix} \phi_1 \\ -\phi_2 \\ \phi_3 \end{pmatrix}$$

and an isospin rotation by angle π about the 3-axis implies

$$\mathrm{e}^{\mathrm{i}\pi T_3} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\pi T_3} = \begin{pmatrix} -\phi_1 \\ \phi_2 \\ -\phi_3 \end{pmatrix} \ ,$$

which tells us that

$$G\phi_i G^{-1} = -\phi_i \ ; \ \ i = 1,2,3 \ .$$

Among other things this tells us that in the isospin symmetry limit a decay like

$$\eta \not\to \pi^+\pi^-\pi^0 \quad ???$$

is strictly forbidden. Note that $\eta^0 \to 2\gamma$ requires $T = 0$ for the η, furthermore $C\eta^0 C^{-1} = \eta^0$ and hence $G\eta^0 G^{-1} = \eta^0$. In fact the decay $\eta \to \pi^+\pi^-\pi^0$ is a $\Delta I = 1$ transition (the pions are in a T=1 state) mediated in leading order (up the electromagnetic isospin violating corrections) by the $\Delta I = 1$ part of the QCD Hamiltonian

$$\mathcal{H}_{\mathrm{QCD}}(x) = \frac{1}{2}(m_u - m_d)\,(\bar{u}u - \bar{d}d)(x) \ ,$$

due to isospin breaking by the $m_d - m_u$ mass difference. Thus the decay is chirally suppressed and would be forbidden in the chiral limit up to even smaller electromagnetic isospin breaking effects:

$$\mathcal{H}_{\mathrm{QED}}(x) = -\frac{1}{2}e^2 \int \mathrm{d}y\, D_{\mu\nu}(x-y)\, T\{j^\mu_{\mathrm{em}}(x)\, j^\nu_{\mathrm{em}}(y)\}$$

where $D_{\mu\nu}(x-y)$ is the photon propagator and $j^\mu_{\mathrm{em}} = j^\mu_3 + \frac{1}{2}j^\mu_Y$ is the electromagnetic current, consisting of an isovector and an isoscalar part. Electromagnetic effects are suppressed by the fact that only the interference part has the right G-parity. The η–decay branching fractions are

$\eta \to$	2γ	$3\pi^0$	$\pi^+\pi^-\pi^0$	$\pi^+\pi^-\gamma$	other
Br	*39.31%*	*32.56 %*	*22.73 %*	*4.6 %*	*2.8 %*
int	QED	QCD	QCD	QED	
	$\not{I}, G$	$\not{I}, \not{G}$	$\not{I}, \not{G}$	$\not{I}, G$	

With $T|\alpha\rangle = t_\alpha|\alpha\rangle$ and $S_{\beta\alpha} = \langle\beta|S|\alpha\rangle$ we find

$$\langle\beta|[T,S]|\alpha\rangle = \langle\beta|TS|\alpha\rangle - \langle\beta|ST|\alpha\rangle = (t_\beta - t_\alpha)\,\langle\beta|S|\alpha\rangle = 0 \ .$$

Therefore

$$\boxed{\text{either} \quad t_\beta = t_\alpha \quad \text{or} \quad \langle\beta|S|\alpha\rangle = 0 \quad .}$$

Such a circumstance is called a **selection rule**. *If T is an absolutely conserved quantity like the charge operator Q or Baryon number B we call it a* **superselection rule**: *e.g. each Hilbert space of states of a fixed charge is totally orthogonal to any other Hilbert space of states of a different charge.*

In the limit of vanishing neutrino masses in the SM **lepton number** L_ℓ *is separately conserved for $\ell = e, \mu$ and τ:*

particle	e^-	e^+	μ^-	μ^+	τ^-	τ^+	γ	*other*
L_e	1	-1	0	0	0	0	0	0
L_μ	0	0	1	-1	0	0	0	0
L_τ	0	0	0	0	1	-1	0	0

As an example, we may consider the process $\mu^- \to e^-\gamma$. In fact $[L_e, S] = 0$ implies that $\langle\mu^-|S|e^-\gamma\rangle = 0$: $\langle\mu^-|[L_e, S]|e^-\gamma\rangle = \langle\mu^-|L_eS|e^-\gamma\rangle - \langle\mu^-|SL_e|e^-\gamma\rangle = (0-1)\,\langle\mu^-|S|e^-\gamma\rangle = 0$ from which the assertion follows. While leptonic flavor transitions are practically forbidden, the corresponding transitions in the quark sector are allowed by quark flavor mixing, as the $\Delta S = 1$ transition $\Lambda^0 \to n\gamma,\ \Sigma^+ \to p\gamma$, etc. with branching fractions of order $\sim 10^{-3}$ (see below).

Back to isospin: in pionic systems (no hypercharge or baryon number) we always have $Q = T_3$ and isospin selection rules coincide with the one of charge conservation. As we discussed above, for the η G-parity seems to provide a good selection rule. Indeed, the η does not decay by strong interactions as it violates isospin. That's why it is so stable, with a width $\Gamma_\eta = 1.30 \pm 0.07$ keV only. Further interesting cases are $\rho^0 \to \pi^+\pi^-$ [$\sim$ 100%] versus $\rho^0 \to \pi^+\pi^-\pi^0$ [Br=$(1.01^{+0.64}_{-0.50}) \cdot 10^{-4}$] in comparison to and $\omega \to \pi^+\pi^-\pi^0$ [89.2 %] versus $\omega \to \pi^+\pi^-$ [$(1.53^{+0.11}_{-0.13})$ %]. These are strong interaction processes taking place in absence of electromagnetic and weak interactions. If isospin is conserved $\rho^0 \to \pi^+\pi^-\pi^0$ as well as $\omega \to \pi^+\pi^-$ are forbidden by G-parity. There occurrence is due to isospin breaking by the quark mass difference $m_u - m_d$, which leads to $\rho-\omega$–mixing (briefly mentioned in Sec. 1.4). Thus the physical ρ and ω are not pure $SU(2)$ states and therefore in fact can decay via the wrong G-parity modes.

What about $\langle\gamma^*|\pi^+\pi^-\rangle$ and $\langle\gamma^*|\pi^0\pi^0\rangle$?

Another example is proton decay $p \to e^+\gamma, e^+\pi^0, \pi^+\pi^0, \cdots$, which is forbidden by B conservation. We know that the matter–antimatter asymmetry in the universe requires B to by violated at scales above about $M_X \sim 10^{16}$ GeV. ✌

④ Use the Young tableaus to construct the meson states in

$$3 \otimes 3^*$$

and the baryon states in

$$3 \otimes 3 \otimes 3\ .$$

The states in the pseudo-scalar meson octet of flavor $SU(3)$ are characterized by the 3^{rd} component of iso-spin and by hypercharge $Y = B + S$ (B baryon number $B = 0$ for mesons, S strangeness $S = 0$ for pions). Display the weight diagram ($I_3 - Y$ plot) of the meson states. How are they composed of u, d and s quarks in the $SU(3)_{\text{flavor}}$ quark model ?

✎
✌

⑤ The structure constants c_{ikl} of a Lie-algebra $[T_i, T_k] = \mathrm{i}\, c_{ikl} T_l$ satisfy the Jacobi identity.

$$c_{ikn}c_{nlm} + \text{ terms cyclic in } (ikl) = 0$$

Use this to show that $(\tilde{T}_i)_{kl} = -\mathrm{i}\, c_{ikl}$ also satisfies the Lie-algebra (adjoint representation).

✎

Cyclic sum is: "$(ikl)+(lik)+(kli)$"$= 0$ and thus Jacobi reads: $c_{ikn}c_{nlm}+c_{lin}c_{nkm}+c_{kln}c_{nim} = 0$. We take the 1st term as the r.h.s of the Lie algebra. In the 2nd and 3rd term: since n is summed over it should be the index of matrix multiplication of two successive $\tilde{T}$'s, i.e no $\tilde{T}_n$ should appear, which may be achieved by antisymmetric permutation of the index n away from its first position in the c_{nkm} etc. and m as a dummy index is good to be kept in the last position. Also only $\tilde{T}_i$ and $\tilde{T}_k$ should appear i.e. I is the other dummy to the left: therefore we may write $c_{ikn}c_{nlm} = \mathrm{i}\, c_{ikn}\,(\tilde{T}_n)_{lm}$, $c_{lin}c_{nkm} = c_{iln}c_{knm} = -\,(\tilde{T}_i)_{ln}\,(\tilde{T}_k)_{nm}$, $c_{kln}c_{nim} = -c_{kln}c_{inm} = (\tilde{T}_k)_{ln}\,(\tilde{T}_i)_{nm}$. Thus

$$\mathrm{i}\, c_{ikn}\,(\tilde{T}_n)_{lm} - (\tilde{T}_i)_{ln}\,(\tilde{T}_k)_{nm} + (\tilde{T}_k)_{ln}\,(\tilde{T}_i)_{nm} = 0 \ ,$$

which is the algebra

$$([\tilde{T}_i, \tilde{T}_k])_{lm} = \mathrm{i}\, c_{ikn}\,(\tilde{T}_n)_{lm} \ .$$

q.e.d. ✌

⑥ Discuss the electromagnetic decays of π^0 and η^0 into photons. Use the fact that the C parity of the photon is $\eta_C^\gamma = -1$. What are the observed decays? What do they imply for the C parities of the neutral pseudoscalar mesons π^0 and η^0 ?. Which decays are strictly forbidden? Which Feynman diagram is responsible for these decays?

✎

C parities of π^0 and η^0:

Since the photon has C–parity $\eta_C^\gamma = -1$ as mentioned earlier, the observed C–conserving electromagnetic decays

$$\pi^0 \to \gamma\gamma \ , \ \ \eta^0 \to \gamma\gamma$$

tell us that

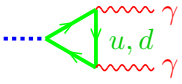
$$\eta_C^{\pi^0} = \eta_C^{\eta^0} = +1 \ .$$

The conservation of C–parity on the other hand implies that

$$\pi^0 \not\to \gamma\gamma\gamma \ , \ \ \eta^0 \not\to \pi^0\gamma \text{ or } \gamma\gamma\gamma$$

are forbidden decays.

As the π^0 is a $\bar{q}q$ bound state the decay is possible via the coupling of the photon to the charged quarks: the leading diagram is the 1–loop diagram

$\pi^0 = (\bar{u}u - \bar{d}d)/\sqrt{2}$ 

: $\mathcal{A}^{\pi^0\gamma\gamma}_{q-\text{loop}}$.

Via PCAC this transition amplitude is related to the axial vector current anomaly, which may be written as an effective Lagrangian (Wess-Zumino Lagrangian see below)

⑦ Which experiments (processes) allow us to determine the intrinsic parity (η_P) of the pions?

⑧ Lepton number L_e is another additive quantum number which is strictly conserved. $L_e(e^-) = 1$ by convention. Determine L_e for the other particles from the observed reactions:

1. $L_e(e^+) = -1,\ L_e(\gamma) = 0$:

$$\begin{aligned} p + e &\rightarrow p + e + \gamma \\ \gamma^* &\rightarrow e^+ + e^- \end{aligned}$$

2. $L_e(\pi^0) = L_e(\pi^\pm) = 0$:

$$\begin{aligned} \pi^0 &\rightarrow 2\gamma,\ \gamma + e^+ + e^- \\ p + \pi^- &\rightarrow n + \pi^0 \\ p + \pi^0 &\rightarrow n + \pi^+ \end{aligned}$$

3. $L_e(\bar{\nu}_e) = -1,\ L_e(\nu_e) = 1$:

$$\begin{aligned} \pi^- &\rightarrow e^- + \bar{\nu}_e \\ \pi^+ &\rightarrow e^+ + \nu_e \end{aligned}$$

From the last two reactions we learn the important result $\nu_e \neq \bar{\nu}_e$!

⑨ Baryon number conservation is responsible for the stability of the proton. By convention $B(p) = 1$, $B(e^-) = 0$. Determine the baryon numbers of particles from the observation of

the following reactions:

a.) Baryons and mesons:

1. $B(\pi^0) = 0$:

$$p + p \quad \rightarrow \quad p + p + \pi^0$$

2. $B(n) = B(p),\ B(\pi^{\pm}) = B(\pi^0) = 0$:

$$\begin{aligned} p + p \quad &\rightarrow \quad p + n + \pi^+ \\ \pi^- + p \quad &\rightarrow \quad n + \pi^0 \end{aligned}$$

3. $B(K^{\pm}) = B(K^0) = 0$:

$$\begin{aligned} K^{\pm} \quad &\rightarrow \quad \pi^{\pm} + \pi^0 \\ K^0 \quad &\rightarrow \quad \pi^+ + \pi^-,\ \pi^+ + \pi^- + \pi^0 \end{aligned}$$

4. $B(\Lambda),\ B(\Sigma) = 1$:

$$\begin{aligned} \pi^- + p \quad &\rightarrow \quad \Lambda^0 + K^0,\ \Sigma^- + K^+ \\ \pi^+ + p \quad &\rightarrow \quad \Sigma^+ + K^+,\ \Sigma^0 + \Lambda^0 \end{aligned}$$

5. $B(\Xi),\ B(\Omega^-) = 1$:

$$K^- + p \quad \rightarrow \quad \Xi^- + K^+,\ \Xi^0 + K^0,\ \Omega^- + K^+ + K^0$$

b.) Antibaryons:

6. $B(\bar{p}) = -1$:

$$p + p \quad \rightarrow \quad p + p + p + \bar{p}$$

7. $B(\bar{B}) = -1$:

$$p + \bar{p} \quad \rightarrow \quad \bar{n} + n,\ \bar{\Lambda}^0 + \Lambda^0,\ \bar{\Sigma}^0 + \Sigma^0,\ \bar{\Sigma}^{\pm} + \Sigma^{\mp},\ \bar{\Xi}^+ + \Xi^-$$

c.) Photon:

8. $B(\gamma) = 0$:

$$p \quad \rightarrow \quad p + \gamma$$

d.) Leptons: All leptons are produced in pairs, $B(e^-) = 0$ by convention.

9. $B(e) = B(\mu) = 0$:

$$\gamma^* \quad \rightarrow \quad e^+ + e^-,\ \mu^+ + \mu^-$$

10. $B(\nu_e) = B(\nu_\mu) = 0$:

$$\begin{aligned} n \quad &\rightarrow \quad p + e^- + \bar{\nu}_e \\ \mu^- \quad &\rightarrow \quad e^- + \bar{\nu}_e + \nu_\mu \\ \mu^+ \quad &\rightarrow \quad e^+ + \nu_e + \nu_\mu^- \\ \pi^- \quad &\rightarrow \quad \mu^- + \bar{\nu}_\mu \\ \pi^+ \quad &\rightarrow \quad \mu^+ + \nu_\mu \end{aligned}$$

✎
✌

A.2 Exercises: Section 2

① Discuss the symmetry breaking of $SU(2)_{\text{flavor}}$ (Isospin) and $SU(3)_{\text{flavor}}$ (Isospin and Strangeness) in the spin 1/2 baryon octet. Comment the decays $\Sigma^0 \to \Lambda\gamma$ and $\Sigma^+ \to p\gamma$ and compare them to the strong decays $\Sigma^- \to \Lambda\pi^-(?)$ and $\Sigma^- \to n\pi^-$. Use the quark model schema for the discussion.

✎

Note the following parallelism between meson and baryon states

Y	T	*Baryons*	*Mesons*
+1	$\frac{1}{2}$	p, n	K^+, K^0
0	*1*	$\Sigma^+, \Sigma^0, \Sigma^-$	π^+, π^0, π^-
-1	$\frac{1}{2}$	Ξ^0, Ξ^-	$\bar{K}^0, K^-$
0	*0*	Λ^0	η^0

Note the hierarchy $m_u < m_d << m_s$ and its influence on the hadron masses. They are the primary source of symmetry breaking and in first place are reflected in the masses.

Spin 1/2 Baryon masses:

Y	T	*Baryon masses in MeV*
+1	$\frac{1}{2}$	$m_p = 937.27203 \pm 0.00008 \quad m_n = 939.56536 \pm 0.00008$
0	*1*	$m_{\Sigma^+} = 1189.37 \pm 0.07 \quad m_{\Sigma^0} = 1192.642 \pm 0.024 \quad m_{\Sigma^-} = 1197.449 \pm 0.030$
-1	$\frac{1}{2}$	$m_{\Xi^0} = 1314.86 \pm 0.20 \quad m_{\Xi^-} = 1321.71 \pm 0.07$
0	*0*	$m_{\Lambda^0} = 1115.683 \pm 0.006$

A key role for the decay patterns play the mass differences, which determine the available decay phase space.

Mass shifts in MeV from replacing quark flavors:

$u \to d$			$d \to s$		
$m_n - m_p$	$=$	1.2933317 ± 0.0000005	$m_{\Sigma^+} - m_p$	$=$	252.1 ± 0.07
$m_{\Sigma^-} - m_{\Sigma^0}$	$=$	4.807 ± 0.035	$m_{\Sigma^0} - m_n$	$=$	253.1 ± 0.02
$m_{\Sigma^0} - m_{\Sigma^+}$	$=$	3.27 ± 0.07	$m_{\Xi^0} - m_{\Sigma^0}$	$=$	122.2 ± 0.20
$m_{\Xi^-} - m_{\Xi^0}$	$=$	6.85 ± 0.21	$m_{\Xi^-} - m_{\Sigma^-}$	$=$	124.3 ± 0.08
$m_{\Sigma^-} - m_{\Lambda^0}$	$=$	81.77 ± 0.03	$m_{\Lambda^0} - m_n$	$=$	176.1 ± 0.01
$m_{\Lambda^0} - m_{\Sigma^+}$	$=$	-73.69 ± 0.07	$m_{\Xi^0} - m_{\Lambda^0}$	$=$	199.2 ± 0.20

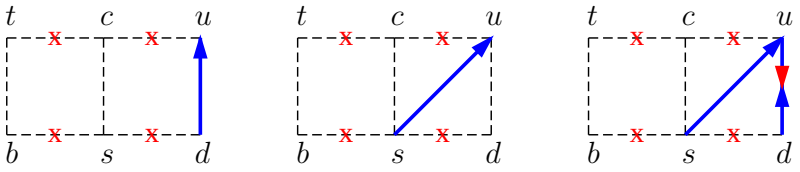


Figure A.13: The relevant CKM transitions for weak Baryon Decays. The horizontal lines correspond to Flavor Changing Neutral Currents (FCNC) which are strictly forbidden (x) in the SM.

We note a surprisingly large mass shift

$$m_{\Sigma^0} - m_{\Lambda^0} = 76.959 \pm 0.023 \ MeV$$

between the singlet T=0 and the octet T=1 states with the same conserved quantum numbers $T_3 = 0, Y = 0$.

Mass differences (phase space) and the different types flavor transitions Fig. A.13 essentially determine the lifetimes. Of course Baryon number must be conserved. For phase space reasons at most one additional hadrons, a pion, can be produced in $\Delta S = 1$ *transitions* $s \to u, d$. *Since FCNC (flavor changing neutral currents) in the SM are forbidden at tree level, most (all except one) decays are second order weak CC (charged current) transitions*

$$\begin{aligned} \mathcal{L}_{CC} &= \frac{g}{\sqrt{2}} \left(J^+_\mu W^{\mu -} + J^-_\mu W^{\mu +} \right) \\ J^-_\mu &= (\bar{u}, \bar{c}, \bar{t}) \, \gamma_\mu \, (1 - \gamma_5) \, U_{\rm CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \end{aligned}$$

with $J^+_\mu = (J^-_\mu)^+$ *and unitary* 3×3 *matrix*

$$U_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}.$$

Fig. A.13 shows the possible transitions for the u, d s subspace. The relevant decay types are depicted in Fig. A.15. $\Delta S = 0$ *transitions* $d \to u$ *only are possible semileptonically via* $d \to u e - \bar{\nu}_e$ *[diagram type a)]. Flavor changing radiative decays* $B \to B' \gamma$ *[diagram type b)] are suppressed by an additional factor* α. *All decays involving charged particle (one the quark level all) may include one or more additional photons, usually called electromagnetic radiative corrections (em RC). Processes where physical charged pions are produced via virtual W's [diagram type f)],* $W^\pm \to \pi^\pm$ *have direct semileptonic counterparts* $W^- \to e^- \bar{\nu}_e, \mu^- \bar{\nu}_\mu$ *or* $W^+ \to e^+ \nu_e, \mu^+ \nu_\mu$, *as* $m_\mu < m_\pi$ *also decays into muons show up, unlike in normal* $d \to u$ β*–decays, where decays into muons are absent because* $m_\mu >> m_d - m_u$. *The neutron is quasi stable due to the small phase space* $m_n - m_p$ *available. All quarks in a hadron are strongly interacting via gluons. Strong interactions like the electromagnetic ones are flavor preserving. In a decay a quark which is inactive with respect to weak and electromagnetic transitions is called a* **spectator quark**. *Decays which just differ by different spectator quarks have similar properties, except from mass and phase space effects.*

Because of the specific structure of the SM quark flavor mixing, a number of transitions are forbidden at order $O(G_F^2)$, *while possible in general at order* $O(G_F^4)$. *Since the latter are dramatically suppressed they usually are called* **forbidden** *transitions. In semileptonic hyperon decays*

proceeding via Fig. A.15 a) type diagrams we obviously must have $\Delta S = \Delta Q$ [where ΔS refers to the hadronic current and ΔQ to the leptonic one]. Since FCNC are forbidden a $s \to q$ transition only is possible by a W^- exchange. Accordingly $\Sigma^+ \to n e^+ \nu_e$ is forbidden [while $\Sigma^- \to n e^- \bar{\nu}_e$ is allowed], another forbidden decay is $\Xi^0 \to \Sigma^- e^+ \nu_e$, here also $\Xi^0 \to \Sigma^- \pi^+$ is absent by kinematics. Note that $\Sigma^+ \to n\pi^+$ is allowed as well as a subsequent decay $\pi^+ \to e^+ \nu_e$, however, overall it is a strongly suppressed $O(G_F^4)$ process. Also $\Delta S = 2$ transitions are forbidden $\Xi^0 \to p\pi^-$ and $\Xi^- \to n\pi^-$.

Baryon lifetimes:

Y	T	*Baryon lifetimes in seconds*
+1	$\frac{1}{2}$	$\tau_p > 2.1 \cdot 10^{29}$ *years* $\quad \tau_n = 885.7 \pm 0.8$
0	*1*	$\tau_{\Sigma^+} = (0.802 \pm 0.003) \cdot 10^{-10} \quad \tau_{\Sigma^0} = (7.4 \pm 0.7) \cdot 10^{-20} \quad \tau_{\Sigma^-} = (1.479 \pm 0.011) \cdot 10^{-10}$
-1	$\frac{1}{2}$	$\tau_{\Xi^0} = (2.90 \pm 0.09) \cdot 10^{-10} \quad \tau_{\Xi^-} = (1.639 \pm 0.015) \cdot 10^{-10}$
0	*0*	$\tau^0_\Lambda = (2.631 \pm 0.020) \cdot 10^{-10}$

Note: the neutral spin 1 meson ρ^0 has a width of about 125 MeV which corresponds to a lifetime $\tau_\rho = 5.266 \cdot 10^{-25}$ sec, which is a typical strong decay lifetime. In fact the decay $\rho^0 \to \pi^+\pi^-$ is a flavor conserving strong decay and no weak or electromagnetic transition is needed to mediate it (see Fig. A.14). Main decay modes (see also Figs. A.13, A.15 and A.16) [q] denotes the inactive spectator quark:

Decay			*Branching fraction*	*transition*	*diagram Fig. A.15*
p	$\to$	stable	–	–	–
n	$\to$	$pe^-\bar{\nu}_e$	*100 %*	β_{du}	*a)*
n	$\to$	$pe^-\bar{\nu}_e\gamma$	$Br = (3.13 \pm 0.35) \cdot 10^{-3}$	*em RC*	*a) +γ's*
Λ^0	$\to$	$n\pi^0$	*35.8 %*	$s \to d[d]$	*c), d)*
Λ^0	$\to$	$p\pi^-$	*63.9 %*	$s \to u[u]$	*e), f)*
Λ^0	$\to$	$n\gamma$	$Br = (1.75 \pm 0.15) \cdot 10^{-3}$	$s \to d$ *rad*	*b)*
Λ^0	$\to$	$pe^-\bar{\nu}_e$	$Br = (8.32 \pm 0.14) \cdot 10^{-4}$	β_{su}	*a)*
Σ^+	$\to$	$p\pi^0$	*51.57 %*	$s \to d[u]$	*c), d)*
Σ^+	$\to$	$n\pi^+$	*48.31 %*	$s \to d[u]$	*c), d)*
Σ^+	$\to$	$p\gamma$	$Br = (1.23 \pm 0.05) \cdot 10^{-3}$	$s \to d$ *rad*	*b)*
Σ^+	$\to$	$n\pi^+\gamma$	$Br = (4.5 \pm 0.5) \cdot 10^{-4}$	*em RC*	*c), d) +γ's*
Σ^+	$\to$	$\Lambda^0 e^+\nu_e$	$Br = (2.0 \pm 0.5) \cdot 10^{-5}$	β^*_{du}	
Σ^0	$\to$	$\Lambda\gamma$	*100 %*	*elmag [M_1]*	*"$q \to q\gamma$"*
Σ^-	$\to$	$n\pi^-$	99.848 ± 0.005 *%*	$s \to u[d]$	*e), f)*
Σ^-	$\to$	$ne^-\bar{\nu}_e$	$Br = (1.02 \pm 0.03) \cdot 10^{-3}$	β_{su}	*a)*
Σ^-	$\to$	$n\pi^-\gamma$	$Br = (4.6 \pm 0.6) \cdot 10^{-4}$	*em RC*	*e), f) +γ's*
Σ^-	$\to$	$\Lambda^0 e^-\bar{\nu}_e$	$Br = (5.73 \pm 0.27) \cdot 10^{-5}$	β_{du}	*a)*
Ξ^0	$\to$	$\Lambda^0\pi^0$	99.925 ± 0.012 *%*	$s \to u[s]$	*e)*
Ξ^0	$\to$	$\Lambda^0\gamma$	$Br = (1.17 \pm 0.07) \cdot 10^{-3}$	$s \to d$ *rad*	*b)*
Ξ^0	$\to$	$\Sigma^0\gamma$	$Br = (3.33 \pm 0.10) \cdot 10^{-3}$	$s \to d$ *rad*	*b)*
Ξ^0	$\to$	$\Sigma^+ e^-\bar{\nu}_e$	$Br = (2.53 \pm 0.08) \cdot 10^{-4}$	β_{su}	*a)*
Ξ^-	$\to$	$\Lambda^0\pi^-$	99.887 ± 0.035 *%*	$s \to u[s]$	*e), f)*
Ξ^-	$\to$	$\Sigma^-\gamma$	$Br = (1.27 \pm 0.23) \cdot 10^{-4}$	$s \to d$ *rad*	*b)*
Ξ^-	$\to$	$\Lambda e^-\bar{\nu}_e$	$Br = (5.36 \pm 0.31) \cdot 10^{-4}$	β_{su}	*a)*
Ξ^-	$\to$	$\Xi^0 e^-\bar{\nu}_e$	$Br < 2.3 \cdot 10^{-3}$	β_{du}	*a)*

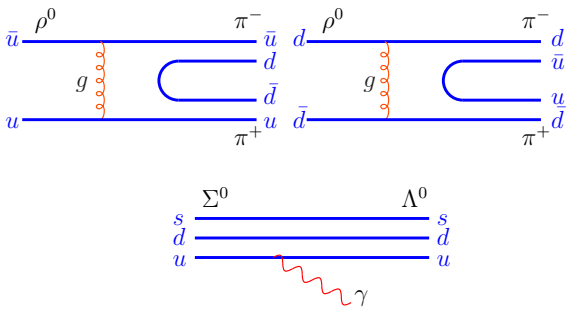


Figure A.14: Examples on non-weak flavor preserving hadron decays. Top: strong decay of the ρ–meson. Bottom: electromagnetic M_1 transition.

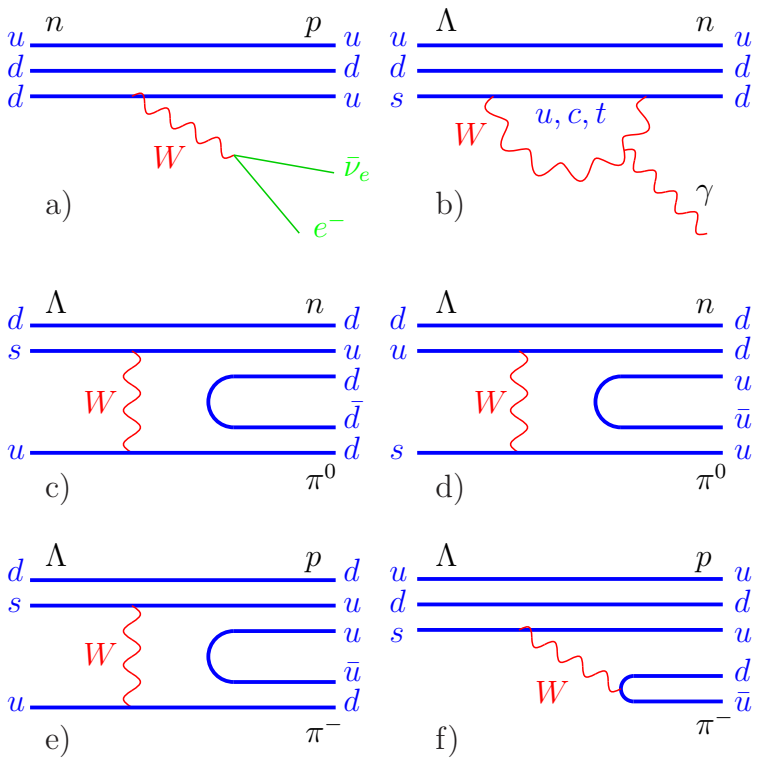


Figure A.15: Various Baryon Decay Schemes a) β–decay $[d, s \to u e^- \bar{\nu}_e]$, b) Penguin radiative decay $[s \to d\gamma]$, c), d) and e) $B \to B' + \pi$ $[s \to d]$, and f) $B \to B' + \pi$ $[s \to u]$.

Note on $\Sigma^0 \to \Lambda^0\gamma$: this is an electromagnetic transition of M_1 type (as the hadrons have no net charge) (see Fig. A.14). The amplitude of this transition is defined by

$$M(\Sigma^0 \to \Lambda^0\gamma) = \sqrt{4\pi\alpha}\, e^*_\mu(k,\lambda)\langle\Lambda^0(k_{\Sigma^0},\sigma_{\Sigma^0})|J^\mu(0)|\Sigma^0(k_{\Lambda^0},\sigma_{\Lambda^0})\rangle, \tag{A.1}$$

where $J^\mu(0)$ is the electromagnetic hadronic current, $e^*_\mu(k,\lambda)$ is a polarization vector of a photon and $\alpha = 1/137.036$ is the fine–structure constant. Due to the $M1$ transition the matrix element of the electromagnetic hadronic current is

$$\langle\Lambda^0(k_{\Sigma^0},\sigma_{\Sigma^0})|J^\mu(0)|\Sigma^0(k_{\Lambda^0},\sigma_{\Lambda^0})\rangle = -\, g_{\Sigma^0\Lambda^0}\, \bar{u}_{\Lambda^0}(k_{\Lambda^0},\sigma_{\Lambda^0})\sigma^{\mu\nu}k_\nu u_{\Sigma^0}(k_{\Sigma^0},\sigma_{\Sigma^0}), \tag{A.2}$$

where $\sigma^{\mu\nu} = (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$, $g_{\Sigma^0\Lambda^0}$ is the effective coupling constant of the transition $\Sigma^0 \to \Lambda^0$, caused by strong low–energy interactions, and $k = k_{\Sigma^0} - k_{\Lambda^0}$, and $\bar{u}_{\Lambda^0}(k_{\Lambda^0},\sigma_{\Lambda^0})$ and $u_{\Sigma^0}(k_{\Sigma^0},\sigma_{\Sigma^0})$ are the bispinors of the Λ^0 and Σ^0 hyperons, normalized by $\bar{u}_Y(k_Y,\sigma_Y)u_Y(k_Y,\sigma_Y) = 2m_Y$ for $Y = \Sigma^0$ and Λ^0.

The width of the $\Sigma^0 \to \Lambda^0\gamma$ decay for unpolarized baryons is proportional to

$$\begin{aligned}
&\frac{1}{2}\sum_{\sigma_{\Lambda^0}=\pm1/2}\sum_{\sigma_{\Sigma^0}=\pm1/2}\sum_{\lambda=\pm1}|M(\Sigma^0 \to \Lambda^0\gamma)|^2 = \\
&= 2\pi\alpha g^2_{\Sigma^0\Lambda^0}\sum_{\lambda=\pm1} e^*_\mu(k,\lambda)e_\alpha(k,\lambda)k_\nu k_\beta\,\mathrm{tr}\{(m_{\Sigma^0}+\hat{k}_{\Sigma^0})\sigma^{\mu\nu}(m_{\Lambda^0}+\hat{k}_{\Lambda^0})\sigma^{\alpha\beta}\} = \\
&= 2\pi\alpha g^2_{\Sigma^0\Lambda^0}\sum_{\lambda=\pm1} e^*_\mu(k,\lambda)e_\alpha(k,\lambda)k_\nu k_\beta\,\mathrm{tr}\{\hat{k}_{\Sigma^0}\sigma^{\mu\nu}\hat{k}_{\Lambda^0}\sigma^{\alpha\beta}\} = 32\pi\alpha\, g^2_{\Sigma^0\Lambda^0}(k\cdot k_{\Sigma^0})(k\cdot k_{\Lambda^0}) = \\
&= 32\pi\alpha\, g^2_{\Sigma^0\Lambda^0}\, m^2_{\Sigma^0}\omega_0^2,
\end{aligned} \tag{A.3}$$

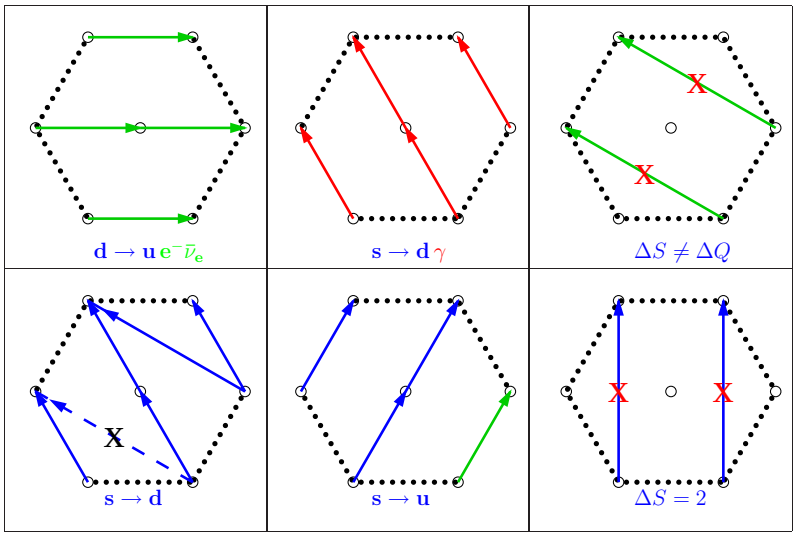


Figure A.16: Types of Transitions in the Baryon Octet. Directions are driven by the quark mass hierarchy $m_s > m_d > m_u$. Only s–decay transitions have enough phase space allowing to produce real pions [blue arrows] (Fig. A.15 diagrams c) to f)). In $d \to u$ generally only semileptonic decays may occur [green arrows] (diagram a)). Flavor changing electromagnetic transition must be second order weak [red arrows] (diagram b)). The flavor neutral $\Sigma \to \Lambda\gamma$ in a pure electromagnetic M_1 (uncharged particles must couple via magnetic moment) decay.

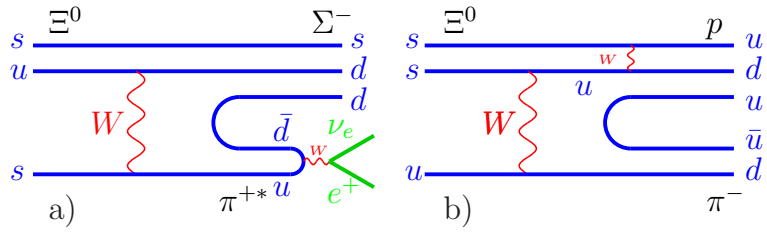


Figure A.17: "Forbidden" [=not allowed at $O(G_F^2)$] Baryon Decays (see text): a) $\Delta S = \Delta Q$ violating, b) $\Delta S = 2$ decay. Being suppressed by an additional factor $G_F^2 \sim 10^{-10}$ these modes in fact are not observable by present standards.

where $\omega_0 = (m^2_{\Sigma^0} - m^2_{\Lambda^0})/2m_{\Lambda^0} = 74.5\,\mathrm{MeV}$ *is the photon energy in the rest frame of the* Σ^0*–hyperon.*

The width of the $\Sigma^0 \to \Lambda^0\gamma$ *decay is equal to*

$$\begin{aligned}\Gamma(\Sigma^0 \to \Lambda^0\gamma) &= \frac{\alpha}{\pi}\frac{g^2_{\Sigma^0\Lambda^0}}{m_{\Sigma^0}}\int \delta^{(4)}(k_{\Lambda^0} + k - k_{\Sigma^0})\,(k\cdot k_{\Sigma^0})^2\,\frac{d^3k_{\Lambda^0}}{E_{\Lambda^0}(\vec{k}_{\Lambda^0})}\frac{d^3k}{\omega(\vec{k})} = \\ &= 4\alpha g^2_{\Sigma^0\Lambda^0}\,\omega_0^3. \qquad \text{(A.4)}\end{aligned}$$

Using the experimental value for the width of the $\Sigma^0 \to \Lambda^0\gamma$ *decay* $\Gamma_{\exp}(\Sigma^0 \to \Lambda^0\gamma) = (8.9 \pm 0.8) \times 10^{-3}\,\mathrm{MeV}$ *we get* $g_{\Sigma^0\Lambda^0} = (8.6 \pm 0.4) \times 10^{-4}\,\mathrm{MeV}^{-1}$*. The effective coupling constant* $g_{\Sigma^0\Lambda^0}$ *can be expressed in terms of the transition magnetic moment* $g_{\Sigma^0\Lambda^0} = \mu_{\Sigma^0\Lambda^0}/2m_N$*, where* $|\mu_{\Sigma^0\Lambda^0}| = 1.61 \pm 0.08$*. Data from PDG.* ✌

② In the Goldstone model of spontaneous symmetry breaking two real fields $\hat{\varphi} = \binom{\varphi_1}{\varphi_2}$ interact by the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\hat{\varphi})^2 + \frac{\mu^2}{2}\hat{\varphi}^2 - \frac{\lambda}{4!}\hat{\varphi}^4$. For $\mu^2 > 0$ let the ground state be given

by $\hat{\varphi}_0 = \binom{0}{v}, v > 0$. Show that the curvature of the potential $\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_k}$ at the ground state $\hat{\varphi}_0$ represents the mass matrix of the system.

✎

The claim is

$$M_{ik} = \left.\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_k}\right|_{\hat{\varphi}=\hat{\varphi}_0} \ .$$

Note that in the symmetric phase V is already diagonal in the fields, depending on $\hat{\varphi}^2$ only. Also we have $\hat{\varphi}_0^2 = v^2$. Therefore

$$\frac{\partial V}{\partial\varphi_i} = \frac{\partial V}{\partial\hat{\varphi}^2}\,\frac{\partial\hat{\varphi}^2}{\partial\varphi_i} = (\frac{\lambda}{6}\,\hat{\varphi}^2 - \mu^2)\,\varphi_i$$

which must vanish at the minimum and this means that the first factor must vanish, because one of the fields φ_i is a non-vanishing constant v at the minimum. Thus μ^2 must be equal to $\lambda v^2/6$. On the other hand

$$M_{ik} = \left.\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_k}\right|_{\hat{\varphi}=\hat{\varphi}_0} = 2\,\frac{\lambda}{6}\varphi_k\varphi_i\Big|_{\varphi_1=0,\varphi_2=v} = \begin{cases} \frac{\lambda v^2}{3} = m_2^2 & ; \quad \varphi_i = \varphi_k = \varphi_2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

which proves the assertion. ✌

③ Write down the Feynman rules for the Goldstone model (see Fig. 2.2).

✎

We denote the shifted fields by φ_1 and φ_2, where, after the shift, both fields have zero vacuum expectation values as expected for normal quantized fields. The vacuum action density must have been subtracted i.e. $\langle 0|\mathcal{L}(x)|0\rangle = 0$. The tadpole constant c_1 is zero at tree level, but must be adjusted to guarantee $\langle 0|\varphi_2(x)|0\rangle = 0$ in higher orders:

$\langle 0|\varphi_2(x)|0\rangle =$ $+$ $= 0$.

Note that for $\mu^2 > 0$ a "quantization" of the fields for before performing a shift on the classical level, would lead to an ill defined perturbative expansion (negative mass square fields [tachyon fields]). The Feynman rules are depicted in Fig. A.18. Note that factorial normalization factors $\frac{1}{n!}$ coming from permutations of identical fields in monomials φ_i^n are not included in the Feynman rule couplings, since diagrams do not differ by taking different permutations, and hence are counted once only.

✌

④ Calculate the Noether currents for the Goldstone model, with respect to the unbroken global symmetry. Discuss how it is possible that a current couples to the vacuum, i.e. why it can bee that for a conserved current we have : $j_i^\mu(x) : |0\rangle \neq 0$.

✎

Under a variation of the field the Lagrangian varies by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\hat{\varphi}}\,\delta\partial_\mu\hat{\varphi} + \frac{\partial\mathcal{L}}{\partial\hat{\varphi}}\,\delta\hat{\varphi} \ ,$$

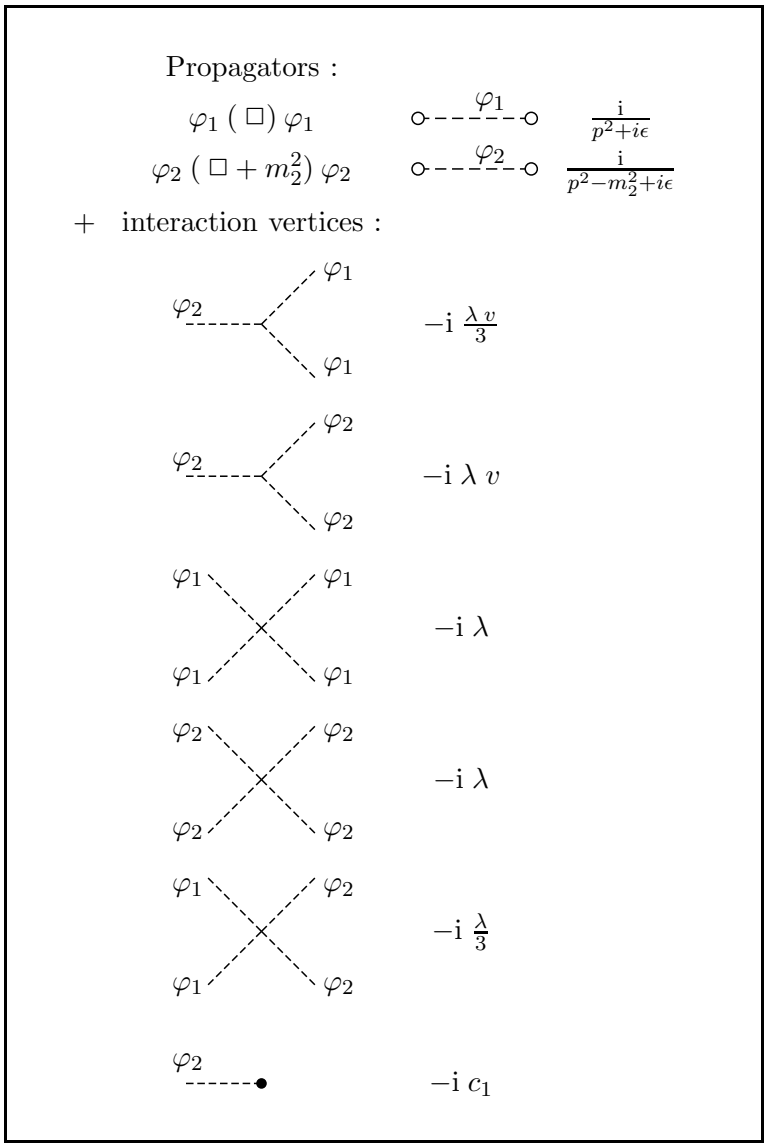


Figure A.18: Feynman rules for the $O(2)$ Goldstone model

and by the Euler-Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \hat{\varphi}} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \hat{\varphi}} \ ,$$

and for global variations $\delta \partial_\mu \hat{\varphi} = \partial_\mu \delta \hat{\varphi}$ *we obtain*

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \partial_\mu \hat{\varphi}} \, \partial_\mu \delta \hat{\varphi} + \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \hat{\varphi}} \right) \delta \hat{\varphi} \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \hat{\varphi}} \, \delta \hat{\varphi} \right) = \partial_\mu \left(\partial^\mu \hat{\varphi} \, \delta \hat{\varphi} \right) \ . \end{aligned}$$

An infinitesimal $\hat{\varphi}$ *rotation (with* $\cos\theta \to 1, \ \sin thetato\theta$ *(*$\theta \ll 1$*) we may write*

$$\delta \hat{\varphi} = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \hat{\varphi} = \theta \begin{pmatrix} \varphi_2 \\ -\varphi_1 \end{pmatrix}$$

and thus

$$\delta\mathcal{L} = \theta\,\partial_\mu\, j^\mu$$

with

$$j^\mu = \left(\begin{array}{cc} \partial^\mu\varphi_1, & \partial^\mu\varphi_2 \end{array}\right)\left(\begin{array}{c} \varphi_2 \\ -\varphi_1 \end{array}\right) = (\partial^\mu\varphi_1)\,\varphi_2 - (\partial^\mu\varphi_2)\,\varphi_1 \equiv \varphi_2\overset{\leftrightarrow}{\partial^\mu}\varphi_1$$

the requested Noether current. Since $\mathcal{L}$ is invariant under rotations we actually have a conserved current

$$\partial_\mu\, j^\mu = 0\ .$$

Spontaneous symmetry breaking (for $\mu^2 > 0$) of the $O(2)$ field rotations requires us to adjust the fields $\varphi_2 = \varphi_2' + v$ while $\varphi_1 = \varphi_1'$, which affects the current in a very interesting way, namely it acquires a linear term:

$$j^\mu = \varphi_2'\overset{\leftrightarrow}{\partial^\mu}\varphi_1' + v\,\partial^\mu\varphi_1$$

The quantized current

$$j^\mu =: \varphi_2'\overset{\leftrightarrow}{\partial^\mu}\varphi_1' : + v\,\partial^\mu\varphi_1$$

obviously creates φ_1 quanta, i.e. Nambu-Goldstone bosons, from the ground state

$$: j^\mu : |0\rangle = \mathrm{i}\,v\int\frac{\mathrm{d}^3p\,p^\mu}{2|\vec{p}|\,(2\pi)^3}\,\mathrm{e}^{\mathrm{i}px}\,|\vec{p}\rangle_1$$

i.e. $: j^\mu : |0\rangle \neq 0$, while in the symmetric phase ($\mu^2 < 0$) , without a linear term in the current, we would have

$$: j^\mu : |0\rangle = 0\ .$$

Actually, for $Q|0\rangle = \int \mathrm{d}^3x : j^0 : |0\rangle$ we obtain

$$\begin{aligned} Q|0\rangle &= \mathrm{i}\,v\int\frac{\mathrm{d}^3p\,|\vec{p}|}{2|\vec{p}|\,(2\pi)^3}\,\mathrm{e}^{\mathrm{i}|\vec{p}|x^0}\int\mathrm{d}^3x\mathrm{e}^{\mathrm{i}\vec{p}\vec{x}}\,|\vec{p}\rangle_1 \\ &= \mathrm{i}\,v\,\frac{1}{2}\,|\vec{0}\rangle_1 \end{aligned}$$

which is a projection to a one Nambu-Goldstone boson state of zero momentum. ✌

A.3 Exercises: Section 3.2

① In the SM the leptonic hypercharge current has the form

$$j_Y^\mu = \frac{1}{2}\left(\bar{\ell}_L\gamma^\mu\ell_L + \bar{\nu}_{\ell L}\gamma^\mu\nu_{\ell L}\right) + \bar{\ell}_R\gamma^\mu\ell_R\ .$$

Write it in terms of vector plus axialvector contributions.

✎
✌

② Calculate the trace and the Feynman integral (3.5) and verify (3.8).

✎
✌

A.4 Exercises: Section 3.3

① The left (L)- and right (R)-handed fields are given by

$$\psi_L = \frac{1-\gamma_5}{2}\psi \ , \ \ \psi_R = \frac{1+\gamma_5}{2}\psi \ .$$

Show that a mass term is given by

$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \ .$$

Discuss helicity mixing for the terms

$$S = \bar{\psi}\psi \ , \ \ P = \bar{\psi}\gamma_5\psi \ , \ \ V^\mu = \bar{\psi}\gamma^\mu\psi \ , \ \ A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \ , \ \ T^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi$$

✎
The chiral fields ψ_R and ψ_L are obtained from a Dirac field by chiral projection $\psi_R = \Pi_+\psi$ and $\psi_L = \Pi_-\psi$ (see footnote p. 42). For the adjoint field $\bar{\psi} \doteq \psi^+\gamma^0$ due to $\gamma^0, \gamma_5 = 0$ we have $\bar{\psi}_R = \bar{\psi}\Pi_-$ and $\bar{\psi}_L = \bar{\psi}\Pi_+$. Therefore, $\bar{\psi}_R\psi_R = 0$ and $\bar{\psi}_L\psi_L = 0$. Furthermore, the $\Pi_\pm$ are eigenoperators of γ_5: $\gamma_5\Pi_\pm = \pm\Pi_\pm$ and $\Pi_\pm\gamma^\mu = \gamma^\mu\Pi_\mp$. Using these properties and the fact that the $\Pi_\pm$ are hermitian projection operators one easily works out

$$\bar{\psi}\Gamma_i\psi = \left(\bar{\psi}_R + \bar{\psi}_L\right)\, \Gamma_i\, \left(\psi_R + \psi_L\right)$$

with Γ_i appropriate for $i = S, P, V^\mu, A^\mu$ and $T^{\mu\nu}$ given above. The results read:

$$\begin{aligned}
S &= \left(\bar{\psi}_R + \bar{\psi}_L\right)\, \left(\psi_R + \psi_L\right) = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R \ , \\
P &= \left(\bar{\psi}_R + \bar{\psi}_L\right)\, \gamma_5\, \left(\psi_R + \psi_L\right) = -\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R \ , \\
V^\mu &= \left(\bar{\psi}_R + \bar{\psi}_L\right)\, \gamma^\mu\, \left(\psi_R + \psi_L\right) = \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_L \ , \\
A^\mu &= \left(\bar{\psi}_R + \bar{\psi}_L\right)\, \gamma^\mu\gamma_5\, \left(\psi_R + \psi_L\right) = \bar{\psi}_R\gamma^\mu\psi_R - \bar{\psi}_L\gamma^\mu\psi_L \ , \\
T^{\mu\nu} &= \left(\bar{\psi}_R + \bar{\psi}_L\right)\, \sigma^{\mu\nu}\, \left(\psi_R + \psi_L\right) = \bar{\psi}_R\sigma^{\mu\nu}\psi_L + \bar{\psi}_L\sigma^{\mu\nu}\psi_R \ .
\end{aligned}$$

Thus S, P and T are helicity flip operators, while V and A are helicity preserving; all gauge interactions are necessarily of the latter type. ✌

② Consider two free fields ψ_1 and ψ_2 with different masses m_1 and m_2. Calculate the divergences of the vector and axial vector currents

$$V^\mu = \bar{\psi}_1\gamma^\mu\psi_2 \quad \text{and} \quad A^\mu = \bar{\psi}_1\gamma^\mu\gamma_5\psi_2$$

$$\partial_\mu V^\mu = ? \qquad \partial_\mu A^\mu = ?$$

What is the consequence of this result for the weak currents ? (Hint: Use the Dirac equations). Comment on the hadronic charged weak current responsible for β–decay by comparing the nucleon current with the quark current.

✎

From the Dirac equation $(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\psi = 0$, *we have* $\gamma^\mu\partial_\mu\,\psi = -\mathrm{i}m\,\psi$ *and taking the adjoint, we have* $\partial_\mu\psi^+\gamma^{\mu+}\gamma^0 = \mathrm{i}m\,\psi^+\gamma^0$ *or, with* $\gamma^0\gamma^{\mu+}\gamma^0 = \gamma^\mu$ *implying* $\gamma^{\mu+}\gamma^0 = \gamma^0\gamma^\mu$ *we get* $\partial_\mu\bar\psi\gamma^\mu = \mathrm{i}m\,\bar\psi$. *Together with* $\{\gamma^\mu,\gamma^\nu\} = 0$ *we easily find*

$$\partial_\mu V^\mu = \mathrm{i}\,(m_1 - m_2)\,\bar\psi_1\psi_2\ ;\ \ \partial_\mu A^\mu = \mathrm{i}\,(m_1 + m_2)\,\bar\psi_1\gamma_5\psi_2$$

and the vector current is conserved only if $\psi_1 = \psi_2$ *(flavor neutral) the axial current is conserved only in the chiral limit of massless fermions.*
The charged weak (V-A) hadronic nucleon current which seems to mediate neutron β*–decay* $n \to p + e^- + \bar\nu_e$ *for the hadrons is (1st family only, CKM mixing discarded))*

$$J^\mu_{\text{nucleon}} = \bar\psi_p\,\gamma^\mu\,(1-\gamma_5)\,\psi_n$$

(thats what it was thought to be before the discovery of the quarks). The true SM weak hadronic current is

$$J^\mu_{\text{hadronic}} = \bar\psi_u\,\gamma^\mu\,(1-\gamma_5)\,\psi_d\ .$$

Note that for the vector part

$$\partial_\mu V^\mu_{\text{nucleon}} = \mathrm{i}\,(m_p - m_n)\,\bar\psi_p\psi_n$$

and

$$\partial_\mu V^\mu_{\text{hadronic}} = \mathrm{i}\,(m_u - m_d)\,\bar\psi_u\psi_d$$

the symmetry braking is of the same order of magnitude: $m_n - m_p \simeq 1.23\ MeV$ *vs.* $m_d - m_u \simeq 1.95\ MeV$ *(using* $m_u \sim 2.35\ MeV, m_d \sim 4.25\ MeV$*). In contrast for the axial current*

$$\partial_\mu A^\mu_{\text{nucleon}} = \mathrm{i}\,(m_p + m_n)\,\bar\psi_p\gamma_5\psi_n$$

compares with

$$\partial_\mu A^\mu_{\text{hadronic}} = \mathrm{i}\,(m_u + m_d)\,\bar\psi_u\gamma_5\psi_d$$

a mismatch by a factor about 1434. The plain absence axial current conservation on the nucleon level lead Nambu in 1960 to the discovery of spontaneous symmetry breaking and the interpretation of the pions as quasi Nambu-Goldstone bosons. ✌

③ Find the form of the quark currents which couple to W and Z and the photon. As a starting point use the $SU(2)_L \otimes U(1)_Y$ covariant derivative form

$$\begin{aligned}\mathcal{L}_q \;=\;& \bar b_R\mathrm{i}\gamma^\mu\left(\partial_\mu + \mathrm{i}\frac{1}{3}g'B_\mu\right)b_R + \bar t_R\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}\frac{2}{3}g'B_\mu\right)t_R \\ & + \bar L_q\mathrm{i}\gamma^\mu\left(\partial_\mu - \mathrm{i}\frac{1}{3}\frac{g'}{2}B_\mu - \mathrm{i}g\frac{\tau_a}{2}W_{\mu a}\right)L_q\end{aligned}$$

Check the correctness of the covariant derivatives with the quantum numbers assigned to the quarks. We have denoted by $L_q = \binom{t}{b}_L$ the left–handed doublet. t_R and b_R are the corresponding right–handed singlets. Quark mixing should be ignored.

✎

Consider the charged current first. In terms of the real Cartesian fields W_i $(i = 1, 2)$ one obtains the charged fields which correspond to the ladder operators of $SU(2)_L$: $W^{\pm} = \frac{1}{\sqrt{2}}(W_1 \mp \mathrm{i}\, W_2)$ such that

$$\frac{g}{2}\sum_{a=1,2}\tau_a W_{\mu a} = \frac{g}{\sqrt{2}}\begin{pmatrix} 0 & W^- \\ W^+ & 0 \end{pmatrix} ,$$

and hence

$$\mathcal{L}_{\mathrm{CC}} = \frac{g}{\sqrt{2}}\left(\bar{t}_L\gamma^\mu b_L\, W^-_\mu + \bar{b}_L\gamma^\mu t_L\, W^+_\mu\right) .$$

The neutral gauge fields mix via a rotation

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} c_w & -s_w \\ s_w & c_w \end{pmatrix}\begin{pmatrix} W_3 \\ B \end{pmatrix} \quad ; \quad \begin{pmatrix} W_3 \\ B \end{pmatrix} = \begin{pmatrix} c_w & s_w \\ -s_w & c_w \end{pmatrix}\begin{pmatrix} Z \\ A \end{pmatrix}$$

with $c_w = \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$ and $s_w = \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$. $\sin^2\theta_W$ is called weak mixing parameter. Note that $\sqrt{g^2+g'^2} = g/c_w$ and $\sqrt{g^2+g'^2}s_w c_w = g s_w = e$ the electric charge unit. For the neutral current we first have (note hypercharges are $Y = 4/3, -2/3, 1/3, 1/3$ for t_R, b_R, t_L, b_L and $Y = 2\,(Q - T_3)$)

$$\begin{aligned}
\mathcal{L}_{\mathrm{NC+em}} &= \frac{1}{2}\bar{t}_R\gamma^\mu\left[2\,Q_t\,g'B_\mu\right]t_R + \frac{1}{2}\bar{b}_R\gamma^\mu\left[2\,Q_b\,g'B_\mu\right]b_R \\
&\quad + \frac{1}{2}\bar{t}_L\gamma^\mu\left[(2\,Q_t - 1)\,g'B_\mu + gW_{\mu 3}\right]t_L + \frac{1}{2}\bar{b}_L\gamma^\mu\left[(2\,Q_b + 1)\,g'B_\mu - gW_{\mu 3}\right]b_L \\
&= Q_t\,\bar{t}\gamma^\mu g'B_\mu t + Q_b\,\bar{b}\gamma^\mu g'B_\mu b - \frac{1}{2}\bar{t}_L\gamma^\mu\left[g'B_\mu - gW_{\mu 3}\right]t_L + \frac{1}{2}\bar{b}_L\gamma^\mu\left[g'B_\mu - gW_{\mu 3}\right]
\end{aligned}$$

and with $gW_3 - g'B = g/c_w\, Z$ and $g'B = -s_w^2\, g/c_w\, Z + s_w c_w\, q/c_w\, A$ we find

$$\mathcal{L}_{\mathrm{NC+em}} = e\sum_{q=t,b} Q_q\bar{q}\gamma^\mu q A_\mu + \frac{g}{c_w}\left[\sum_{q=t,b} -Q_q\, s_w^2\,\bar{q}\gamma^\mu q Z_\mu + \sum_{q=t,b} T_q\,\bar{q}\gamma^\mu\,\frac{(1-\gamma_5)}{2}\,q Z_\mu\right] .$$

Thus

$$\mathcal{L}^{q,\mathrm{neutral}}_{\mathrm{matter,\ int}} = e j^\mu_{\mathrm{em}} A_\mu + \frac{g}{\cos\theta_w}\, J^\mu_Z Z_\mu$$

where

$$j^\mu_{\mathrm{em,\ had}} = \sum_q Q_q\bar{\psi}_q\gamma^\mu\psi_q$$

is the hadronic electromagnetic current and

$$J^\mu_Z = J^\mu_3 - \sin^2\theta_W\, j^\mu_{\mathrm{em}}$$

weak neutral current. ✌

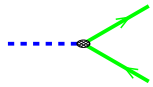
④ Give numerical values for the Higgs couplings to the various particles. Calculate the decay width of the Higgs into a fermion pair and estimate the branching ratios for the different flavors. Discuss the dependence of the Higgs mass for the range $0 < m_H < 2M_W$. Compare the Higgs width with the width of known particles.

✎

In the unitary gauge the SM Yukawa Lagrangian takes the "trivial" from

$$\mathcal{L}_{\text{Yukawa}} = -\sum_f m_f\,\bar{\psi}_f\psi_f\,\left(1+\frac{H}{v}\right)$$

where m_f is the fermion mass and v the Higgs vacuum expectation value (with $\sqrt{2}\,G_F = 1/v^2$). The Higgs decay matrix element is given by



$$= \mathcal{M}[H \to f(q_1,r_1)\bar{f}(q_2,r_2)] \quad = \quad \frac{m_f}{v}\,\bar{u}(q_1,r_1)\,v(q_2,r_2)\ .$$

The transition rate is then

$$\sum_{r_1,r_2}|\mathcal{M}[H \to f(q_1,r_1)\bar{f}(q_2,r_2)]|^2 \quad = \quad \sqrt{2}G_F m_f^2 \sum_{r_1,r_2}\bar{u}(q_1,r_1)_\alpha\,v(q_2,r_2)_\alpha\,\bar{v}(q_2,r_2)_\beta\,u(q_1,r_1)_\beta$$
$$= \sqrt{2}G_F m_f^2\,(\not{q}_1 + m_f)_{\alpha\beta}\,(\not{q}_2 - m_f)_{\beta\alpha} = \sqrt{2}G_F m_f^2\,4\,((q_1\cdot q_2) - m_f^2)$$

The decay rate is

$$\Gamma_{Hf\bar{f}} = \frac{1}{16\pi m_H}\sqrt{1-\frac{4m_f^2}{m_H^2}}\ \sum|T_{12}|^2$$

and with $q_1q_2 = \frac{1}{2}(m_H^2 - 2m_f^2)$ we have $4\,(q_1q_2 - m_f^2) = 2\,(m_H^2 - 4m_f^2)$ and hence

$$\Gamma_{Hf\bar{f}} = \frac{\sqrt{2}G_F m_f^2}{8\pi}\left(1-\frac{4m_f^2}{m_H^2}\right)^{\frac{3}{2}}\ m_H$$

The direct Higgs mass limit from LEP is $m_H \gtrsim 114$ GeV. If $m_H < 2M_W \sim 160$ GeV the Higgs decays into all SM leptons and quarks, except for the top (as $m_t \sim 173$ GeV), preferably into the heaviest particle accessible, which then is the b quark.

The couplings are given by the lepton and quark masses in units of $v = 246.221(1)$ GeV: thus the strengths are ordered according to the masses (in GeV) $m_e = 0.000511, m_u = 0.003, m_d = 0.006, m_s = 0.105, m_\mu = 0.1057, m_c = 1.25, m_\tau = 1.777, m_b = 4.25$. We have $\sum N_{cf}m_f^2 \simeq 62.08$ GeV^2 [with threshold corrections 61.66 GeV^2] and with $G_F = G_\mu = 1.16637(1)\cdot 10^{-5}$ GeV^{-2} we obtain, neglecting the threshold factors which are very close to unity for all light fermions,

$$\Gamma_{Hf\bar{f}}/m_H = 4.05\cdot 10^{-5}$$

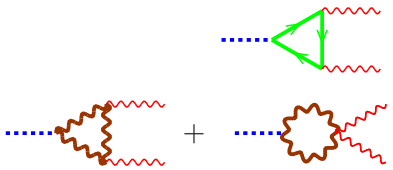
or with $m_H = 120[160]$ GeV a width $\Gamma_{Hf\bar{f}} \simeq 4.9[6.5]$ MeV which is much smaller than e.g. the ρ meson width $\Gamma_\rho \simeq 145$ MeV. The branching fractions are $\text{Br}_e = 4.2\cdot 10^{-9}, \text{Br}_u = 4.4\cdot 10^{-7}, \text{Br}_d = 1.8\cdot 10^{-6}, \text{Br}_s = 5.4\cdot 10^{-4}, \text{Br}_\mu = 1.8\cdot 10^{-4}, \text{Br}_c = 0.0760, \text{Br}_\tau = 0.0511, \text{Br}_b = 0.872$.

Addendum: a relatively light Higgs of mass below $2M_W$ mainly decays into 2 b hadron jets. At the LHC such decays suffer from huge background which is expected to bury all hadronic Higgs

decay events. The most promising decay channel for the Higgs discovery at a hadron collider is $H \to \gamma\gamma$, which has a unique signature but a very low rate. The decay rate is

$$\Gamma_{H\gamma\gamma} = \frac{\sqrt{2} G_F\, \alpha^2\, m_H^3}{256\, \pi^3} \left| \sum_f N_{cf} e_f^2\, \mathcal{A}_{f-\mathrm{loop}}^{H\gamma\gamma} + \mathcal{A}_{W-\mathrm{loop}}^{H\gamma\gamma} \right|^2 \quad ,$$

with 1–loop amplitudes



$$: \quad \mathcal{A}_{f-\mathrm{loop}}^{H\gamma\gamma} = 2x\,[1 + (1-x)\, f(x)] \quad ,$$

$$: \quad \mathcal{A}_{W-\mathrm{loop}}^{H\gamma\gamma} = -x\,[3 + 2/x + 3\,(2-x)\, f(x)] \quad ,$$

where $x = 4m/m_H^2$ with m the mass of the particle in the loop, $m = m_f$ or M_W, and

$$f(x = 1/\tau) = \begin{cases} \arcsin^2 \sqrt{\tau} & \text{for } \tau \leq 1 \\ -\frac{1}{4} \left[\ln \frac{\sqrt{\tau} + \sqrt{\tau - 1}}{\sqrt{\tau} - \sqrt{\tau - 1}} - \mathrm{i}\, \pi \right]^2 & \text{for } \tau > 1 \ . \end{cases}$$

For $m_H = 120[160]$ GeV one obtains $\Gamma_{H\gamma\gamma} = 1.7 \cdot 10^{-4}[5.1 \cdot 10^{-2}]$ MeV . The branching fraction is $2.7 \cdot 10^{-5}[7.8 \cdot 10^{-3}]$ only, which compares to the τ–pair branching fraction of 5.1% . Note that $H\gamma\gamma$ branching fraction increases strongly with the Higgs mass and close to the W–pair threshold reaches almost 1% . ✌

A.5 Exercises: Section 3.4

① Write the PCAC relation in terms of physical parameters (masses and decay constants of hadrons) and try to understand the Goldberger-Treiman (GT) relation [Phys. Rev. 110 (1958) 1178, ibid. 111 (1958) 354]

$$g_A\, (m_p + m_n)/2 = F_\pi\, g_{\pi pn}$$

where m_p and m_n are the proton and neutron mass, respectively, and $g_{\pi pn}$ the effective pion nucleon coupling. The other couplings show up in the matrix elements for neutron β–decay

$$\mathcal{M}(n \to p + e^- + \bar{\nu}_e) = \frac{G_F}{\sqrt{2}} \left[\bar{u}_p\, (f_V\, \gamma_\lambda - g_A \gamma_\lambda \gamma_5)\, u_n\, L^\lambda \right]$$

and π–decay

$$\mathcal{M}(\pi^- \to \mu^- + \bar{\nu}_\mu) = \frac{G_F}{\sqrt{2}} \left(-\mathrm{i}\, F_\pi p_\lambda\, L^\lambda \right) \ .$$

By $L^\lambda = \bar{u}_\ell \gamma^\lambda\, (1 - \gamma_5)\, v_{\nu_\ell}$ we denoted the leptonic "current", p is the pion momentum and G_F is the Fermi constant.

✎

For the hadron states we use the notation introduced in Sect. 1.5. The GT relation attempts to relate the nucleon current matrix element $\langle p|J^{+\lambda}(0)|n\rangle$ to the hadronic pion decay matrix element $\langle 0|J^{+\lambda}(0)|\pi\rangle$: thus

$$\bar{u}_p\, (f_V \gamma^\lambda - g_A \gamma^\lambda \gamma_5)\, u_n \quad \Leftrightarrow \quad -\mathrm{i}\, F_\pi\, p^\lambda$$

with the pion decay constant F_π. The vector and axialvector couplings f_V and g_A may be taken as separate parameters here. We first consider the divergence of the weak nucleon current

$$\begin{aligned}\partial_\lambda\,\bar\psi_p\,(f_V\gamma^\lambda - g_V\gamma^\lambda\gamma_5)\,\psi_n &= +\mathrm{i}\,f_V\,(m_p-m_n)\,\bar\psi_p\psi_n - \mathrm{i}\,g_A\,(m_p+m_n)\,\bar\psi_p\gamma_5\psi_n \\ &\simeq -\mathrm{i}\,g_A\,(m_p+m_n)\,\bar\psi_p\gamma_5\psi_n \quad \text{in the isospin limit} \quad m_n=m_p=m_N\ ,\end{aligned}$$

which is dominated by the divergence of the axial current, and which we may compare with the PCAC relation[47]

$$\partial_\lambda J_5^{+\lambda}(x) = F_\pi m_\pi^2\,\varphi^+(x)$$

which relates the divergence of the axial nucleon current

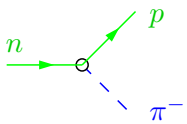
$$J_5^{+\lambda}(x) = \frac{1}{\sqrt{2}}\,\bar\psi_p\,\gamma^\lambda\gamma_5\,\psi_n(x)$$

to the pion field $\varphi^+(x)$. The latter is a composite field with the appropriate quantum numbers as an interpolating field for the pion state. In the spirit of the LSZ reduction formalism we may reduce the pion field by relating it to its source $j_5^{\pi^+}$ [this means amputation of the pion leg] via

$$j_5^{\pi^+}(x) \doteq \left(\Box + m_\pi^2\right)\,\varphi^+(x)\ .$$

The matrix element (the factor $\sqrt{2}$ is a convention)

$$\langle p(p')|j_5^{\pi^+}(0)|n(p)\rangle = \mathrm{i}\,\sqrt{2}\,g(q^2)\,\bar u_p(p')\,\gamma_5\,u_n(p)$$



with $q = p - p'$ defines a pion nucleon form factor, where $g_{\pi pn} = g(m_\pi^2)$ is the experimentally determined effective coupling [see e.g. Arndt et al., Phys.Rev.C74:045205,2006]

$$\frac{g_{\pi pn}^2}{4\pi} \simeq 13.76 \pm 0.01\ .$$

Alternatively, applying PCAC with $\partial_\mu \to \mathrm{i}q_\mu$, we may write

$$\begin{aligned}\langle p(p')|j_5^{\pi^+}(0)|n(p)\rangle &= (m_\pi^2 - q^2)\,\langle p(p')|\varphi^+(0)|n(p)\rangle \\ &= \mathrm{i}\,q_\mu\,\frac{(m_\pi^2-q^2)}{F_\pi m_\pi^2}\langle p(p')|J_5^{+\mu}(0)|n(p)\rangle\ .\end{aligned}$$

The latter matrix element is given by

$$\langle p(p')|J_5^{+\mu}(0)|n(p)\rangle = \frac{1}{\sqrt{2}}\bar u_p(p')\,\left\{\gamma^\mu\gamma_5\,a_A(q^2) + q^\mu\,\gamma_5 h_A(q^2)\right\}\,u_n(p)\ ,$$

for the divergence

$$\langle p(p')|\partial_\mu J_5^{+\mu}(0)|n(p)\rangle = \frac{\mathrm{i}}{\sqrt{2}}\bar u_p(p')\,\left\{\not{q}\gamma_5\,a_A(q^2) + q^2\,\gamma_5 h_A(q^2)\right\}\,u_n(p)\ ,$$

[47] We use the notation introduced in Sect.3.2.2. In the PCAC relation

$$\partial^\mu A_\mu^a(x) = F_\pi m_\pi^2\,\varphi^a(x)\ .$$

we use the convention in which $F_\pi = 92.4$ MeV. Experimental papers often use $f_\pi = \sqrt{2}\,F_\pi \sim 132$ MeV. The charged fields we obtain as

$$A_\mu^+ = \left(A_\mu^1 + \mathrm{i}\,A_\mu^2\right)/\sqrt{2}\,;\ \ \varphi^+ = \left(\varphi^1 + \mathrm{i}\,\varphi^2\right)/\sqrt{2}\ .$$

such that, applying the Dirac equation,

$$\langle p(p')|\partial_\mu J_5^{+\mu}(0)|n(p)\rangle = \frac{\mathrm{i}}{\sqrt{2}}\bar{u}_p(p')\,\left\{(m_p+m_n)\,\gamma_5\,a_A(q^2)+q^2\,\gamma_5 h_A(q^2)\right\}\,u_n(p)\ .$$

thus

$$\langle p(p')|j_5^{\pi^+}(0)|n(p)\rangle \;\;=\;\; \mathrm{i}\,\frac{(m_\pi^2-q^2)}{\sqrt{2}F_\pi m_\pi^2}\,\left\{(m_p+m_n)\,g_A(q^2)+q^2\,h_A(q^2)\right\}\bar{u}_p(p')\,\gamma_5\,u_n(p)\ .$$

and, using the definition of the effective pion nucleon coupling, obtain the identity

$$\sqrt{2}\,g(q^2)=\frac{m_\pi^2-q^2}{\sqrt{2}\,F_\pi m_\pi^2}\,\left\{2m_N\,g_A(q^2)+q^2\,h_A(q^2)\right\}\ ,$$

where m_N is the nucleon mass $m_N=(m_p+m_n)/2$. Assuming form factors to have a small variation in the range $0\le q^2\le m_\pi^2$, such that we may take as an approximation

$$g_{\pi pn}=g(m_\pi^2)\simeq g(0)\ ,\ \text{and}\ g_A=g_A(0)\ ,$$

one obtains

$$\boxed{F_\pi\,g_{\pi pn}=m_N\,g_A}$$

which is the Goldberger-Treiman relation. Numerically we have $g_A/g_V=-1.2695\pm0.0029$, $g_{\pi pn}=13.150\pm0.005$, $F_\pi=92.21\pm0.20$ MeV [$f_\pi=130.4\pm0.04\pm0.2$ PDG 2008] and $m_N=(m_p+m_n)/2=938.92$ MeV which yields $F_\pi\,g_{\pi pn}(m_N\,g_A)\simeq1.017$. Thus the GT relation is satisfied at the 2% level, quite good as a leading approximation.

In fact the smoothness assumption made by Goldberger-Treiman is not evident, because the induced pseudoscalar term h_A actually may be expected to have a pole. According to Nambu's spontaneous chiral symmetry breaking scenario, where the chiral quark flavor group $SU(2)_V\otimes SU(2)_A$ is broken down to $SU(2)_V$, the proper hadronic current is the quark current

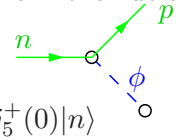
$$J_5^{+\lambda}(x)=\frac{1}{\sqrt{2}}\,\bar{\psi}_u\,\gamma^\lambda\gamma_5\,\psi_d(x)$$

which in the chiral limit of massless quarks is conserved. Indeed, for $m_\pi^2\to0$ we realize axial current conservation

$$\partial_\lambda J_5^{+\lambda}(x)=0\ ,$$

by the PCAC relation. The latter for m_π finite requires a pion pole in the nucleon matrix element:

$$\begin{aligned}\langle p|\partial_\lambda J_5^{+\lambda}(0)|n\rangle &= C\,m_\pi^2\,\langle p|\phi^+(0)|n\rangle\\ &= C\,\frac{m_\pi^2}{(p_p-p_n)^2-m_\pi^2}\,\langle p|j_5^+(0)|n\rangle\end{aligned}$$



where $\langle p|j_5^+(0)|n\rangle$ is given above. By comparison of the two evaluations of the PCAC relation we find

This requires

$$2m_N\,g_A(q^2)+q^2h_A(q^2)=0\ .$$

i.e. $h_A(q^2)$ must exhibit a pole in q^2, which corresponds to the pseudoscalar Nambu-Goldstone pion pole:

$$\lim_{q^2\to 0} h_A(q^2) = -\frac{2\,F_\pi g_{\pi pn}(0)}{q^2}\ ,$$

which in the limit $m_\pi \to 0$ again yields the GT relation, but this time based on the spontaneous chiral symmetry breaking pattern.

Models proposed for inferring the proper chiral structure of the hadronic currents and implementing the CVC and the PCAC relations are the σ model and the Nambu–Jona-Lasinio model.

The σ–model of Gell-Mann–Levy: Ψ is the nucleon doublet field $\Psi = \binom{\psi_p}{\psi_n}$, $\vec{\phi}$ is the pseudoscalar pion triplet field and σ a scalar auxiliary field:

$$\begin{aligned}\mathcal{L}^{(\sigma)} &= \bar\Psi\left(\mathrm{i}\gamma^\mu\partial_\mu - g_0\left(\sigma + \mathrm{i}\,\vec\tau\cdot\vec\phi\,\gamma_5\right)\right)\Psi + \frac{1}{2}\left(\partial_\mu\vec\phi\right)^2 + \frac{1}{2}\left(\partial_\mu\sigma\right)^2 \\ &\quad -\frac{1}{2}\mu_0^2\left(\vec\phi^{\,2}+\sigma^2\right) - \lambda_0\left(\vec\phi^{\,2}+\sigma^2 - \frac{1}{4f_0^2}\right)^2 - \frac{\mu_0^2}{2f_0}\,\sigma\end{aligned}$$

$$J_5^\mu = \frac{1}{2}\,\bar\Psi\,\gamma^\mu\gamma_5\vec\tau\,\Psi + \left(\sigma\partial^\mu\vec\phi - \vec\phi\partial^\mu\sigma\right)$$

$$\partial_\mu J_5^\mu = \frac{\mu_0^2}{2f_0}\vec\phi$$

The Nambu–Jona-Lasinio model:

$$\mathcal{L}^{(\mathrm{NJL})} = \bar\Psi_0\left(\mathrm{i}\gamma^\mu\partial_\mu - M_0\right)\Psi_0 + g_0\left[\bar\Psi_0\Psi_0\bar\Psi_0\Psi_0 - \bar\Psi_0\gamma_5\vec\tau\Psi_0\cdot\bar\Psi_0\gamma_5\vec\tau\Psi_0\right]$$

For $M_0 = 0$ the NJL Lagrangian is chirally symmetric under $SU(2)_L \otimes SU(2)_R$, the symmetry is spontaneously broken and $\langle 0|\bar\Psi_0\Psi_0|0\rangle \neq 0$. By the shift

$$\bar\Psi_0\Psi_0 = \bar\Psi\Psi + \langle 0|\bar\Psi_0\Psi_0|0\rangle$$

a nucleon field satisfying $\langle 0|\bar\Psi\Psi|0\rangle = 0$ may be achieved, whereby the shifted field acquires the nucleon mass

$$M = -g_0\,\langle 0|\bar\Psi_0\Psi_0|0\rangle$$

entirely due to the interactions of the nucleons. The spontaneous breakdown of the chiral symmetry $SU(2)_L \otimes SU(2)_R \to SU(2)_V$ implies the existence of the triplet of pseudoscalar pions (π^+, π^0, π^-). For $M_0 = 0$ they are true Nambu-Goldstone bosons. The non-vanishing pion mass requires a non-vanishing M_0 of the order of a few MeV.

Addendum:
In pion physics G–parity plays an important role. it allows to single out so called "first–class" and "second–class" currents, among the hypercharge conserving weak currents. The G–parity operator is given by (C charge conjugation, I_2 isospin generator)

$$\mathcal{G} \doteq C\,\mathrm{e}^{\mathrm{i}\,\pi\,I_2}$$

and one easily checks that currents may be odd [first–class] or even [second–class] under $\mathcal{G}$

$$\begin{aligned}\mathcal{G}J_5^\mu\mathcal{G}^{-1} &= -J_5^\mu \quad \text{first}-\text{class current} \\ \mathcal{G}J_5^\mu\mathcal{G}^{-1} &= +J_5^\mu \quad \text{second}-\text{class current}\end{aligned}$$

✌

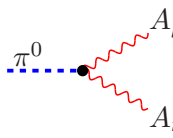
② Calculate the decay rate for $\pi^0 \to \gamma\gamma$ using the effective $O(p^4)$ coupling

$$\mathcal{L}_{\rm WZW} = \frac{\alpha}{\pi}\frac{N_c}{12F_\pi}\left(\pi^0 + \frac{1}{\sqrt{3}}\,\eta_8 + 2\sqrt{\frac{2}{3}}\,\eta_0\right)\,\tilde{F}_{\mu\nu}F^{\mu\nu}\ ,$$

which is the Wess-Zumino-Witten Lagrangian. The latter reproduces the ABJ anomaly on the level of the hadrons. π^0 is the neutral pion field, F_π the pion decay constant ($F_\pi = 92.4\,\mathrm{MeV}$). The pseudoscalars η_8, η_0 are mixing into the physical states η, η'.

✎

Let us first write down the Feynman rule for the $\pi^0\gamma(q_1,\mu)\gamma(q_2,\nu)$ vertex: first we note that $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In momentum space $\partial_\mu \to \mathrm{i}q_\mu$. We have $\tilde{F}_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = \frac{1}{2}\times 2\times 2\times \varepsilon_{\mu\nu\rho\sigma}\,\partial^\mu A^\nu\,\partial^\rho A^\sigma = -2\varepsilon_{\mu\nu\alpha\beta}\,\partial^\alpha A^\mu\,\partial^\beta A^\nu$ as the other terms of the field strength tensors just yield a factor 2 each, by relabeling of contracted indices. In momentum space the coefficient of the vertex (a factor $\frac{1}{2}$ is commonly included for Bose symmetry of the two photons) $\frac{1}{2}\,\tilde{\pi}^0(p)\,\tilde{A}(q_1,\mu)\,\tilde{A}(q_2,\nu)$ is $\frac{\alpha}{\pi}\frac{N_c}{12F_\pi}\times(\mathrm{i})^2\,(-2)\,\varepsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta\times 2$ (the last factor compensates for the $\frac{1}{2}$ included in the field monomial). We thus have the Feynman rule:



$$= \frac{\alpha}{\pi}\frac{N_c}{12F_\pi}\,4\varepsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta$$

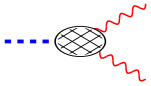
In the matrix element the photon field will be represented by the polarization vector of an outgoing photon $A^\mu \to \varepsilon^{\mu*}(q,\lambda)$. Thus the matrix element $\langle\gamma(q_1,\lambda_1),\gamma(q_2,\lambda_2)|\pi^0(p)\rangle$ of the WZW vertex in momentum space reads (for antisymmetrically contracted $F^{\mu\nu}$ use $F^{\mu\nu} = 2\partial^\mu A^\nu$ a factor two for Bose symmetry)

$$2\,\frac{\alpha}{\pi}\frac{N_c}{12F_\pi}\,\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\ \times\ -4\,q_1^\mu\varepsilon^{\nu*}(1)q_2^\alpha\varepsilon^{\beta*}(2) = \frac{\alpha}{\pi}\frac{N_c}{12F_\pi}\,4\varepsilon^{\mu\nu\alpha\beta}\,\varepsilon_\mu^*(1)\,\varepsilon_\nu^*(2)\,q_{1\alpha}q_{2\beta}\ ,$$

in agreement with the above Feynman rule. It is customary to define a $\pi^0\gamma^*\gamma^*$ form factor $\mathcal{F}$ by

$$\begin{aligned}\mathcal{A}_{\mu\nu}(\pi^0\to\gamma^*\gamma^*) &= \mathrm{i}\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}q\cdot x}\langle 0|T\{j_\mu(x)j_\nu(0)\}|\pi^0(p)\rangle\\ &= \varepsilon_{\mu\nu\alpha\beta}\,q^\alpha p^\beta\,\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2,q^2,(p-q)^2)\ ,\end{aligned}$$

with $j_\mu(x)$ the electromagnetic current. The pion is on-shell, but the photons are in general off-shell. Then



$$= \mathcal{M}[\pi^0\to\gamma(q_1,\lambda_1)\gamma(q_2,\lambda_2)] = e^2\,\mathcal{A}_{\mu\nu}(\pi^0\to\gamma\gamma)\,\varepsilon_\mu^*(q_1,\lambda_1)\,\varepsilon_\nu^*(q_2,\lambda_2)$$
$$= e^2\,\varepsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}\mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2,0,0)\,\varepsilon_\mu^*(1)\,\varepsilon_\nu^*(2)\ .$$

The WZW fromfactor (a constant) corresponds to the formfactor in the chiral limit $m_\pi^2 \to 0$ and for real photons:

$$M^{\rm WZW}_{\pi^0\gamma\gamma} = e^2\,\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0,0) = \frac{e^2N_c}{12\pi^2F_\pi} = \frac{\alpha}{\pi F_\pi} \approx 0.025\ \mathrm{GeV}^{-1}$$

where we used $F_\pi \sim 92.4$ MeV and the color factor is $N_c = 3$. The transition rate into photons of arbitrary polarization is

$$\begin{aligned}\sum_{\lambda_1\lambda_2}|\mathcal{M}_{\pi^0\gamma\gamma}|^2 &= e^4\,g^{\mu\rho}g^{\nu\sigma}\varepsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta\varepsilon_{\rho\sigma\alpha'\beta'}q_1^{\alpha'}q_2^{\beta'}\,|\mathcal{F}|^2\\ &= e^4\times -2\,\left(q_1^2q_2^2-(q_1\cdot q_2)^2\right)\,|\mathcal{F}|^2 = \frac{m_\pi^4}{2}\,|e^2\mathcal{F}|^2\ .\end{aligned}$$

where we utilized, as usual, the completeness relation $\sum_\lambda \varepsilon^{\mu *}(q,\lambda)\varepsilon^\rho(q,\lambda) = -g^{\mu\rho} + q^\mu f^\rho + q^\rho f^\mu$ *with* f *some arbitrary vector which must drop out from the transition amplitude (gauge invariance). We furthermore used* $\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma\alpha'\beta'} = -2\,(\delta^{\alpha'}_\alpha\,\delta^{\beta'}_\beta - \delta^{\beta'}_\alpha\,\delta^{\alpha'}_\beta)$, $(q_1\cdot q_2) = \frac{1}{2}(p^2 - q_1^2 - q_2^2)$ *and* $p^2 = m_\pi^2, q_1^2 = 0, q_2^2 = 0$.

The total 2–body decay width into two massless particles is (a factor 1/2 is required for identical particles)

$$\Gamma_{\pi^0\gamma\gamma} = \frac{1}{32\pi m_\pi}\sum_{\lambda_1\lambda_2}|\mathcal{M}_{\pi^0\gamma\gamma}|^2$$

and using the WZW amplitude, we obtain

$$\Gamma^{\rm WZW}_{\pi^0\gamma\gamma} = \frac{1}{64\pi}\,m_\pi^3\,|M^{\rm WZW}_{\pi^0\gamma\gamma}|^2 = \frac{\alpha^2\,m_\pi^3}{64\pi^3\,F_\pi^2}\ ,$$

where $|M_{\pi^0\gamma\gamma}|$ *was given above. Using the experimental* $\pi^0\to 2\gamma$ *width, we may calculate*

$$|M^{\rm exp}_{\pi^0\gamma\gamma}| = \sqrt{64\pi\Gamma_{\pi^0\gamma\gamma}/m_\pi^3} = 0.025\pm 0.001\ GeV^{-1}$$

in excellent agreement with the WZW prediction. ✌

A.6 Exercises: Section 4.1

① Show that the Maxwell equation $\partial_\mu F^{\mu\nu} = 0$ as a field equation for the vector potential takes the form

$$\Box A^\nu(x) - \partial^\nu\left(\partial_\mu A^\mu(x)\right) = 0\ .$$

Show that $A^\nu(x)$ is not determined by this equation because the operator $\Box g^{\mu\nu} - \partial^\mu\partial^\nu$ has no inverse. Hint.: $\varphi_\mu = \partial_\mu\alpha(x)$, $\alpha(x)$ an arbitrary scalar function, is a solution of the above equation with eigenvalue 0.

✎

We have $\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \Box A^\nu - \partial^\nu\left(\partial_\mu A^\mu\right) = \left(\Box g^{\mu\nu} - \partial^\nu\partial^\mu\right)A_\mu = 0$ *and* $\left(\Box g^{\mu\nu} - \partial^\mu\partial^\nu\right)\,\partial_\nu\alpha(x) = \partial^\mu\,\Box\alpha(x) - \partial^\mu\,\Box\alpha(x) \equiv 0$ *whatever* $\alpha(x)$ *is (but assumed to be differentiable). As claimed the operator* $\Box g^{\mu\nu} - \partial^\mu\partial^\nu$ *has no inverse, because it has a zero eigenvalue.*

We may try to solve for the propagator in momentum space directly. The kernel is $K^{\mu\nu}(p) = p^2 g^{\mu\nu} - p^\mu p^\nu$ *while the propagator should be of the form* $D_{\nu\rho}(p) = \left(A\,g_{\nu\rho} + B\,\frac{p_\nu p_\rho}{p^2}\right)\frac{1}{p^2}$, *and a solution of*

$$K^{\mu\nu}(p)\,D_{\nu\rho}(p) = \delta^\mu_\rho\ .$$

$$\begin{aligned} K^{\mu\nu}(p)\,D_{\nu\rho}(p) &= A\,\delta^\mu_\rho - A\,\frac{p^\mu p_\rho}{p^2} + B\frac{p^\mu p_\rho}{p^2} - B\,\frac{p^\mu p_\rho}{p^2} \\ &= A\,\delta^\mu_\rho - A\,\frac{p^\mu p_\rho}{p^2} \stackrel{?}{=} \delta^\mu_\rho\ , \end{aligned}$$

which has no solution, as it should satisfy $A = 1$ *and* $A = 0$ *at the same, a contradiction.* ✌

② Show that for a massive spin 1 field the Proca equation

$$\left(\Box + m^2\right) A^\nu(x) - \partial^\nu \left(\partial_\mu A^\mu(x)\right) = 0$$

implies $\partial_\mu A^\mu(x) \equiv 0$ automatically. Comment on the number of degrees of freedom ! Show that the Proca equation is the Euler-Lagrange equation of the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \;\; ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \;\; .$$

Discuss the invariance properties of $\mathcal{L}$ under gauge transformations.

✎

Taking the derivative ∂_ν of the Proca equation as stated above, we obtain

$$\left(\Box + m^2\right) \left(\partial_\nu A^\nu(x)\right) - \Box \left(\partial_\mu A^\mu(x)\right) = 0$$

which means

$$m^2 \left(\partial_\nu A^\nu(x)\right) = 0$$

and since $m \neq 0$ we must have $\partial_\nu A^\nu(x) = 0$ as claimed. It means there is automatically no scalar component in A^μ which solves the Proca equation. As a massive spin 1 field has 3 physical degrees of freedom the 2 transversal plus one longitudinal, the Proca field does not exhibit any unphysical ghost degrees of freedom.

The Euler-Lagrange equation reading

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = \frac{\partial \mathcal{L}}{\partial A_\nu}$$

yields

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} \frac{\partial F_{\rho\sigma}}{\partial \partial_\mu A_\nu} = -\frac{1}{4}\, 2\, F^{\rho\sigma} \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma\right) = -\frac{1}{2} \left(F^{\mu\nu} - F^{\nu\mu}\right) = -F^{\mu\nu}$$

while

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{1}{2}\, m^2\, 2\, A^\nu = m^2 A^\nu$$

and hence

$$-\partial_\mu F^{\mu\nu} = m^2 A^\nu$$

and inserting $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ directly yields the Proca equation.

While under a gauge transformation $A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha(x)$ as in QED $F^{\mu\nu}$ and hence the kinetic term of the Proca Lagrangian is invariant, the mass term $A^\mu A_\mu$ is obviously not gauge invariant. ✌

③ Prove that under local gauge transformations

$$\psi \to \mathrm{e}^{-\mathrm{i}e\alpha(x)} \psi \;, \quad A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha(x)$$

the covariant derivative $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$ has the property: $D_\mu \psi$ transforms identical to ψ and $\bar{\psi} \Gamma D_\mu \psi$ is gauge invariant provided D_μ commutes with the 4 by 4 matrix Γ.

✎

If $\psi \to \mathrm{e}^{-\mathrm{i}e\alpha(x)}\psi$ *and* $A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x)$ *then*

$$\begin{aligned} D_\mu\psi &\to (\partial_\mu - \mathrm{i}eA_\mu + \mathrm{i}e\,(\partial_\mu\alpha(x)))\ \mathrm{e}^{-\mathrm{i}e\alpha(x)}\psi \\ &= \mathrm{e}^{-\mathrm{i}e\alpha(x)}\,(\partial_\mu - \mathrm{i}\,e(\partial_\mu\alpha(x)) - \mathrm{i}eA_\mu + \mathrm{i}e(\partial_\mu\alpha(x)))\ \psi \\ &= \mathrm{e}^{-\mathrm{i}e\alpha(x)}\,(\partial_\mu - \mathrm{i}eA_\mu)\ \psi = \mathrm{e}^{-\mathrm{i}e\alpha(x)}\,D_\mu\psi \end{aligned}$$

q.e.d.

Invariance of $\bar\psi\Gamma D_\mu\psi$ *requires* $\mathrm{e}^{\mathrm{i}e\alpha(x)}\Gamma\mathrm{e}^{-\mathrm{i}e\alpha(x)} = \Gamma$, *which for an Abelian transformation is trivial. For chiral transformations this becomes non-trivial. Gauge invariance is obtained only if* $[\gamma^0\Gamma, \gamma_5] = 0$. *For non-Abelian transformations* $U = U(x)$ *gauge invariance requires* $[\Gamma, T_i] = 0$ *for all generators* T_i. *Indeed with* $D_\mu = \partial_\mu - \mathrm{i}\,g\,\boldsymbol{V}_\mu$ *and* $\boldsymbol{V}_\mu = \sum_i T_i V_{i\mu}$ *we have* $[\Gamma, D_\mu] = -\mathrm{i}\,g\,\sum_i V_{i\mu}[\Gamma, T_i]$ *and thus* $[\Gamma, D_\mu] = 0$ *implies* $[\Gamma, T_i] = 0$, *which implies* $[\Gamma, U] = 0$. ✌

A.7 Exercises: Section 4.2

① If $\boldsymbol{V}_\mu = \frac{\mathrm{i}}{g}U^{-1}(x)\,(\partial_\mu U_x)$ the field is called a " pure gauge" field. Show that in this case $\boldsymbol{G}_{\mu\nu} \equiv 0$.

✎

The non-Abelian field strength tensor has been obtained from calculating the commutator of two covariant derivatives:

$$\begin{aligned} [D_\mu, D_\nu]_{il} &= -ig\sum_j (T_j)_{il}\,(\partial_\mu V_{j\nu} - \partial_\nu V_{j\mu}) \\ &\quad -g^2\sum_{j',\,j''} \left[T_{j'}, T_{j''}\right]_{il} V_{j'\mu}V_{j''\nu} \\ &= -ig\sum_j (T_j)_{il}G_{j\mu\nu} = -ig\ (\boldsymbol{G}_{\mu\nu})_{il}\ . \end{aligned}$$

This directly translates into a relation for elements of the Lie algebra (sums $\sum_i T_i X_i$*). We easily read off that the field strength tensor can be written in as:*

$$\boldsymbol{G}_{\mu\nu} = \partial_\mu\,\boldsymbol{V}_\nu - \partial_\nu\,\boldsymbol{V}_\mu - \mathrm{i}\,g\,[\boldsymbol{V}_\mu, \boldsymbol{V}_\nu]$$

Inserting the "pure gauge" field, we have

$$\begin{aligned} \mathrm{i}\,g\,\boldsymbol{G}_{\mu\nu} &= \left\{-\partial_\mu[U_x^{-1}\,(\partial_\nu U_x)] + \partial_\nu[U_x^{-1}\,(\partial_\mu U_x)] - \left[U_x^{-1}\,(\partial_\mu U_x)\,, U_x^{-1}\,(\partial_\nu U_x)\right]\right\} \\ = \ & -(\partial_\mu U_x^{-1})(\partial_\nu U_x) - U_x^{-1}(\partial_\mu\partial_\nu U_x) + (\partial_\nu U_x^{-1})(\partial_\mu U_x) + U_x^{-1}(\partial_\nu\partial_\mu U_x) \\ & -U_x^{-1}(\partial_\mu U_x)U_x^{-1}(\partial_\nu U_x) + U_x^{-1}(\partial_\nu U_x)U_x^{-1}(\partial_\mu U_x)\ . \end{aligned}$$

From $\partial_\mu(U_x^{-1}U_x) = 0$ *we find* $(\partial_\mu U_x^{-1})U_x = -U_x^{-1}(\partial_\mu U_x)$ *or* $(\partial_\mu U_x^{-1}) = -U_x^{-1}(\partial_\mu U_x)U_x^{-1}$. *Inserting this and using that* ∂_μ *and* ∂_ν *commute, we easily find that* $\boldsymbol{G}_{\mu\nu}$ *vanishes for any* U. *Note that because of the non-linearity of the problem the signs of* $\boldsymbol{V}_\mu = \frac{\mathrm{i}}{g}U^{-1}(x)\,(\partial_\mu U_x)$ *is crucial. The opposite sine would not yield* $\boldsymbol{G}_{\mu\nu} \equiv 0$. *Also note that the "pure gauge" field can be represented as a gauge transformation of the zero field.* ✌

② Show that a mass term $\frac{M^2}{2}\sum_i V_{i\mu}V_i^\mu$ cannot be locally gauge invariant.

✎

In order to check gauge invariance in general , i.e. beyond infinitesimal transformations, it is simplest to write the fields in terms of elements of the Lie-algebra. Using the orthogonality $Tr\,T_iT_j = \frac{1}{2}\,\delta_{ij}$ *of the generators we obviously may write the mass term as*

$$M^2\,Tr\,\boldsymbol{V}_\mu\boldsymbol{V}^\mu$$

A general gauge transformation reads

$$\boldsymbol{V}_\mu \to \boldsymbol{V}_\mu{}' = U(x)\left(\boldsymbol{V}_\mu - \frac{\mathrm{i}}{g}U^{-1}(\partial_\mu U)\right)U^{-1}(x)\ .$$

and hence

$$\begin{aligned} & Tr\,\boldsymbol{V}_\mu\boldsymbol{V}^\mu \\ \to\ & Tr\,U_x\left(\boldsymbol{V}_\mu - \frac{\mathrm{i}}{g}U^{-1}(\partial_\mu U)\right)U_x^{-1}U_x\left(\boldsymbol{V}^\mu - \frac{\mathrm{i}}{g}U^{-1}(\partial^\mu U)\right)U_x^{-1} \\ =\ & Tr\left(\boldsymbol{V}_\mu - \frac{\mathrm{i}}{g}U^{-1}(\partial_\mu U)\right)\left(\boldsymbol{V}^\mu - \frac{\mathrm{i}}{g}U^{-1}(\partial^\mu U)\right)\ , \end{aligned}$$

which is not gauge invariant. ✌

③ Show that $\mathcal{L}_{\mathrm{YM}}$ can be written in the form of a trace

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{Tr}\,(\boldsymbol{G}_{\mu\nu}\boldsymbol{G}^{\mu\nu})$$

✎

By the normalization adopted for the generators: $Tr\,T_iT_j = \frac{1}{2}\,\delta_{ij}$ *we have*

$$Tr\,(\boldsymbol{G}_{\mu\nu}\boldsymbol{G}^{\mu\nu}) = Tr\sum_{ij}T_iT_jG_{\mu\nu\,i}G_j^{\mu\nu} = \frac{1}{2}\sum_{ij}\delta_{ij}G_{\mu\nu\,i}G_j^{\mu\nu} = \frac{1}{2}\sum_i G_{\mu\nu\,i}G_i^{\mu\nu}\ .$$

Therefore,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,Tr\,(\boldsymbol{G}_{\mu\nu}\boldsymbol{G}^{\mu\nu}) = -\frac{1}{4}\,G_{\mu\nu\,i}G_i^{\mu\nu}$$

the usual representation. ✌

④ Prove the validity of the "homogeneous Maxwell equation" or "Bianchi-identity":

$$D_\rho\,\boldsymbol{G}_{\mu\nu} + \ \text{terms cyclic in}\ (\rho\mu\nu) \equiv 0$$

✎

✌

A.8 Exercises: Section 5

① Spin 1 propagators are symmetric second rank Lorentz tensors proportional to $g^{\mu\nu}$ and $k^\mu k^\nu$, where k is the four momentum of the spin 1 boson. Show that

$$T^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \; ; \;\; L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}$$

are projection tensors to the **transversal** and the **longitudinal** field components. Verify the projector property of T and L. Use the decomposition

$$K^{\mu\nu} = A\, T^{\mu\nu} + B\, L^{\mu\nu}$$

for the bilinear part of the $\mathcal{L}_{0G} = G_\mu K^{\mu\nu} G_\nu$ of the free part of the Lagrangian (what are A and B in case of the gluon), calculate the propagator (inverse of the kernel K) using the projection technique. Formulate the condition under which the inverse exists.

② Given the Yang-Mills Lagrangian in coordinate space derive the triple gluon coupling

$$-\mathrm{i} g_s c_{ijk} \left\{ g^{\mu\nu} \, (p_2 - p_1)^\rho \; + \; g^{\mu\rho} \, (p_1 - p_3)^\nu \; + \; g^{\nu\rho} \, (p_3 - p_2)^\mu \right\}$$

and the quartic gluon coupling

$$-g_s^2 \left\{ c_{nij} c_{nkl} \, (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + c_{nik} c_{njl} \, (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + c_{nil} c_{njk} \, (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \right\}$$

in momentum space.

③ According to Faddeev and Popov the fatal violation of gauge invariance by the gauge fixing term can be avoided by taking into account the functional determinant obtained in the functional integral under a gauge transformation of the fields (integration variables). If we define the functional integral as follows, with a Faddeev-Popov determinant,

$$\int \mathcal{D} G_{\mu a} \mathrm{Det} \left(\frac{\delta C_a}{\delta \omega_b} \right) \, \mathrm{e}^{\mathrm{i} \int \left(\mathcal{L}_{\mathrm{inv}} - \frac{1}{2\xi} C_a^2 \right) \mathrm{d}^4 x}$$

one easily checks that now the functional integral is independent on the specific choice of the gauge function C_a. Check this!

By introducing anticommuting scalars, the FP ghost fields $\bar{\eta}_a$ and η_a, we may represent the FP-determinant as a Berezin integral over Grassmann variables (algebra of anticommuting c-numbers)

$$\mathrm{Det} \left(\frac{\delta C_a}{\delta \omega_b} \right) = \int \mathcal{D}\eta \mathcal{D}\bar{\eta} \, \mathrm{e}^{\mathrm{i} \int \mathcal{L}_{\mathrm{FP}} \mathrm{d}^4 x}$$

with

$$\mathcal{L}_{\mathrm{FP}} = \bar{\eta}_a M_{ab} \eta_b \; ; \;\; M_{ab} \doteq \frac{\delta C_a}{\delta \omega_b} \;\; .$$

Check that this agrees with the expression adopted above.

✎
✌

④ **Becchi-Rouet-Stora (BRS) symmetry:** given the "quasi invariant" effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} \ ,$$

the local gauge invariance of the functional integral (path integral quantization)

$$\int \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta}\, \mathrm{e}^{\mathrm{i}\int \mathcal{L}_{\text{eff}}\mathrm{d}^4x} \tag{A.5}$$

yields relations between Green functions, the Slavnov-Taylor (ST) identities. They generalize the Ward-Takahashi (WT) identities, relations between Green functions which derive from global symmetries. In contrast to canonical quantization, which relies on the non gauge-invariant splitting into a free and an interacting part of the Lagrangian, the path integral formulation work with the complete effective Lagrangian.

The ST-identities provide the tool needed for proofs of

i) gauge invariance
ii) unitarity
iii) renormalizability

of the S-matrix. ST-identities may be obtained from the BRS-symmetry of $\mathcal{L}_{\text{eff}}$.

Convince yourself that the BRS procedure (described in the following) to control gauge invariance indeed works. The idea behind BRS-symmetry is to dispose of the as yet undefined transformation properties of the FP-ghost fields $\bar{\eta}$ and η such that

$$\delta^{\text{BRS}}\mathcal{L}_{\text{eff}} = 0 \ . \tag{A.6}$$

In order to achieve this it is natural to demand the following relations to hold:

$$\begin{aligned}
&i) \quad \delta\mathcal{L}_{\text{inv}} &&= 0 \\
&ii) \quad \delta\mathcal{L}_{\text{GF}} &&= -\tfrac{1}{\xi}C_a M_{ab}\omega_b \\
&iii) \quad \delta\mathcal{L}_{\text{FP}} &&= \delta\bar{\eta}M_{ab}\eta_b + \bar{\eta}_a\delta(M_{ab}\eta_b) \\
&\qquad\qquad &&= -\delta\mathcal{L}_{\text{GF}} \\
&iv) \quad \mathcal{D}G\mathcal{D}\eta\mathcal{D}\bar{\eta} \quad \text{invariant} \ .
\end{aligned}$$

A solution for this set of conditions may be obtained as follows:
1) Introduce anticommuting global $c-$number variables $\delta\lambda$, $\delta\lambda$ anticommuting with η and $\bar{\eta}$, and identify $\omega_b = \eta_b\delta\lambda$. Thus

- $\delta G_a = D_{ab}\eta_b\delta\lambda$

where G_a can be a gauge field, a scalar or a Fermi field with $D_{ab}\eta_b$ given by

$$\frac{\delta G_{a\mu}}{\delta\omega_b}\eta_b \doteq D_{\mu ab}\eta_b$$

in case of the gauge field.

2) Assume η to transform according to the regular representation, thus

- $\delta\eta_a = -\frac{1}{2} g f_{abc}\eta_b\eta_c\delta\lambda$

where a permutation symmetry factor 1/2 (antisymmetry of f_{abc} and anticommutativity of the η's) has been taken into account. 1) and 2) imply $\delta(M_{ab}\eta_b) = 0$. We thus take the freedom to choose: 3) The field $\bar{\eta}$ transforms as

- $\delta\bar{\eta}_a = -\frac{1}{\xi} C_a\delta\lambda$

such that conditions i) to iii) are satisfied. One can show iv) to be true for the above set of transformations which define the BRS-transformation.

ST-identities:

The *BRS* invariance of $\mathcal{L}_{\text{eff}}$ allows a simple derivation of the ST-identities. Performing a change of integration variables in the functional integral does not change the value of the integral. See my TASI Lectures *Renormalizing the Standard Model* for more details.

✎

✌

A.9 Exercises: Section 6

① Prove the relations (6.4) and (6.5) for the d dimensional Dirac algebra.

✎

The γ's–contractions can all be worked out using the Dirac algebra: 1) $\gamma_\alpha\gamma^\alpha = g_{\alpha\beta}\,\gamma^\beta\gamma^\alpha = g_{\alpha\beta}\,\frac{1}{2}\,\left(\gamma^\beta\gamma^\alpha + \gamma^\alpha\gamma^\beta\right) = g_{\alpha\beta}\,\frac{1}{2}\,2\,g^{\alpha\beta}\,\mathbf{1} = d\,\mathbf{1}$

where besides the Dirac algebra we have used the symmetry of the metric tensor for a symmetrization and $g^\alpha_\alpha = d$ in d dimensions.

2) $\gamma_\alpha\gamma^\mu\gamma^\alpha = -\gamma^\mu\gamma_\alpha\gamma^\alpha + 2\,\delta^\mu_\alpha\gamma^\alpha == -d\,\gamma^\mu + 2\,\gamma^\mu$

3) $\gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\alpha = \gamma^\mu\gamma^\nu\gamma_\alpha\gamma^\alpha + 2\,\delta^\mu_\alpha\gamma^\nu\gamma^\alpha - 2\,\delta^\nu_\alpha\gamma^\mu\gamma^\alpha = \gamma^\mu\gamma^\nu\,d + 2\,\gamma^\nu\gamma^\mu - 2\,\gamma^\mu\gamma^\nu$

4) $\gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\alpha = -\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha\gamma^\alpha + 2\,\underbrace{\gamma^\nu\gamma^\rho}\,\gamma^\mu - 2\,\gamma^\mu\,\underbrace{\gamma^\rho\gamma^\nu} + 2\,\gamma^\mu\gamma^\nu\gamma^\rho = (4-d)\,\gamma^\mu\gamma^\nu\gamma^\rho - 2\,\gamma^\rho\gamma^\nu\gamma^\mu$

applying the Dirac algebra to the underbraced terms yields the given result, in the given (non-unique) form.

etc.

The traces are obtained using the Dirac algebra in conjunction with the cyclicity property:

1) In $d = 2n$ (n=integer) dimensions the representation of the Dirac algebra is composed of a tensor product of $n = d/2$ Pauli matrices $\sigma_{i_1}\otimes\sigma_{i_2}\otimes\cdots\otimes\sigma_{i_n}$ such that the dimension of the representation is $Tr\,(\mathbf{1}) = 2^n = 2^{d/2}$, which provides a natural extension to arbitrary dimensions.

2) $Tr\;\gamma^\mu = 0$ is a basic property of the γ–matrices, being hermitian and traceless.

3) $Tr\;\gamma^\mu\gamma^\nu = \frac{1}{2}\;(Tr\;\gamma^\mu\gamma^\nu + Tr\;\gamma^\nu\gamma^\mu) = \frac{1}{2}\,Tr\gamma^\mu,\gamma^\nu = g^{\mu\nu}\;Tr(\mathbf{1})$

4) $Tr\;\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = Tr\;\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu = -Tr\;\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + 2\,g^{\mu\sigma}\;Tr\;\gamma^\nu\gamma^\rho - 2\,g^{\mu\rho}\;Tr\;\gamma^\nu\gamma^\sigma + 2\,g^{\mu\nu}\;Tr\;\gamma^\rho\gamma^\sigma$

therefore

$Tr\;\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = (g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\nu}g^{\rho\sigma})\;Tr(\mathbf{1})$

as claimed.

5) γ–odd traces: $Tr\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n-1}} = Tr\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n-1}}\gamma_5\gamma_5 = Tr\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n-1}}\gamma_5$ *anticommuting the first γ_5 through all γ^{μ_i}'s to the right we end up with* $Tr\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n-1}} = (-)^{2n-1}\,Tr\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n-1}}$ *which means* $Tr\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n-1}} = 0$ *for all integer $n \geq 1$. This argument is based strictly on the anti–commutativity of γ_5: $\{\gamma_5,\gamma^\mu\} = 0$, which is granted in any case in $d = 4$. In dimensional regularization, one eventually has to give up the anti–commutativity of γ_5 ('t Hooft, Veltman 1972 [1], see also F. Jegerlehner 2001 [2]). See next exercise.* ✌

② Show that the γ_5–odd traces are not cyclic in the d dimensional Dirac algebra. Note that for a mathematically well defined set of matrices a trace trivially is cyclic. In particular prove the claim (6.6). Hint: calculate $\mathrm{Tr}\,\gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma_5$ and join the connected γ's such that one may use $\gamma_\alpha\gamma^\alpha = d$. Note that there are two ways to join these γ's.

✎

$X^{\mu\nu\rho\sigma} = \mathrm{Tr}\,\gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma_5$ may be computed in two ways: 1) anticommute γ^α with γ_5 to the right and use cyclicity of the trace we obtain $X^{\mu\nu\rho\sigma} = -\mathrm{Tr}\,\gamma^\alpha\gamma_\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 = -d\,E^{\mu\nu\rho\sigma}$; 2) anticommute γ_α through $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ to the right until it meets γ^α, which yields $2\,\delta^\mu_\alpha\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha - 2\,\delta^\nu_\alpha\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\alpha + 2\,\delta^\rho_\alpha\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\alpha - 2\,\delta^\sigma_\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\alpha + \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\alpha\gamma^\alpha = 2\,\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu - 2\,\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu + 2\,\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho - 2\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + d\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$. Finally, we are anticommuting the γ–matrices into the common order. Note that at this stage the $g^{\mu\nu}$ terms from the Dirac algebra do not contribute to the trace as all γ_5 odd traces with less than 4 γ's vanish. Thus we obtain

$$-d\,E^{\mu\nu\rho\sigma} = -8\,E^{\mu\nu\rho\sigma} + d\,E^{\mu\nu\rho\sigma}$$

or

$$(4-d)\,\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 = 0$$

and hence

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 = 0 \quad \text{if} \quad d \neq 4$$

in case we assume γ_5 to be anticommuting when $d \neq 4$. Note that the fact of the existence of the ABJ anomaly must be understood as a proof that no regularization exists which satisfies all symmetries which seem to be there at the classical level. This is a no–go theorem: it is not possible to find a regulatization which preserves all symmetries (gauge and chiral) we would like to preserve in first place (in this context we should mention the Nielsen and Ninomiya theorem 1981 concerning lattice regularizations [3]). ✌

③ Prove the relations (6.39) and derive the corresponding relations for the tensor integrals B_1, B_{21}, B_{22} as well as for the three point amplitudes C_{1i} and C_{2i}. Verify (6.46).

✎

✌

④ Calculate $B_0(m_1, m_2; p^2)$ performing the 1-dimensional integral (6.36) in terms of elementary functions.

✎

Loop integrals typically lead to logarithms and at higher orders to polylogarithms (dilogarithm[=Spence function], trilogarithm, ...). These are multi–valued analytic functions and one of the major complications is the proper specification of the sheet which is appropriate for a given physical application. The unique answer in physics is dictated by the $\mathrm{i}\varepsilon$–prescription in the Feynman propagators. In Feynman diagram calculations one commonly chooses the logarithm to have the cut along the negative axis. This implies that

$$\ln(ab) = \ln a + \ln b + \eta(a,b)$$

with

$$\eta(a,b) = 2\pi\,\mathrm{i}\,\{\Theta(-Im\,a)\Theta(-Im\,b)\Theta(Im\,ab) - \Theta(Im\,a)\Theta(Im\,b)\Theta(Im\,-ab)\}$$

The following simple rules apply:

$$\ln ab = \ln a + \ln b \quad \text{if } Im\,a \text{ and } Im\,b \text{ have different sign ,}$$

$$\ln a/b = \ln a - \ln b \quad \text{if } Im\,a \text{ and } Im\,b \text{ have the same sign .}$$

If A and B are real we have

$$\ln(AB - \mathrm{i}\varepsilon) = \ln(A - \mathrm{i}\varepsilon') + \ln(B - \mathrm{i}\varepsilon/A)$$

where ε' has the same sign as ϵ.

*The scalar two–point function is given by Eq.(**??**):*

$$B_0(m_1,m_2;p^2) - Reg = -\int_0^1 \mathrm{d}x\,\ln(x\,m_1^2 + (1-x)\,m_2^2 - x\,(1-x)\,p^2 - \mathrm{i}\varepsilon)\ .$$

The argument of the logarithm is a quadratic form in the integration variable x and we may decompose it in its roots. From $q(x) = x^2\,p^2 + x\,(m_1^2 - m_2^2 - p^2) + m_2^2 - \mathrm{i}\varepsilon$ which has the zeros

$$x_\pm = -(m_1^2 - m_2^2 - p^2)/(2p^2) \pm \sqrt{((m_1^2 - m_2^2 - p^2)^2 - 4m_2^2p^2)/(4p^4)}$$

Note that the roots have imaginary parts of different signs such that we may write

$$\ln\left[p^2\,(x-x_+)\,(x-x_-)\right] = \ln(p^2 - \mathrm{i}\varepsilon) + \ln(x-x_+) + \ln(x-x_-)$$

The integral is then easily obtained. The primitive of $\ln(x-a)$ is $(x-a)\,\ln(x-a) - x$ and thus

$$\begin{aligned}
\int_0^1 \mathrm{d}x\,\ln q(x) &= \left[x\,\ln(p^2 - \mathrm{i}\varepsilon) + (x-x_+)\,\ln(x-x_+) + (x-x_-)\,\ln(x-x_-) - 2x\right]\Big|_0^1 \\
&= \ln(p^2 - \mathrm{i}\varepsilon) + (1-x_+)\,\ln(1-x_+) + x_+\,\ln(-x_+) \\
&\quad + (1-x_-)\,\ln(1-x_-) + x_-\,\ln(-x_-) - 2 \\
&= \ln(p^2 - \mathrm{i}\varepsilon) + \sum_{i=\pm}\left[\ln(1-x_i) - x_i\,\ln\frac{x_i - 1}{x_i} - 1\right]\ .
\end{aligned}$$

Note: in the form $-s\,x\,(1-x) + m_1^2\,x + m_2^2\,(1-x)$ the $\mathrm{i}\varepsilon$–prescription form the Feynman propagator is $m_i^2 \to m_i^2 - \mathrm{i}\varepsilon$ for s translates to $s \to s + \mathrm{i}\varepsilon$. Thus when $m_1 = m_2 = 0$ we may write $q(x) = -x\,(1-x)\,p^2 - \mathrm{i}\varepsilon = (-p^2 + \mathrm{i}\varepsilon)\,x\,(1-x)$ (since $x\,(1-x)$ is positive in the integration range $\mathrm{i}\varepsilon$ in $q(x)$ is equivalent to $x\,(1-x)\,\mathrm{i}\varepsilon$). The integral reads

$$\ln(-p^2 + \mathrm{i}\varepsilon) + [x\ln x - x - (1-x)\,\ln(1-x) - x]_0^1 = \ln(-p^2 + \mathrm{i}\varepsilon) - 2\ .$$

If $p^2 = 0$ $q(x) = x\,(m_1^2 - m_2^2) + m_2^2 - \mathrm{i}\varepsilon$ has one root only Special cases:

$$\begin{aligned}
B_0(0,m;s) &= 1 - A_0(m)/m^2 + \frac{m^2 - s}{s}\,\ln\left(1 - \frac{s}{m^2}\right) \\
B_0(m,m,s) &= \\
B_0(0,0;s) &= Reg + 2 - \ln(-s - \mathrm{i}\varepsilon) \\
B_0(0,m;0) &= = -A_0(m)/m^2 \\
B_0(m,m;0) &= = -\left[1 + A_0(m)/m^2\right] \\
B_0(m_1,m_2;0) &= = -\left[A_0(m_1) - A_0(m_2)\right]/(m_1^2 - m_2^2)
\end{aligned}$$

✌

A.10 Exercises: Section 7

① Calculate the group invariants $C_2(G)$ and $C_2(R)$ using the known properties of the $SU(n)$ groups.

✎

1) *General consideration: for any irreducible representation R of dimension $d(R)$ we have the quadratic Casimir invariants $T(R)$ and $C_2(R)$ (i.e. an operator commuting with all the generators T_i^R $(i = 1, \cdots, r)$ of the group, $r = d(G)$ the order or dimension of the group G [$r = n^2 - 1$ for $SU(n)$]). $T(R)$ is defined by*

$$Tr\, T_i^R T_k^R = \sum_{a,b=1}^{d(R)} \left(T_i^R\right)_{ab} \left(T_k^R\right)_{ba} = \delta_{ik}\, T(R) \ ,$$

and $C_2(R)$ by

$$\left(T_i^R T_i^R\right)_{ab} = \sum_{i=1}^{r} \sum_{c=1}^{d(R)} \left(T_i^R\right)_{ac} \left(T_i^R\right)_{cb} = \delta_{ab}\, C_2(R) \ .$$

Taking the trace of the second form using the first and the fact that for square matrices $Tr\,(A + B) = Tr\, A + Tr\, B$, we have

$$d(G)\, T(R) = d(R)\, C_2(R) \quad \text{or} \quad C_2(R) = \frac{d(G)}{d(R)}\, T(R) \ .$$

2) *For $G = SU(n)$ and $R = F$ the fundamental representation using the last relation we immediately obtain*

$$C_F = C_2(F) = \frac{n^2 - 1}{n}\, T_F \ ,$$

where $T_F = T(F)$ is fixed by the standard normalization

$$Tr\, T_i T_k = \delta_{ik}\, T(R) \ ; \ \ T(R) = \frac{1}{2}$$

of the $SU(n)$ generators. Hence

$$C_2(R) = C_F = \frac{n^2 - 1}{2n}$$

in the fundamental representation.

3) *$C_2(G)$ corresponds to $C_2(R)$ for the adjoint (regular) representation $R = A$ of dimension $d(R) = d(G) = r$ and is defined correspondingly by*

$$f_{ijl}\, f_{jlk} = \delta_{ik}\, C_2(G) \ ,$$

which we may write, with $f_{ijl} = \mathrm{i}\,(\tilde{T}_i)_{jl}$ and the antisymmetry of f_{ijl}, as

$$\mathrm{i}\, \left(\tilde{T}_i\right)_{jl} \cdot (-\mathrm{i})\, \left(\tilde{T}_k\right)_{lj} = Tr\, \tilde{T}_i \tilde{T}_k = \delta_{ik}\, C_2(G) \ .$$

Again this trace is diagonal, i.e., proportional to δ_{ik}, and $C_2(G)$ provides the normalization of the generators in the adjoint representation. In other words

$$C_2(G) = C_A = T_A \quad \text{as} \quad d(R) = d(G) \quad \text{for} \quad R = A \ .$$

Again a simple consequence of the general relation given under **1)**. *However, we do not yet know the normalization T_A of the adjoint generators. Once the normalization of the generators in the fundamental representation is fixed, the normalization of the adjoint generators is fixed by the Lie algebra, actually $T_A = g\,T_F$. But what is g (in group theory called Coxeter number, $n\,g$ is the number of roots).*

In fact since we know (as inferred from the normalization equation) that all generators have the same normalization, we need compute only one, and we may choose the simplest possibility: calculate f_{rkl} where $r = n^2 - 1$ and $T_r = \frac{1}{2}\lambda_r$ with λ_r the last of the diagonal λ–matrices Eq. (1.1) in the fundamental representation. The commutator of T_r vanishes in the upper left $n-1$–dimensional sub-matrix where it just is the unit matrix. Thus the only non-vanishing elements are those which have a 1 or $-\mathrm{i}$ in the n-th column. These elements have the form

$$[T_r, T_{\hat{k}}] = \frac{1}{2}\sqrt{\frac{2}{n(n-1)}}\ (1 - (-(n-1)))\ \mathrm{i}\,T_{l_{\hat{k}}} = \mathrm{i}\,f_{r\hat{k}l_{\hat{k}}}\,T_{l_{\hat{k}}} \ ,$$

where, if $T_{\hat{k}}$ is chosen to be the generator having a 1 in the n-th column, then $T_{l_{\hat{k}}}$ is fixed to be the generator which has a $-\mathrm{i}$ in the same position. Thus we have $n-1$ possible pairings $[kl]$ with values

$$f_{r[kl]} = \sqrt{\frac{n}{2(n-1)}} \ .$$

The same we obtain for the $n-1$ possibilities when we replace 1 by $-\mathrm{i}$ in the n-th column on the left-hand side. This we have to take in quadrature, and we have to sum over all $2(n-1)$ possible non-vanishing terms: thus

$$\sum_{[kl]} f^2_{r[kl]} = 2(n-1)\,\frac{n}{2(n-1)} = n \ .$$

Therefore we obtain the known result

$$C_A = T_A = n$$

q.e.d. ✌

② Derive from our one-loop QCD results the corresponding ones for QED. Discuss main differences.

✎
✌

③ Calculate the integral $C_0(m_g, m, m; m^2, k^2, m^2)$ in the regime $m_g^2 \ll m^2, |k^2|$ using the representation (6.38). Discuss typical properties of the result.

✎

We may use a cyclic permutation of masses and momenta and consider $C_0(m, m, \sqrt{\delta}; s, m^2, m^2)$ *with* $\delta = m_g^2$ *infinitesimal and* $s = k^2$*. Then* $a = m^2, b = s, c = -s, d = -\delta, e = 0$ *and* $f = \delta$*. Starting from the representation*

$$C_0(m_1, m_2, m_3; s_1, s_2, s_3) = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \frac{1}{D(x,y)}$$

we may rescale the variable as $y = xy'$ *to obtain*

$$C_0(m_1, m_2, m_3; s_1, s_2, s_3) = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y' \frac{x}{D(x, y = xy')} \ .$$

Thus the bi-quadratic form $D(x, y')$ *form in the denominator of the last integral then reads*

$$D(x,y) = A\,x^2 + \delta\,(1-x) \quad \text{with} \quad A = s\,y^2 - s\,y + m^2 \ ,$$

where y *here and in the following denotes* y'*. The integration over* x *may now be performed. To this end we write, using a partial fraction decomposition,*

$$\begin{aligned} \frac{x}{A\,x^2 + \delta\,(1-x)} &= \frac{1}{A}\,\frac{x}{(x-x_1)(x-x_2)} = \frac{1}{A}\,\frac{1}{x_1 - x_2}\left(\frac{x}{x-x_1} - \frac{x}{x-x_2}\right) \\ &= \frac{1}{A}\,\frac{1}{x_1 - x_2}\left(\frac{x_1}{x-x_1} - \frac{x_2}{x-x_2}\right) \end{aligned}$$

with $x_1 + x_2 = x_1\,x_2 = \delta/A$*. For the roots we find*

$$x_{1,2} = \pm \mathrm{i}\sqrt{\frac{\delta}{A}} + O\left(\frac{\delta}{A}\right)$$

Up to a factor $1/A$ *the integral is then given by*

$$\begin{aligned} & \frac{x_1}{x_1 - x_2}\,\ln\frac{1-x_1}{x_1} - \frac{x_2}{x_1 - x_2}\,\ln\frac{1-x_2}{x_2} \\ = \ & \frac{1}{2}\,\ln\frac{(1-x_1)(1-x_2)}{x_1 x_2} + O\left(\sqrt{-\frac{\delta}{A}}\right) = -\frac{1}{2}\,\ln\frac{\delta}{A} + O\left(\sqrt{-\frac{\delta}{A}}\right) \ . \end{aligned}$$

As a result of the x *integration we have*

$$\int_0^1 \frac{x}{Ax^2 + \delta\,(1-x)} = \frac{-1}{2A}\,\ln\frac{\delta}{A} + O\left(\sqrt{-\frac{\delta}{A}}\right) \ .$$

We now may perform the remaining y *integration. With appropriate function* $A(y)$*, this result just given also holds if we let the two non-zero masses be different. The function* A *then reads*

$$A(y) = -s\,y\,(1-y) + m_1^2\,(1-y) + m_2^2\,y - \mathrm{i}\varepsilon = s\,(y - y_1)\,(y - y_2) \ .$$

As the coefficient of $-s$ *is real positive in the integration range we may give* s *a small imaginary part* $s \to s + \mathrm{i}\varepsilon$*, which is equivalent to* $m_i^2 \to m_i^2 - \mathrm{i}\varepsilon$*. For space–like* s*,* $-s = Q^2 > 0$ A *is positive and has no real roots. In any case the two roots* y_i *have opposite signs such that we may write*

$$\ln A(y) = \ln(s + \mathrm{i}\varepsilon) + \ln(y - y_1) + \ln(y - y_2)$$

and use the partial fraction decompositions

$$\frac{1}{(y-y_1)(y-y_2)} = \frac{1}{y_1-y_2}\left(\frac{1}{y-y_1} - \frac{1}{y-y_2}\right) \ .$$

The integral to be calculated is then

$$\begin{aligned} C_0 &= \int\limits_0^1 \mathrm{d}y\, \frac{-1}{2A}\, \ln\frac{\delta}{A} \\ &= \frac{-1}{2\,s\,(y_1-y_2)} \int\limits_0^1 \mathrm{d}y\, \left(\frac{1}{y-y_1} - \frac{1}{y-y_2}\right) \left(\ln\frac{\delta}{s+\mathrm{i}\varepsilon} - \ln(y-y_1) - \ln(y-y_2)\right) \ . \end{aligned}$$

The different types of integrals needed are

$$\begin{aligned} \int\limits_0^1 \mathrm{d}y\, \frac{1}{y-y_i} &= \ln\frac{y_i-1}{y_i} \ , \\ \int\limits_0^1 \mathrm{d}y\, \frac{1}{y-y_i}\, \ln(y-y_i) &= \int\limits_{-y_i}^{1-y_i} \mathrm{d}y\, \frac{1}{y}\, \ln(y) \\ &= \frac{1}{2}\, \ln^2 y \Big|_{-y_i}^{1-y_i} = \frac{1}{2}\, \left(\ln^2(1-y_i) - \ln^2(-y_i)\right) \ , \end{aligned}$$

and for $i \neq k$

$$\begin{aligned} & \int\limits_0^1 \mathrm{d}y\, \frac{1}{y-y_i}\, \ln(y-y_k) \\ = & \int\limits_0^1 \mathrm{d}y\, \frac{1}{y-y_i}\, \left\{\ln\left(\frac{y-y_k}{y_i-y_k}\right) + \ln(y_i-y_k)\right\} \\ = & \int\limits_{-y_i}^{1-y_i} \mathrm{d}y\, \frac{1}{y}\, \left\{\ln\left(\frac{y+y_i-y_k}{y_i-y_k}\right) + \ln(y_i-y_k)\right\} \\ = & \int\limits_{-y_i}^{1-y_i} \mathrm{d}y\, \frac{1}{y}\, \ln\left(1+\frac{y}{y_i-y_k}\right) + \ln\frac{y_i-1}{y_i}\, \ln(y_i-y_k) \end{aligned}$$

The integral we write as

$$\int\limits_{-y_i}^{1-y_i} \mathrm{d}y\, \cdots = \int\limits_0^{1-y_i} \mathrm{d}y\, \cdots - \int\limits_0^{-y_i} \mathrm{d}y\, \cdots$$

and rescale the "upper" limit to one: $y = (1-y_i)\,y'$ and $y = -y_i\,y'$, repectively. Thereby $\mathrm{d}y/y$ remains invariant. This way we obtain

$$\int\limits_0^{1-y_i} \mathrm{d}y\, \frac{1}{y}\, \ln\left(1+\frac{y}{y_i-y_k}\right) = \int\limits_0^1 \mathrm{d}y\, \frac{1}{y}\, \ln\left(1+\frac{1-y_i}{y_i-y_k}y\right) = -\mathrm{Sp}\left(-\frac{1-y_i}{y_i-y_k}\right)$$

and

$$\int\limits_0^{-y_i} \mathrm{d}y\,\frac{1}{y}\,\ln\left(1+\frac{y}{y_i-y_k}\right) = \int\limits_0^{1} \mathrm{d}y\,\frac{1}{y}\,\ln\left(1+\frac{-y_i}{y_i-y_k}y\right) = -\mathrm{Sp}\left(\frac{y_i}{y_i-y_k}\right)\ .$$

Finally, adding up all terms we arrive at the somewhat clumsy expression

$$\begin{aligned} C_0(m_1,m_2,m_g;s,m_2^2,m_1^2) = \frac{-1}{2\,s\,(y_1-y_2)} \quad & \left\{ \ln\frac{m_g^2}{s+\mathrm{i}\varepsilon}\left(\ln\frac{y_1-1}{y_1}-\ln\frac{y_2-1}{y_2}\right)\right. \\ & -\frac{1}{2}\left(\ln^2(1-y_1)-\ln^2(-y_1)-\ln^2(1-y_2)+\ln^2(-y_2)\right) \\ & -\ln(y_1-y_2)\,\ln\frac{y_1-1}{y_1}+\ln(y_2-y_1)\,\ln\frac{y_2-1}{y_2} \\ & +\mathrm{Sp}(\frac{y_1-1}{y_1-y_2})-\mathrm{Sp}(\frac{y_1}{y_1-y_2}) \\ & \left.-\mathrm{Sp}(\frac{y_2-1}{y_2-y_1})+\mathrm{Sp}(\frac{y_2}{y_2-y_1})\right\} \end{aligned}$$

For $m_1=m_2=m$ (as in QED or QCD) we obtain

$$C_0(m,m,m_g;s,m^2,m^2) = \frac{1}{s}\left[F_1(y)\,\ln\frac{m_g^2}{s}+F_2(y)\right]$$

where, with $y=4m^2/s$ and $r=\sqrt{1-y}$ we have

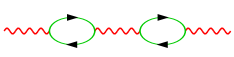
$$F_1(y) = \frac{1}{r}\,\ln\frac{r+1}{r-1}$$

$$\begin{aligned} F_2(y) \;=\; & \frac{1}{2r}\left\{-\ln\frac{r+1}{r-1}\left[\ln-\frac{r+1}{2}+\ln\frac{r-1}{2}+2\,\ln r\right]\right. \\ & \left.+2\,\mathrm{Sp}\left(\frac{r+1}{2r}\right)-2\,\mathrm{Sp}\left(\frac{r-1}{2r}\right)\right\} \end{aligned}$$

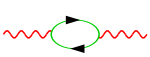
✌

④ Elaborate the following statement [from 't Hooft and Veltman's Diagrammar].
Statement: *Order-by-order renormalization is* **not** *equivalent to "throwing away ε–poles and their residues" of the unrenormalized S–matrix.*

Illustrate this for the following example of a double lepton–loop contribution to the photon self–energy:



Up to a factor $p^2\,g^{\mu\nu}-p^\mu p^\nu$, the irreducible self–energy bubble may be written in the form

 $:= f(p^2) = \dfrac{1}{d-4}\,f_1(p^2)+f_2(p^2)+(d-4)\,f_3(p^3)\ .$

What is the correct two–loop contribution from the factorizable double–loop above?

✎

Without performing a subtraction, the contribution from the two–loop diagram is

$$\left(f(p^2)\right)^2 = \left(\frac{1}{d-4}\right)^2 f_1^2 + 2\,\frac{f_1 f_2}{d-4} + f_2^2 + 2\,f_1 f_3 + O(d-4)\ .$$

If we now throw away the pole parts (MS–scheme) the result for $d = 4$ is

$$f_2^2 + 2\,f_1 f_3\ .$$

This result is **wrong** *because it violates unitarity.*

Suppose now the correct order–by–order subtraction. After the one–loop treatment one has

The physical result in $d = 4$ is just f_2. Including the proper counterterms at two loops, we find

This result can be shown to be consistent with unitarity. ✌

Exercises: Section ??

⑤ Calculate the exact Born cross section for $e^+e^- \to \mu^+\mu^-$, by replacing the hadronic tensor $h_{\mu\nu}$ by the leptonic one $\ell_{\mu\nu}$ of the muon. What is the difference to $e^+e^- \to e^+e^-$?

✎

✌

⑥ Verify that the subtraction changes the DR of the form (8.13), assumed to converge or supplemented with a cut–off, to (8.14), which we call a subtracted DR.

✎

Consider the DR

$$\hat{\Pi}'_\gamma(q^2) = \frac{1}{\pi}\int_{s_0}^{\Lambda^2} \mathrm{d}s\,\frac{Im\,\hat{\Pi}'_\gamma(s)}{s - q^2 - \mathrm{i}\varepsilon}\ ,$$

with $s_0 > 0$, which does not have a limit as $\Lambda \to \infty$ for the photon vacuum polarization, because in QCD $Im\,\hat{\Pi}'_\gamma(s) = R(s)/12\pi)$ and $R(s) \to 5$ as $s \to \infty$. Thus $Im\,\hat{\Pi}'_\gamma(s) \to$ constant as $s \to \infty$ and the integral is logarithmically divergent: (assuming $q^2 < s_0$ for simplicity)

$$\hat{\Pi}'_\gamma(q^2) \sim \frac{R(\infty)}{12\pi^2}\,\ln\frac{\Lambda^2 - q^2}{s_0 - q^2}\ .$$

Since the divergence is logarithmic one subtraction suffices to render the intergal finite. We perform a subtraction at $q^2 = 0$ (charge renormalization in the classical low energy limit; a

subtraction at any other fixed q^2 like $q^2 = m^2$ or the like would remove the UV singularity as well). At $q^2 = 0$ we have

$$\hat{\Pi}'_\gamma(0) = \frac{1}{\pi} \int_{s_0}^{\Lambda^2} \mathrm{d}s \, \frac{Im\, \hat{\Pi}'_\gamma(s)}{s - \mathrm{i}\varepsilon} \ ,$$

here the $\mathrm{i}\varepsilon$ *prescription has no effect because there is no pole in $s > s_0$. The required subtraction yields*

$$\begin{aligned} \hat{\Pi}'_\gamma(q^2) - \hat{\Pi}'_\gamma(0) &= \frac{1}{\pi} \int_{s_0}^{\Lambda^2} \mathrm{d}s \, Im\, \hat{\Pi}'_\gamma(s) \left\{ \frac{1}{s - q^2 - \mathrm{i}\varepsilon} - \frac{1}{s} \right\} \\ &= \frac{q^2}{\pi} \int_{s_0}^{\Lambda^2} \mathrm{d}s \, \frac{Im\, \hat{\Pi}'_\gamma(s)}{s\,(s - q^2 - \mathrm{i}\varepsilon)} \ . \end{aligned}$$

The last version of the integral is absolutely concergent above the pole at $s = q^2$, and the limit $\Lambda \to \infty$ exists and can be taken in the integral. This proves the finitenes of the subtracted DR. q.e.d. ✌

⑦ Discuss the positivity properties of the functional $\dot{P}(s)$ defined as the derivative of

$$P(s) = \fint_{s_0}^{\infty} \mathrm{d}s' \, \frac{R(s')}{s'(s' - s)} \ ,$$

where $R(s)$ is a positive ordinary function. As a distribution (or linear functional) $P(s)$ has a derivative. Formally,

$$\dot{P}(s) = \frac{\mathrm{d}P(s)}{\mathrm{d}s} = \fint \mathrm{d}s' \, \frac{R(s')}{(s' - s)^2} \ .$$

is an integral over an integrand which is almost everywhere positive.

✎
✌

References

[1] G. 't Hooft, M. J. G. Veltman, Nucl. Phys. B **44** (1972) 189.

[2] F. Jegerlehner, Eur. Phys. J. C **18** (2001) 673 [arXiv:hep-th/0005255].

[3] H. B. Nielsen, M. Ninomiya, Phys. Lett. B **105** (1981) 219; Nucl. Phys. B **185** (1981) 20; Erratum: Nucl. Phys. B **195** (1982) 541; Nucl. Phys. B **193** (1981) 173.

Index

B–mesons, 232
B–violation, 16
$U(1)$-axial current, 42
$U(n)_V \otimes U(n)_A$, 45
Υ–resonances, 230
σ-model, 30

Abelian gauge field, 68
adjoint representation, 4
Adler–function, 194, 199
Adler-Bell-Jackiw anomaly, 89
analyticity, 98
anomaly cancellation, 45, 47
anti–screening, 89
asymptotic freedom, 18, 89, 138, 188
asymptotic symmetry, 19
axial-vector anomaly, 42, 44

baryon asymmetry, 16
baryon number
    conservation, 89
bottomonium, 223
broken symmetry, 19

Casimir invariants, 271
causality, 98
charmed mesons, 228
charmonium, 223
chiral
    currents, 61
    perturbation theory, 61–65, 89, 204
    symmetry, 89
    symmetry breaking, 61
chiral fields, 42
chiral group, 18, 19, 45
chiral limit, 31
chiral symmetry, 38, 42
chiral transformations, 42, 43
CKM–matrix, 51
color, 38, 87
color confinement, 38
color singlets, 38
confinement, 18, 85, 89
conformal invariance, 196
conjugate representation, 5
covariant derivative, 72, 77, 79, 87, 201
current
    conserved, 61
    partially conserved, 61
current quarks, 61
current–field identity, 203
currents, 82
curvature tensor, 82
CVC, 61

decoupling by hand, 164
dilatation current, 196
dimension of a representation, 6
dimensional transmutation, 152
dispersion relation, 182
duality
    quark–hadron, 188
dynamical symmetries, 84
dynamical symmetry breaking, 31
Dyson
    series, 116, 125
    summation, 183

electroweak theory, 74
e^+e^- cross–section
    in pQCD, 189–193
equations of motion, 82
equivalence principle, 77
equivalent representation, 5
Euclidean field theory, 98
exponentiation
    Coulomb singularity, 191

Faddeev-Popov
    ghosts, 88
    term, 88
Feynman propagator, 98
Feynman rules
    QCD, 88
    sQED, 201
field strength tensor
    non–Abelian, 87
fine structure constant
    effective, 185
first–class currents, 313
flavor mixing, 51
flavor mixing pattern, 57

G–parity, 291, 313
gauge
    coupling, 87
    fixing, 87

group, 87
parameter, 87
gauge copies, 70
gauge fields, 42
gauge fixing, 69
gauge fixing term, 70
gauge invariance, 68
gauge orbits, 70
gauge parameter, 70
gauge potential, 73
gauge principles, 68
generators of a group, 1
glueballs, 19
gluon radiation, 278
gluons, 18, 87, 189
jet, 194
Goldstone bosons, 26, 28, 64
Goldstone model, 34
Goldstone theorem, 26
GOR relation, 64

hadronization, 193
hadrons, 38
Heisenberg model, 26
Higgs
mechanism, 89

imaginary time, 98, 99
integral
contour, 98, 181
form factor, 101
self–energy, 101
tadpole, 101
interaction
final state, 214
strong, 87
internal symmetry, 1
invariance
scale, 196
irreducible representation, 5

Jacobi identity, 2
jet shape analysis, 273
jets, 194, 269
gluon jet, 194

ladder operators, 2
lattice QCD, 89
lepton number, 293
lepton-quark duality, 45
Lie algebra, 2
local gauge invariance, 68

Majorana neutrino, 18
mass renormalization
quarks, 114
matter fields, 39
minimal
substitution, 201
minimal couplings, 78
minimal substitution, 73, 79
gauge potential, 73
minimal subtraction, 94, 95, 114
scheme, 101

Nambu-Jona-Lasinio model, 30
neutrino–oscillations, 17
non–perturbative effects, 195
non-Abelian field strength tensor, 80
non-Abelian gauge fields, 79

one loop integrals
scalar, 100, 101
one–particle irreducible, 117
OPE, 64, 195
operator product expansion , see OPE 64
order of a group, 1
order parameter, 26
OZI–rule, 251

parallel displacement, 77
particles
unstable, 205
partons, 194
PCAC, 61
permutations, 5
perturbation theory
chiral (CHPT), 61
potential models, 233
production threshold, 102
pseudo Goldstone boson, 20
pure Yang-Mills theory, 81

QCD, 5, 16, 74, 85
gauge structure, 270
renormalization group, 146
QED
Noether theorem, 71
Quantum Chromodynamics, 87
quantum chromodynamics, 18
quantum flavordynamics (QFD), 74
quantum numbers: color, 18
quark

condensates, 65, 195
quark flavor mixing, 38
quark mixing, 51
quark parton model, 189
quarkonia
decays, 251
quarkonium, 223
hyperfine splitting, 263
quarks, 87, 189

radiative corrections, 111
rank of a group, 2
reducible representation, 5
renormalizability, 87
renormalization
scale, 101
wave function, 183
representation of a group, 4
resonance, 194
Breit-Wigner, 209
narrow width, 209
$\rho - \omega$ mixing, 209
Roy equation, 214
running charge, 183

Schwarz reflection principle, 181
second–class currents, 313
selection rule, 293
self–energy
quark, 111
SM, 74
space–like, 185
spin-statistics crisis, 38
spontaneous symmetry breaking, 26
Standard Model, 38, 74
Sterman-Weinberg jets, 272
symmetry class, 5

tensor
decomposition, 94
energy momentum, 196
integral, 103
vacuum polarization, 117
tensor product of representations, 5
theorem
Cauchy's, 181
optical, 179, 202
Thomson limit, 183
threshold, 194
triangle anomaly, 47

unimodular group, 1
unitary group, 1
unitary groups, 1

Van Royen - Weisskopf formula, 252–254
VMD model, 201

wave function renormalization, 115
weak hypercharge, 40
weak interaction
quantum numbers, 40
weak isospin, 39
weight diagram, 2
Wess-Zumino-Witten Lagrangian, 65
Wess-Zumino-Witten Lagrangian, 314
Wick
rotation, 98
width
finite, 205

Yang-Mills fields, 79
Young tableau, 5
Yukawa interaction, 51
Yukawa sector, 51